TWMS J. App. and Eng. Math. V.15, N.10, 2025, pp. 2421-2438

# EXTENSIVE EXPLORATION OF MULTI-TERM HYBRID FUNCTIONAL EQUATION VIA HYBRID DIFFERENTIAL FEEDBACK CONTROL

A. M. A. EL-SAYED[1], SH. M. AL-ISSA[2,3*], H. H. G. HASHEM[1], I. H. KADDOURA[2,3], A. A. NAJDI[2], §

Abstract. This paper investigates the existence of solutions for a multidimensional hybrid functional equation with multiple delays, incorporating differential feedback control. The focus is on finding well-defined, continuous, and bounded solutions on the semi-infinite interval. To establish the existence of these solutions, we employ measures of noncompactness associated with a specified modulus of continuity within the space $BC(\mathbb{R}_+)$. Furthermore, we derive sufficient conditions to ensure the asymptotic stability of the solutions to this integral equation. An illustrative example is provided to demonstrate the applicability of the theoretical results.

Keywords: multi-term hybrid functional equation, existence of continuous solution, a measure of noncompactness; Darbofixed-point theorem, differential feedback control.

AMS Subject Classification: 74H10; 45G10

## 1. Introduction

Incorporating control variables into such problems is increasingly important, especially in modeling real-world phenomena, where unexpected disruptions can significantly impact ecosystems. In ecology, a critical issue is understanding whether an ecosystem can withstand temporary disturbances that may alter biological characteristics, such as survival rates. These disruptions are modeled through control variables, which represent external disturbances. In [10], Chen derived averaged conditions for the persistence and

[1] Faculty of Science, Department of Mathematics, Alexandria University, Alexandria, Egypt.
e-mail: amasayed@alexu.edu.eg; ORCID no.https://orcid.org/0000-0001-7092-7950.
e-mail: hindhghashem@gmail.com; ORCID no. https://orcid.org/0000-0002-9921-832X.
[2] Faculty of Arts and Sciences, Department of Mathematics, Lebanese International University, Saida, Lebanon.
e-mail: shorouk.alissa@liu.edu.lb; ORCID no. https://orcid.org/0000-0002-1488-0817.
e-mail: issam.kaddoura@liu.edu.lb; ORCID no. https://orcid.org/0000-0003-3627-6941.
e-mail: Ali Najdi 32310039@students.liu.edu.lb ; ORCID no. https://orcid.org/0009-0004-4225-5762.
[3] Faculty of Arts and Sciences, Department of Mathematics, The International University of Beirut, Beirut, Lebanon.
* Corresponding author.
§ Manuscript received: September 29, 2024; accepted: February 14, 2025.

global attractivity of a nonautonomous feedback-controlled Lotka-Volterra system using an appropriately constructed Lyapunov functional.

The significance of feedback control in functional equations has also been explored in various contexts. Nasertayoob et al. [11] studied a class of nonlinear functional-integral equations involving feedback control, demonstrating the existence of solutions that are both asymptotically stable and globally attractive by applying the measure of noncompactness and Darbo's fixed point theorem. Furthermore, under suitable conditions, the existence of positive periodic solutions for nonlinear neutral delay population systems with feedback control has been investigated [12], with the proof relying on the fixed-point theorem for strict-set-contraction operators.

El-Sayed et al. [7] considered a functional integral equation involving a control parameter that satisfies a multivalued feedback control, while [8] extended these results to study nonlinear functional integral equations constrained by fractal feedback control.

In this article, we establish the existence and asymptotic stability of solutions for a class of multidimensional functional equations, which take the form

$$\begin{aligned}\frac{\mu(\sigma)-\eta_1(\sigma,\mu(\sigma))}{\eta_2(\sigma,\mu(\sigma))} &= g_1(\sigma, v(\sigma), \frac{\mu(\varphi(\sigma))-\eta_1(\sigma,\mu(\varphi(\sigma)))}{\eta_2(\sigma,\mu(\varphi(\sigma)))}) \qquad (1)\\ &\times g_2\Bigg(\sigma, \frac{\mu(\phi_1(\sigma))-\eta_1(\sigma,\mu(\phi_1(\sigma)))}{\eta_2(\sigma,\mu(\phi_1(\sigma)))}\int_0^{\psi_1(\sigma)} h_1(\sigma,\tau,\frac{\mu(\tau)-\eta_1(\tau,\mu(\tau))}{\eta_2(\tau,\mu(\tau))})\ d\tau,\\ &...., \frac{\mu(\phi_k(\sigma))-\eta_1(\sigma,\mu(\phi_k(\sigma)))}{\eta_2(\sigma,\mu(\phi_k(\sigma)))}\int_0^{\psi_k(\sigma)} h_k(\sigma,\tau,\frac{\mu(\tau)-\eta_1(\tau,\mu(\tau))}{\eta_2(\tau,\mu(\tau))})\ d\tau\Bigg),\end{aligned}$$

subject to a hybrid differential feedback control described by

$$\frac{dv_x(\sigma)}{d\sigma} = -\lambda v_x(\sigma) + f\Bigg(\sigma, v_x(\sigma), \frac{\mu(\sigma)-\eta_1(\sigma,\mu(\sigma))}{\eta_2(\sigma,\mu(\sigma))}\Bigg),\ v_0 = v_x(0),\ \lambda \geq 0,\ \sigma \in \mathbb{R}_+. \qquad (2)$$

In our approach, we utilize the method of measures of noncompactness to prove the existence of solutions for this multidimensional functional equation in the space of bounded continuous functions on the positive real line, denoted $BC(\mathbb{R}_+)$. The analysis primarily relies on Darbo's fixed point theorem [6] and techniques involving the measure of noncompactness.

The measure of noncompactness and Darbo's fixed point theorem have proven to be effective tools for analyzing nonlinear functional-integral equations, which are pertinent to various fields, including vehicular traffic theory, biology, and queuing theory [4]. The application of noncompactness measures in the Banach space $BC(\mathbb{R}_+)$, which consists of all bounded and continuous functions on the positive real line, was pioneered by J. Banaś [5].This approach has been widely employed to establish the existence of asymptotically stable solutions for integral and quadratic integral equations [4, 5]. Further discussions on solvability in the half-axis domain can be found in [1, 2].

The structure of the article is as follows: Section 2 provides essential background information and elaborates on the concept of the measure of noncompactness. In Section 3, we introduce and demonstrate an existence theorem for multidimensional functional equations with multiple delays, along with an analysis of the asymptotic stability of the solutions. Lastly, Section 4 presents an illustrative example to support the theoretical results.

## 2. Background and Auxiliary Facts

In this section, we review key foundational concepts and auxiliary information related to the theory of noncompactness in Banach spaces. Before proceeding with our discussion on the space $BC(\mathbb{R}_+)$, we first define the function space in which solutions exist.

Let $E$ be a Banach space equipped with the norm $||\cdot||$. We consider the space of bounded continuous functions on $\mathbb{R}_+$, denoted by $BC(\mathbb{R}_+)$, which consists of all functions $u : \mathbb{R}_+ \to \mathbb{R}$ such that $u(\sigma)$ is continuous and there exists a constant $M > 0$ satisfying $|u(\sigma)| \leq M$ for all $\sigma \in \mathbb{R}_+$. The norm in this space is given by:

$$||u|| = \sup_{\sigma\in\mathbb{R}_+} |u(\sigma)|,$$

which ensures that $(BC(\mathbb{R}_+), ||\cdot||)$ is a complete normed space.

We now define the Hausdorff measure of noncompactness, denoted by $\chi(X)$, for a nonempty, bounded subset $X \subset E$:

$$\chi(X) = \inf\{\epsilon > 0 : X \text{ admits a finite } \epsilon\text{-net in } E\}.$$

We then extend this to the space of bounded continuous functions, $BC(\mathbb{R}_+)$, which was investigated in earlier studies. Let $X \subset BC(\mathbb{R}_+)$ be a nonempty, bounded subset, and let $T > 0$ be a given constant. For any function $v \in X$ and for $\epsilon > 0$, we define the modulus of continuity over the interval $[0, T]$ as:

$$\omega^T(v, \epsilon) = \sup\{|v(\sigma) - v(\tau)| : \sigma, \tau \in [0, T], |\sigma - \tau| \leq \epsilon\}.$$

Additional notations related to the measure of noncompactness are given as follows:

$$\omega^T(X, \epsilon) = \sup\{\omega^T(v, \epsilon) : v \in X\}, \quad \omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X, \epsilon),$$

and

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

It has been shown that the measure of noncompactness can be expressed as:

$$\chi(X) = \frac{1}{2}\omega_0(X).$$

Furthermore, we define the diameter of the set $X$ at a given time $\sigma$ as:

$$\text{diam } X(\sigma) = \sup\{|u(\sigma) - v(\sigma)| : u, v \in X\}.$$

The measure of noncompactness for functions in the space $BC(\mathbb{R}_+)$ has been explored in detail in various works. Additionally, we introduce a function $\mu$, which is defined on the family of subsets $\mathcal{M}_{BC(\mathbb{R}_+)}$, as follows:

$$\mu(X) = \omega_0(X) + \lim_{\sigma\to\infty} \sup \text{diam } X(\sigma).$$

Next, we present a fixed-point result, known as Darbo's fixed point theorem, which will be instrumental in proving the existence of solutions in subsequent sections (see [17]).

**Theorem 2.1.** *Let $M$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$, and let $F : M \to M$ be a continuous operator satisfying $\mu(F(X)) \leq l\mu(X)$ for any subset $X \subset M$, where $l \in [0, 1)$ is a constant. Then, the operator $F$ has a fixed point in $M$.*

## 3. Multi-term Equation with Multiple Delays

Consider the equation

$$x(\sigma) = \frac{\mu(\sigma) - \eta_1(\sigma, \mu(\sigma))}{\eta_2(\sigma, \mu(\sigma))}, \tag{3}$$

from which any solution of the functional equation (1) can be expressed as:

$$\mu(\sigma) = \eta_1(\sigma, x(\sigma)) + x(\sigma)\eta_2(\sigma, \mu(\sigma)), \quad \sigma \geq 0, \tag{4}$$

where $x(\sigma)$ satisfies the following multidimensional equation:

$$x(\sigma) = g_1\left(\sigma, v_x(\sigma), x(\varphi(\sigma))\right) g_2\Bigg(\sigma, x(\phi_1(\sigma)) \int_0^{\psi_1(\sigma)} h_1(\sigma, \tau, x(\tau))\, d\tau, \dots, \tag{5}$$
$$\dots, x(\phi_k(\sigma)) \int_0^{\psi_k(\sigma)} h_k(\sigma, \tau, x(\tau))\, d\tau\Bigg), \quad \sigma \in [0.\infty),$$

accompanied by the differential feedback control condition:

$$\frac{dv_x(\sigma)}{d\sigma} = -\lambda v_x(\sigma) + f\left(\sigma, v_x(\sigma), x(\sigma)\right), \quad v_0 = v_x(0), \quad \lambda \geq 0. \tag{6}$$

We analyze the multidimensional functional equation (5) under the following assumptions:

(i) The functions $\varphi, \phi_i, \psi_i : [0, \infty) \to [0, \infty)$ (for $i = 1, 2, \dots, k$) are continuous and satisfy the condition $\varphi(\sigma), \phi_i(\sigma), \psi_i(\sigma) \leq \sigma$.

(ii) The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable in its arguments. Additionally, there exists a bounded measurable function $n : [0, \infty) \to \mathbb{R}$ and a constant $m > 0$ such that:

$$|f(\sigma, x_1, y_1) - f(\sigma, x_2, y_2)| \leq m\left(|x_1 - x_2| + |y_1 - y_2|\right),$$

$$|f(\sigma, x, y)| \leq |n(\sigma)| + m\left(|x| + |y|\right),$$

with $|n(\sigma)| = |f(\sigma, 0, 0)|$, and

$$\lim_{\sigma\to\infty} \int_0^\sigma |n(s)|\, ds = 0, \quad \sup_{\sigma\in\mathbb{R}_+} \int_0^\sigma |n(s)|\, ds \leq N.$$

(iii) The function $g_2 : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition:

$$|g_2(\sigma, x_1, \dots, x_k) - g_2(\sigma, y_1, \dots, y_k)| \leq l \sum_{i=1}^k |x_i - y_i|$$

for all $\sigma \in \mathbb{R}_+$ and $x_i, y_i \in \mathbb{R}$. Additionally, the function $g_2(\sigma, 0, \dots, 0)$ belongs to $BC(\mathbb{R}_+)$, with

$$|g_2(\sigma, x_1, \dots, x_k)| \leq l \sum_{i=1}^k |x_i| + G_2(\sigma),$$

where $G_2(\sigma) = |g_2(\sigma, 0, \dots, 0)| \in BC(\mathbb{R}_+)$ and

$$\lim_{\sigma\to\infty} G_2(\sigma) = 0, \quad G_2 = \sup_{\sigma\in\mathbb{R}_+} G_2(\sigma) < \infty.$$

(iv) The function $g_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, with a positive constant $\eta$, such that:

$$|g_1(\sigma, x_1, x_2) - g_1(\sigma, y_1, y_2)| \leq \eta(|x_1 - y_1| + |x_2 - y_2|)$$

for all $\sigma \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Moreover, $g_1(\sigma, 0, 0) \in BC(\mathbb{R}_+)$, and

$$|g_1(\sigma, x_1, x_2)| \leq \eta(|x_1| + |x_2|) + G_1(\sigma),$$

where $G_1(\sigma) = |g_1(\sigma, 0, 0)| \in BC(\mathbb{R}_+)$, with

$$\lim_{\sigma\to\infty} G_1(\sigma) = 0, \quad G_1 = \sup_{\sigma\in\mathbb{R}_+} G_1(\sigma) < \infty.$$

(v) The functions $h_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ (for $i = 1, 2, \ldots, k$) are Carathéodory functions, measurable in $\sigma \in \mathbb{R}_+$ for all $x \in \mathbb{R}$, and continuous in $x$ for all $\sigma \in \mathbb{R}_+$. Additionally, there exist measurable and bounded functions $\alpha_i, \beta_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$|h_i(\sigma, \tau, x)| \leq \alpha_i(\sigma, \tau) + \beta_i(\sigma, \tau)|x|,$$

for $\sigma, \tau \in [0, \infty)$, and

$$\lim_{\sigma\to\infty} \int_0^\sigma \alpha_i(\sigma, \tau)\, d\tau = 0, \quad \sup_{\sigma\in[0,T]} \int_0^\sigma \alpha_i(\sigma, \tau)\, d\tau = \alpha_i,$$

$$\lim_{\sigma\to\infty} \int_0^\sigma \beta_i(\sigma, \tau)\, d\tau = 0, \quad \sup_{\sigma\in[0,T]} \int_0^\sigma \beta_i(\sigma, \tau)\, d\tau = \beta_i.$$

Let $\alpha = \sup\{\alpha_i\}$ and $\beta = \sup\{\beta_i\}$.

(vi) There exists a positive solution $r$ of the following equation:

$$nlk(\aleph_2 + 1)\beta r^3 + (G_1lk\beta + nlk(\aleph + \aleph_1 + \aleph_3)\beta + nlk(\aleph_2 + 1)\alpha)\, r^2$$
$$+ [(G_1lk\alpha + nG_2(\aleph_2 + 1) + nlk(\aleph + \aleph_1 + \aleph_3)\alpha) - 1]r + G_1G_2 + nG_2(\aleph + \aleph_1 + \aleph_3) = 0,$$

such that

$$\eta\ (\Delta+1)\big[\ G_2+l\ k\ r|[\alpha+\beta\ r]\big]+l\ k\ [G_1+\eta\ (\aleph+\aleph_1+\aleph_2\ r+\aleph_3+r)][\alpha+\beta\ r]<1.$$

**Lemma 3.1.** *Let assumption (ii) hold, if a solution to the differential feedback control variable in (2) exists, it can be expressed as the solution of the following integral equation:*

$$v_x(\sigma) = v_0\ e^{-\lambda\sigma} + \int_0^\sigma e^{-\lambda(\sigma-\tau)} f(\tau, v(\tau), x(\tau))\ d\tau, \quad v_0 = v_x(0), \tag{7}$$

*where $v_x(\sigma)$ is the solution.*

*Proof.* Starting from the differential feedback control (2), we have:

$$\frac{dv_x(\sigma)}{d\sigma} = -\lambda v_x(\sigma) + f(\sigma, v(\sigma), x(\sigma)).$$

Multiplying both sides by $e^{\lambda\sigma}$, we get:

$$e^{\lambda\sigma}\frac{dv_x(\sigma)}{d\sigma} + \lambda e^{\lambda\sigma} v_x(\sigma) = e^{\lambda\sigma} f(\sigma, v(\sigma), x(\sigma)).$$

Then

$$\frac{d}{d\sigma}\left(v_x(\sigma)e^{\lambda\sigma}\right) = e^{\lambda\sigma} f(\sigma, v(\sigma), x(\sigma)).$$

Integrating both sides with respect to $\sigma$:

$$v_x(\sigma)e^{\lambda\sigma} = v_x(0) + \int_0^\sigma e^{\lambda\tau} f(\tau, v(\tau), x(\tau))\ d\tau.$$

Thus, solving for $v_x(\sigma)$:

$$v_x(\sigma) = v_0\, e^{-\lambda\sigma} + \int_0^\sigma e^{-\lambda(\sigma-\tau)} f(\tau, v(\tau), x(\tau))\, d\tau,$$

as expressed in (7).

Now, for any real-valued function $x(\sigma)$, the solution $v_x(\sigma)$ is bounded. If $v_0 > 0$, we have:

$$\begin{aligned}
|v_x(\sigma)| &\le |v_0| e^{-\lambda\sigma} + \int_0^\sigma e^{-\lambda(\sigma-\tau)} \left[m(|v_x(\tau)| + |x(\tau)|) + n(\tau)\right] d\tau \\
&\le \aleph + \sup_{\sigma\in I}\left(\int_0^\sigma e^{-\lambda(\sigma-\tau)} m|v_x(\tau)| d\tau\right) + \sup_{\tau\in I}\left(\int_0^\sigma e^{-\lambda(\sigma-\tau)} m\|x\| d\tau\right) \\
&+ \sup_{\tau\in I}\left(\int_0^\sigma e^{-\lambda(\sigma-\tau)} n(\tau) d\tau\right) \\
&\le \aleph + \aleph_1 + \aleph_2 r + \aleph_3,
\end{aligned}$$

where the constants $\aleph, \aleph_1, \aleph_2, \aleph_3$ are defined as:

$$\begin{aligned}
\lim_{\sigma\to\infty} v_0\, e^{-\lambda\sigma} &= 0, \quad \sup_{\sigma\in\mathbb{R}_+} v_0\, e^{-\lambda\sigma} = \aleph, \\
\lim_{\sigma\to\infty} \int_0^\sigma e^{-\lambda(\sigma-\tau)} m|v_x(\tau)| d\tau &= 0, \quad \sup_{\sigma\in\mathbb{R}_+} \int_0^\sigma e^{-\lambda(\sigma-\tau)} m|v_x(\tau)| d\tau = \aleph_1, \\
\lim_{\sigma\to\infty} \int_0^\sigma e^{-\lambda(\sigma-\tau)} m\|x\| d\tau &= 0, \quad \sup_{\sigma\in\mathbb{R}_+} \int_0^\sigma e^{-\lambda(\sigma-\tau)} m\|x\| d\tau = \aleph_2, \\
\lim_{\sigma\to\infty} \int_0^\sigma e^{-\lambda(\sigma-\tau)} n(\tau) d\tau &= 0, \quad \sup_{\sigma\in\mathbb{R}_+} \int_0^\sigma e^{-\lambda(\sigma-\tau)} n(\tau) d\tau = \aleph_3.
\end{aligned}$$

Additionally, for two distinct functions $x_1(\sigma)$ and $x_2(\sigma)$, the difference in solutions satisfies:

$$\begin{aligned}
|v_{x_1}(\sigma) - v_{x_2}(\sigma)| &\le \int_0^\sigma e^{-\lambda(\sigma-\tau)} \left|f(\tau, v_{x_1}(\tau), x_1(\tau)) - f(\tau, v_{x_2}(\tau), x_2(\tau))\right| d\tau \\
&\le m \int_0^\sigma e^{-\lambda\tau} \left(|v_{x_1}(\tau) - v_{x_2}(\tau)| + |x_1(\tau) - x_2(\tau)|\right) d\tau \\
&\le \frac{m}{\lambda}\left(\|v_{x_1} - v_{x_2}\| + \|x_1 - x_2\|\right).
\end{aligned}$$

Thus, the bound for the difference between solutions is:

$$\|v_{x_1} - v_{x_2}\| \le \Delta\|x_1 - x_2\|,$$

where $\Delta = \frac{m}{\lambda(1-\ m/\lambda)}$ and $m/\lambda < 1$. □

**Theorem 3.1.** *Assume that conditions (i)-(vi) are satisfied. Then the multidimensional functional equation* (5) *admits at least one solution $x = x(\sigma)$ in the space $BC(\mathbb{R}_+)$.*

*Proof.* Let us define the operator $\mathbb{F}$ as follows:

$$\begin{aligned}
&\mathbb{F}x(\sigma) = g_1(\sigma, v_x(\sigma), x(\varphi(\sigma))) \\
&\quad\times g_2\left(\sigma, x(\phi_1(\sigma)) \int_0^{\psi_1(\sigma)} h_1(\sigma,\tau,x(\tau))\, d\tau, \ldots, x(\phi_k(\sigma)) \int_0^{\psi_k(\sigma)} h_k(\sigma,\tau,x(\tau))\, d\tau\right),
\end{aligned}$$

$\sigma \in \mathbb{R}_+$. By assumptions $(iii)$ and $(iv)$, the functions $g_1(\sigma, v_x, x)$ and $g_2(\sigma, x_1, x_2, \ldots, x_k)$ are continuous, and thus, $\mathbb{F}x \in C(\mathbb{R}_+, \mathbb{R})$.

We will now show that for some $r > 0$, the operator $\mathbb{F}$ maps the ball $B_r$ into itself. Specifically, we have

$$\begin{aligned}\big|\mathbb{F}x(\sigma)\big| &= \bigg| g_1(\sigma, v_x(\sigma), x(\varphi(\sigma)))\\ &\times g_2\bigg(\sigma, x(\phi_1(\sigma))\int_0^{\psi_1(\sigma)} h_1(\sigma,\tau,x(\tau))\,d\tau, \dots, x(\phi_k(\sigma))\int_0^{\psi_k(\sigma)} h_k(\sigma,\tau,x(\tau))\,d\tau\bigg)\bigg|\\ &\le \left[|g_1(\sigma,0,0)| + n\left(|v_x(\sigma)| + |x(\varphi(\sigma))|\right)\right]\\ &\times \left[|g_2(\sigma,0,\dots,0)| + l\sum_{i=1}^{k}|x(\phi_i(\sigma))|\int_0^{\psi_i(\sigma)}\left(\alpha_i(\sigma,\tau)+\beta_i(\sigma,\tau)|x(\tau)|\right)d\tau\right]\\ &\le \left[G_1 + n\left(\aleph + \aleph_1 + \aleph_2 r + \aleph_3 + r\right)\right]\left[G_2 + lrk(\alpha+\beta r)\right] = r.\end{aligned}$$

Considering assumption $(vi)$ and the estimate above, we conclude that the operator $\mathbb{F}$ maps the ball $B_r$ into itself, provided there exists a positive solution $r = r_0$ to the equation

$$\begin{aligned}&nlk(\aleph_2+1)\beta r^3 + (G_1 lk\beta + nlk(\aleph+\aleph_1+\aleph_3)\beta + nlk(\aleph_2+1)\alpha)\,r^2\\ &+ [(G_1lk\alpha + nG_2(\aleph_2+1) + nlk(\aleph+\aleph_1+\aleph_3)\alpha) - 1]r + G_1G_2 + nG_2(\aleph+\aleph_1+\aleph_3) = 0.\end{aligned}$$

Next, we show that the operator $\mathbb{F}$ is continuous on the ball $B_r$. Let $\epsilon > 0$ and take $x, y \in B_r$ such that $\|x - y\| \le \epsilon$. For $\sigma \in \mathbb{R}_+$, we have:

$$\begin{aligned}\big|\mathbb{F}x(\sigma) - \mathbb{F}y(\sigma)\big| &\le \eta\left(|v_x(\sigma) - v_y(\sigma)| + |x(\varphi(\sigma)) - y(\varphi(\sigma))|\right)\\ &\cdot\left(G_2 + l\sum_{i=1}^{k}|x(\phi_i(\sigma))|\int_0^{\psi_i(\sigma)}\left(\alpha_i(\sigma,\tau)+\beta_i(\sigma,\tau)\|x\|\right)d\tau\right)\\ &+ l\left(G_1 + \eta\left(|v_y(\sigma)| + \|y\|\right)\right)\\ &\cdot\sum_{i=1}^{k}\left(|x(\phi_i(\sigma)) - y(\phi_i(\sigma))|\int_0^{\psi_i(\sigma)}\left(\alpha_i(\sigma,\tau)+\beta_i(\sigma,\tau)\|x\|\right)d\tau\right)\\ &+ \sum_{i=1}^{k}|y(\phi_i(\sigma))|\int_0^{\psi_i(\sigma)}\big|\ h_i(\sigma,\tau,x(\tau)) - h_i(\sigma,\tau,y(\tau))\ \big|d\tau\bigg].\end{aligned}$$

Consider the following two cases:

**Case i)**: Let $T > 0$ be chosen such that for $\sigma \ge T$, the following inequalities hold:

$$k\ r\int_0^{\sigma}[\alpha_i(\sigma,\tau) + \beta_i(\sigma,\tau)\|x\|]d\tau \le \epsilon_1 \text{ and } k\ r\int_0^{\sigma}[\alpha_i(\sigma,\tau) + \beta_i(\sigma,\tau)\|y\|]d\tau \le \epsilon_2.$$

We then obtain

$$\begin{aligned}&|\mathbb{F}(x(\sigma)) - \mathbb{F}(y(\sigma))| \le \eta(\Delta+1)\|x-y\|\left[G_2 + lkr\int_0^{\sigma}[\alpha_i(\sigma,\tau)+\beta_i(\sigma,\tau)r]d\tau\right]\\ &+ l\left[G_1 + \eta(\aleph+\aleph_1+\aleph_2 r+\aleph_3+r)\right]k\|x-y\|\int_0^{\sigma}[\alpha_i(\sigma,\tau)+\beta_i(\sigma,\tau)r]d\tau\\ &+ kr\int_0^{\sigma}\alpha_i(\sigma,\tau)+\beta_i(\sigma,\tau)[\|x\|+\|y\|]d\tau\\ &\le \left(\eta(\Delta+1)\left[G_2 + lkr(\alpha_i+\beta_i r)\right] + l\left[G_1+\eta(\aleph+\aleph_1+\aleph_2 r+\aleph_3+r)\right]k(\alpha_i+\beta_i r)\right)\\ &\le \epsilon + \epsilon_1 + \epsilon_2.\end{aligned}$$

**Case ii)**: For $\sigma \le T$, define the function $\omega^T(h_i, \epsilon)$ for $i = 1, 2, \dots, k$, where for $\epsilon > 0$, we have

$$\omega^T(h_i,\epsilon) = \sup\left\{|h_i(\sigma,\tau,x(\tau)) - h_i(\sigma,\tau,y(\tau))| : \sigma,\tau \in [0,T], \|x-y\| \le \epsilon\right\}.$$

Using the uniform continuity of $h$, we know that $\omega^T(h,\epsilon)\to 0$ as $\epsilon\to 0$. Therefore, by the above estimates, we get

$$\begin{aligned}
&|\mathbb{F}(x(\sigma))-\mathbb{F}(y(\sigma))| \le \eta(\Delta+1)\|x-y\|\left[G_2+lkr(\alpha+\beta r)\right]\\
&+l\left[G_1+\eta(\aleph+\aleph_1+\aleph_2 r+\aleph_3+r)\right]k\left[\|x-y\|(\alpha+\beta r)+kr\omega^T(h_i,\epsilon)\right]\\
&\le\left[\eta(\Delta+1)\left[G_2+lkr(\alpha+\beta r)\right]+l\left[G_1+\eta(\aleph+\aleph_1+\aleph_2 r+\aleph_3+r)\right]k(\alpha+\beta r)\right]\epsilon\le\epsilon.
\end{aligned}$$

**Conclusion**: From cases (i) and (ii), we conclude that the operator $\mathbb{F}$ is continuous on the ball $B_r$, mapping $B_r$ into itself.

Now, consider a non-empty subset $X$ of $B_r$. Fix $\epsilon>0$ and choose $x\in X$ and $\sigma_1,\sigma_2\in\mathbb{R}_+$ such that $|\sigma_2-\sigma_1|\le\epsilon$. Without loss of generality, assume $\sigma_1\le\sigma_2$. Then

$$\begin{aligned}
& \mid \mathbb{F}x(\sigma_2)-\mathbb{F}x(\sigma_1)\mid\\
\le\; & \Big| g_1(\sigma_2,v_x(\sigma_2),x(\varphi(\sigma_2))\\
\times\; & g_2(\sigma_2,x(\phi_1(\sigma_2))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\tau)))\int_0^{\psi_k(\sigma_2)}h_k(\sigma_2,\tau,x(\tau))d\tau\\
-\; & g_1(\sigma_1,v_x(\sigma_1),x(\varphi(\sigma_1)))\\
\times\; & g_2(\sigma_2,x(\phi_1(\sigma_2))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\tau)))\int_0^{\psi_k(\sigma_2)}h_k(\sigma_2,\tau,x(\tau))d\tau\Big|\\
+\; & \Big| g_1(\sigma_1,v_x(\sigma_1),x(\varphi(\sigma_1)))\\
\times\; & g_2(\sigma_2,x(\phi_1(\sigma_2))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\tau)))\int_0^{\psi_k(\sigma_2)}h_k(\sigma_2,\tau,x(\tau))d\tau\\
-\; & g_1(\sigma_1,v_x(\sigma_1),x(\varphi(\sigma_1)))\\
\times\; & g_2(\sigma_1,x(\phi_1(\sigma_1))\int_0^{\psi(\sigma_1)}h_1(\sigma_1,\tau,x(\tau))d\tau,....,x(\phi_k(\tau)))\int_0^{\psi_k(\sigma_1)}h_k(\sigma_1,\tau,x(\tau))d\tau\Big|\\
\le\; & \big|g_1(\sigma_2,v_x(\sigma_2),x(\varphi(\sigma_2)))-g_1(\sigma_1,v_x(\sigma_1),x(\varphi(\sigma_1)))\big|\\
\times\; & \big|g_2(\sigma_2,x(\phi_1(\sigma_2))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\sigma)))\int_0^{\psi_k(\sigma)}h_k(\sigma_2,\tau,x(\tau))d\tau\big|\\
+\; & |g_1(\sigma_1,v_x(\sigma_1),x(\varphi(\sigma_1)))|\\
\cdot\; & \Big[\big|g_2(\sigma_2,x(\phi_1(\sigma_2))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\sigma_2)))\int_0^{\psi_k(\sigma_2)}h_k(\sigma_2,\tau,x(\tau))d\tau\\
-\; & g_2(\sigma_1,x(\phi_1(\sigma_1))\int_0^{\psi_1(\sigma_1)}h_1(\sigma_1,\tau,x(\tau))d\tau,....,x(\phi_k(\sigma_1)))\int_0^{\psi_k(\sigma_1)}h_k(\sigma_1,\tau,x(\tau))d\tau\big|\ \Big]\\
\le\; & \big[|g_1(\sigma_2,v_x(\sigma_2),x(\varphi((\sigma_2)))-g_1(\sigma_1,v_x(\sigma_2),x(\varphi(\sigma_2)))|\\
+\; & |g_1(\sigma_1,v_x(\sigma_2),x(\varphi(\sigma_2)))-g_1(\sigma_1,v_x(\sigma_1),x(\varphi(\sigma_1)))|\big]\\
\times\; & \Big[|g_2(\sigma_2,0,0,....,0)|+l\sum_{i=1}^{k}|x(\phi_i(\sigma_2))|\int_0^{\psi_i(\sigma_2)}|h_i(\sigma_2,\tau,x(\tau))|d\tau\Big]\\
+\; & \big[|g_1(\sigma_1,0,0)|+\eta(|v_x(\sigma_1)|+|x(\varphi((\sigma_1))|\big]\\
\times\; & \Big[\big|g_2(\sigma_2,x(\phi_i((\sigma_2)))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\sigma_2)))\int_0^{\psi_k(\sigma_2)}h_k(\sigma_2,\tau,x(\tau))d\tau\\
-\; & g_2(\sigma_1,x(\phi_1(\sigma_2))\int_0^{\psi_1(\sigma_2)}h_1(\sigma_2,\tau,x(\tau))d\tau,....,x(\phi_k(\sigma_2)))\int_0^{\psi_k(\sigma_2)}h_k(\sigma_2,\tau,x(\tau))d\tau)\big|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad g_2(\sigma_1, x(\phi_i((\sigma_2))) \int_0^{\psi_1(\sigma_2)} h_1(\sigma_2, \tau, x(\tau))d\tau, ...., x(\phi_k(\sigma_2))) \int_0^{\psi_k(\sigma_2)} h_k(\sigma_2, (\tau, x(\tau))d\tau) \\
&- \quad g_2(\sigma_1, x(\phi_i(\sigma_1)) \int_0^{\psi_1(\sigma_1)} h_1(\sigma_1, \tau, x(\tau))d\tau, ...., x(\phi_k(\sigma_1))) \int_0^{\psi_k(\sigma_1)} h_k(\sigma_1, \tau, x(\tau))d\tau) \Big| \Big] \\
&\le \quad [\theta_{g_1}(\delta) + \eta(|v_x(\sigma_2) - v_x(\sigma_1)| + |x(\varphi(\sigma_2)) - x(\varphi(\sigma_1))|)] \\
&\times \quad \big[\, G_2 + l \sum_{i=1}^{k} |x(\phi_i(\sigma_2))| \int_0^{\psi_i(\sigma_2)} [\alpha_i(\sigma, \tau) + \beta_i(\sigma, \tau)|x(\tau))|]d\tau \big] \\
&+ \quad [G_1 + \eta(|v_x((\sigma_1))| + |x(\varphi(\sigma_1))|)] \\
&\times \quad \big[\theta_{g_2}(\delta) + \ l \ \sum_{i=1}^{k} [|x(\phi_i(\sigma_2)) \int_0^{\psi_i(\sigma_2)} h_i(\sigma_2, \tau, x(\tau))d\tau - x(\phi_i(\sigma_1)) \int_0^{\psi_i(\sigma_1)} h_i(\sigma_1, \tau, x(\tau))d\tau|] \\
&\le \quad [\theta_{g_1}(\delta) + \eta\theta_{v_x}(\delta) + \eta|x(\varphi(\sigma_2)) - x(\varphi(\sigma_1))|] \\
&. \quad \big[\, G_2 + l \sum_{i=1}^{k} |x(\phi_i(\sigma_2))| \int_0^{\psi_i(\sigma_2)} [\alpha_i(\sigma, \tau) + \beta_i(\sigma, \tau)|x(\tau))|]d\tau \big] \\
&\qquad + [G_1 + \eta(\aleph + \aleph_1 + \aleph_2 \ r + \aleph_3 + |x(\varphi(\sigma_1))|))] \\
&\times \quad \big[\theta_{g_2}(\delta) + l \ \sum_{i=1}^{k} |x(\phi_i(\sigma_2))| \int_0^{\psi_i(\sigma_2)} [h_i(\sigma_2, \tau, x(\tau)) - h_i(\sigma_1, \tau, x(\tau))]d\tau \\
&+ \quad l \ \sum_{i=1}^{k} |x(\phi_i(\sigma_2)) - x(\phi_i(\sigma_1))| \int_0^{\psi_i(\sigma_1)} |h_i(\sigma_1, \tau, x(\tau))|d\tau \\
&+ \quad l \ \sum_{i=1}^{k} \big|x(\phi_i(\sigma_2))\big| \int_{\psi_i(\sigma_1)}^{\psi_i(\sigma_2)} |h_i(\sigma_1, \tau, x(\tau))|d\tau \big] \\
&\le \quad [\theta_{g_1}(\delta) + \eta\theta_{v_x}(\delta) + \eta|x(\varphi(\sigma_2)) - x(\varphi(\sigma_1))|] \\
&. \quad \big[\, G_2 + l \sum_{i=1}^{k} |x(\phi_i(\sigma_2))| \int_0^{\sigma_2} [\alpha_i(\sigma, \tau) + \beta_i(\sigma, \tau)|x(\tau))|]d\tau \big] \\
&+ \quad [G_1 + \eta(\aleph + \aleph_1 + \aleph_2 \ r + \aleph_3 + |x(\varphi(\sigma_1))|)]. \Big[\theta_{g_2}(\delta) + l \sum_{i=1}^{k} |x(\phi_i(\sigma_2))| \int_0^{\psi_i(\sigma_2)} \omega_{h_i}^T(x, \epsilon) \ d\tau \\
&+ \quad l \sum_{i=1}^{k} |x(\phi_i(\sigma_2)) - x(\phi_i(\sigma_1))| \int_0^{\sigma_1} [\alpha_i(\sigma_2, \tau) + \beta_i(\sigma_2, \tau)|x(\tau))|]d\tau \\
&+ \quad l \sum_{i=1}^{k} \big|x(\phi_i(\sigma_1))\big| \int_{\sigma_1}^{\sigma_2} \ [\alpha_i(\sigma_2, \tau) + \beta_i(\sigma_2, \tau)|x(\tau))|]d\tau \Big] \\
&\le \quad [\theta_{g_1}(\delta) + \eta\theta_{v_x}(\delta) + \eta|x(\varphi(\sigma_2)) - x(\varphi(\sigma_1))|]. \big[\, G_2 + l \sum_{i=1}^{k} \|x\| \int_0^{\sigma_2} [\alpha_i(\sigma, \tau) + \beta_i(\sigma, \tau)\|x\|]d\tau \big] \\
&+ \quad [G_1 + \eta(\aleph + \aleph_1 + \aleph_2 \ r + \aleph_3 + \|x\|)] \big[\theta_{g_2}(\delta) + l \ kr\omega_{h_i}^T(x, \epsilon) \\
&+ \quad l \ \sum_{i=1}^{k} \ |x(\phi_i(\sigma_2)) - x(\phi_i(\sigma_1))| \int_0^{\sigma_1} [\alpha_i(\sigma_2, \tau) + \beta_i(\sigma_2, \tau)\|x\|]d\tau \\
&+ \quad \ l \ \sum_{i=1}^{k} \|x\| \int_{\sigma_1}^{\sigma_2} \ [\alpha_i(\sigma_2, \tau) + \beta_i(\sigma_2, \tau)\|x\|]d\tau \\
&\le \quad [\theta_{g_1}(\delta) + \eta\theta_{v_x}(\delta) + \eta \ \omega^T(x, \omega^T(\varphi, \epsilon))] \big[\, G_2 + l \ r \ k \ [\alpha + \beta \ r]\big] \\
&+ \quad [G_1 + \eta(\aleph + \aleph_1 + \aleph_2 r + \aleph_3 + r)] \big[\theta_{g_2}(\delta) + l \sum_{i=1}^{k} r\omega_{h_i}^T(x, \epsilon) + l \sum_{i=1}^{k} \omega^T(x, \omega^T(\phi_i, \epsilon))[\alpha_i + \beta_i r]\big],
\end{aligned}
$$

where for $\sigma_1, \sigma_2 \in I,\ \sigma_1 < \sigma_2,$ we denote

$\theta_{g_1}(\delta) = \sup\{|g_1(\sigma_2, x) - g_1(\sigma_1, x)| :\ |\sigma_2 - \sigma_1| < \delta,\ |x| \leq r\},$

$\theta_{g_2}(\delta) = \sup\{|g_2(\sigma_2, x_1, x_2, .., x_k) - g_2(\sigma_1, , x_1, x_2, .., x_k)| :\ |\sigma_2 - \sigma_1| < \delta,\ |x_i| \leq r\},$

$\theta_{v_x}(\delta) = \sup\{|v_x(\sigma_2) - v_x(\sigma_1)| :\ |\sigma_2 - \sigma_1| < \delta\}$

We therefore arrive at the following estimate

$$\begin{aligned} \omega^T(\mathbb{F}x, \epsilon) \ &\leq\ [\theta_{g_1}(\delta) + \theta_{v_x}(\delta) + \eta\ \omega^T(x, \omega^T(\varphi, \epsilon))] \\ &.\ \ \big[\ G_2 + l\ r\ k\ [\alpha + \beta\ r]\big] +\ [G_1 + \eta(\aleph + \aleph_1 + \aleph_2\ r + \aleph_3 +\ r)] \\ &\times\ \ \Big[\theta_{g_2}(\delta) + l\sum_{i=1}^{k} r\omega^T_{h_i}(x, \epsilon) + l\sum_{i=1}^{k} \omega^T(x, \omega^T(\phi_i, \epsilon))\ [\alpha_i + \beta_i\ r]\Big], \end{aligned} \tag{8}$$

then based on the functions $g_1, g_2 : [0, \infty) \times B_r \to \mathbb{R}$, are uniform continuity, assumptions (iii) and (iv), we have concluded $\theta_{g_1}(\delta)\ \theta_{g_2}(\delta), \theta_{v_x}(\delta)\ \to 0,$ as $\delta \to 0$. Moreover, it is obvious that $\omega^T(\phi_i, \epsilon) \to 0,\ (i = 1, 2, .., k)$ as $\epsilon \to 0$, and $\omega^T(\varphi_i, \epsilon) \to 0, as\ \epsilon \to 0$. Thus, linking the established facts with the estimate (8) we get

$$\begin{aligned} w_0^T(\mathbb{F}X) \ &\leq\ \big[\eta\ (G_2 + lrk(\alpha + \beta r)) \\ &+\ \ lk[G_1 + \eta(\aleph + \aleph_1 + \aleph_2 r + \aleph_3 + r)](\alpha + \beta r)\big] w_0^T(X). \end{aligned} \tag{9}$$

Consequently, we obtain

$$\begin{aligned} w_0(\mathbb{F}X) \ &\leq\ \big[\eta\ (G_2 + lrk(\alpha + \beta r)) \\ &+\ \ lk[G_1 + \eta(\aleph + \aleph_1 + \aleph_2 r + \aleph_3 + r)](\alpha + \beta r)\big]\ w_0(X). \end{aligned} \tag{10}$$

In the following, we take a non-empty set $X \subset B_r$.Then for any $x,\ y \in X$, and fixed $\sigma \geq 0$, we obtain

$$\begin{aligned} &|\ \mathbb{F}x(\sigma) - \mathbb{F}y(\sigma)\ | \\ &\eta\ (\Delta + 1) \sup\{|x(\sigma) - y(\sigma)|,\ x, y \in X\}\ \big[(G_2 + \sum_{i=1}^{k} l\ \|x\| \int_0^\sigma [\alpha_i + \beta_i\|x\|]ds\big] \\ &+ l[G_1 + \eta(\aleph + \aleph_1 + \aleph_2 r + \aleph_3 + \|y\|)] \\ &.\Big[\sum_{i=1}^{k} l \sup\{|x(\phi_i(\sigma)) - y(\phi_i(v))|,\ x, y \in X\} \int_0^\sigma [\alpha_i + \beta_i\|x\|]d\tau \\ &+ \sum_{i=1}^{k} \|y\| \int_0^\sigma \big|\ h_i(\sigma, \tau, x(\tau)) - h_i(\sigma, \tau, y(\tau))\ \big|d\tau\Big]. \end{aligned}$$

Hence, we can easily derive the following inequality:

$$\begin{aligned} \mathrm{diam}(\mathbb{F}X)(\sigma) \leq\ &\eta\ (\Delta + 1)\mathrm{diam}X(\sigma)\big[\ G_2 + l\ k\ r|[\alpha + \beta\ r]\big] \\ &+\ l\ k\ [G_1 + \eta(\aleph + \aleph_1 + \aleph_2\ r + \aleph_3 +\ r)]\big[\mathrm{diam}X(\sigma)[\alpha + \beta\ r]\ + k\ r\omega^T(h, \epsilon)\big]. \end{aligned}$$

Now, considering our assumptions, we get the following estimate:

$$\begin{aligned} &\lim_{\sigma\to\infty} \sup\ \mathrm{diam}\mathbb{F}X(\sigma) \leq\ (\eta\ (\Delta + 1)\big[\ G_2 + l\ k\ r|[\alpha + \beta\ r]\big] \\ &+ l\ k\ [G_1 + \eta\ (\aleph + \aleph_1 + \aleph_2\ r + \aleph_3 + r)][\alpha + \beta\ r]) \lim_{\sigma\to\infty} \sup\ \mathrm{diam}X(\sigma) \end{aligned}$$

$$\lim_{\sigma\to\infty} \sup\ \mathrm{diam}\mathbb{F}X(\sigma) \leq c\ \lim_{\sigma\to\infty} \sup\ \mathrm{diam}X(\sigma), \tag{11}$$

where we denoted

$$c = \eta\ (\Delta + 1)\big[\ G_2 + l\ k\ r|[\alpha + \beta\ r]\big] + l\ k\ [G_1 + \eta\ (\aleph + \aleph_1 + \aleph_2\ r + \aleph_3 + r)][\alpha + \beta\ r].$$

Clearly, given assumption (vi), we know that $c < 1$.

Finally, linking (11) and (10) and the definition of the measure of noncompactness $\mu$, we derive the following inequality

$$\mu(\mathbb{F}X) \leq c\ \mu(X). \tag{12}$$

Now, taking into account the condition that

$$\begin{aligned} c &= \eta\ (\Delta + 1)\big[\ G_2 + l\ k\ r[\alpha + \beta\ r]\big] \\ &\quad + l\ k\ [G_1 + \eta\ (\aleph + \aleph_1 + \aleph_2\ r + \aleph_3 + r)][\alpha + \beta\ r] < 1, \end{aligned}$$

and Theorem 2.1, we deduce that the operator $\mathbb{F}$ has a fixed point x in the ball $B_r$. Obviously $x$ is a solution of the functional integral equation (5). Moreover, taking into account that the image of the space $BC(\mathbb{R}_+)$ under the operator $\mathbb{F}$ is contained in the ball $B_r$ we infer that the set Fix $\mathbb{F}$ of all fixed points of $\mathbb{F}$ is contained in Br . Obviously, the set Fix $\mathbb{F}$ contains all solutions of Eq. (5), we conclude that the set Fix $\mathbb{F}$ belongs to the family $ker\mu$. Now, taking into account the description of sets belonging to $ker\mu$ (given in Section 2) we deduce that all solutions of Eq. (5) are globally asymptotically stable. This completes the proof. □

3.1. **Applications and Special cases.** In this section, we will present several specific cases that are highly relevant for the qualitative analysis of certain functional integral equations. These examples illustrate the significance of these equations in modeling real-world problems, wherein they find applications across various scientific fields.

(1) Consider the case where $g_1(\sigma, v_x, x) = 1$. In this scenario, the multidimensional functional equation (5) simplifies to the following form:

$$x(\sigma) = g_2\left(\sigma, x(\phi_1(\sigma)) \int_0^{\psi_1(\sigma)} h_1(\sigma,\tau,x(\tau))d\tau, \ldots, x(\phi_k(\sigma)) \int_0^{\psi_k(\sigma)} h_k(\sigma,\tau,x(\tau))d\tau\right), \tag{13}$$

where, under the conditions outlined in Theorem 3.1, the multidimensional functional equation (13) admits at least one asymptotically stable solution $x \in BC(\mathbb{R}_+)$.

(2) Now, if we take $g_2(\sigma, x_1, x_2, \ldots, x_k) = \prod_{i=1}^{k} x_i$, this modifies the multidimensional functional equation (13) to:

$$x(\sigma) = \prod_{i=1}^{k} x(\phi_i(\sigma)) \int_0^{\psi_i(\sigma)} h_i(\sigma,\tau,x(\tau))\ d\tau. \tag{14}$$

Under the conditions outlined in Theorem 3.1, this updated functional equation (14) also admits at least one asymptotically stable solution $x \in BC(\mathbb{R}_+)$.

The current study investigates the existence and uniqueness of solutions to a product of $k$-quadratic integral equations. Quadratic integral equations are of particular interest as they find applications in fields such as astrophysics, radiative transfer theory, and neutron transport [13].

Furthermore, it is notable that various classes of quadratic integral equations have been extensively studied within $L_p$ spaces [15] utilizing the measure of non-compactness analysis coupled with Darbo's fixed-point theorem under different sets of assumptions.

Additionally, the study of the product of two or more operators is crucial, as highlighted by Medved and Brestovanska [15, 16]. Their work focuses on the Banach algebras of continuous functions, employing distinct techniques for proof and analysis in comparison to our approach. This diversity in methodology underscores the importance of exploring different mathematical frameworks for solving integral equations and enhances our understanding of their behavior in various contexts.

Notably, our research extends the findings in [7, 8, 10, 12]. Unlike Chen [10], who explored feedback control for ecological models, our work provides a generalized framework for multidimensional systems with delays and control parameters. Building on Nasertayoob [12], which focused on single-variable functional integral equations, our study investigates the multidimensional case with differential feedback control on semi-infinite intervals. El-Sayed et al. [7] and Hashem et al. [8] presented initial results on multidimensional systems and feedback control, while our work includes comprehensive stability analysis and practical examples to demonstrate applicability.

3.2. **Existence of Solutions for Functional Equations (4).** In this section, we examine the existence of solutions $u \in BC(\mathbb{R}_+)$ for the functional equation (4):

$$u(\sigma) = \eta_1(\sigma, x(\sigma)) + x(\sigma)\eta_2(\sigma, x(\sigma)), \quad \sigma \geq 0.$$

We make the following assumptions:

(vii) The functions $\eta_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\eta_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous and satisfy the following Lipschitz conditions:

$$\begin{aligned} &|\eta_1(\sigma, x) - \eta_1(\sigma, y)| \leq l_1|x - y|, \quad |\eta_2(\sigma, x) - \eta_2(\sigma, y)| \leq l_2|x - y|, \\ &\forall(\sigma, x), (\sigma, y) \in \mathbb{R}_+ \times \mathbb{R}, \quad l_1, l_2 > 0. \end{aligned} \tag{15}$$

(viii) The constants $l_1$ and $l_2$ satisfy the condition $l_1 + Ml_2 < 1$.

From inequality (15), we can deduce:

$$|\eta_1(\sigma, x)| - |\eta_1(\sigma, 0)| \leq |\eta_1(\sigma, x) - \eta_1(\sigma, 0)| \leq l_1|x|,$$
$$|\eta_1(\sigma, x)| \leq |\eta_1(\sigma, 0)| + l_1|x|,$$

and

$$|\eta_1(\sigma, x)| \leq |\alpha_1(\sigma)| + l_1|x|.$$

Similarly, we obtain:

$$|\eta_2(\sigma, x)| \leq |\alpha_2(\sigma)| + l_2|x|.$$

**Theorem 3.2.** *Let the assumptions of Theorems 3.1 hold. If assumptions $(vii)$ and $(viii)$ are satisfied and $x \in BC(\mathbb{R}_+)$ is a solution of either (5), then there exists at least one solution $u \in BC(\mathbb{R}_+)$ of the functional equation (4).*

**Proof.** Let $u \in BC(\mathbb{R}_+)$ and define the set $B_\rho$ as

$$B_\rho = \{u \in BC(\mathbb{R}_+) : \|u\| \leq \rho\}, \quad \rho = \frac{\|\alpha_1\| + M\|\alpha_2\|}{1 - (l_1 + Ml_2)}.$$

We define the operator $A$ as follows:

$$Au(\sigma) = \eta_1(\sigma, u(\sigma)) + x(\sigma)\eta_2(\sigma, u(\sigma)), \quad \sigma \geq 0.$$

Let $u \in BC(\mathbb{R}_+)$ and $M = \sup_{\sigma \in \mathbb{R}_+} |x(\sigma)|$. Then, for all $\sigma \geq 0$, we have

$$|Au(\sigma)| \leq |\alpha_1(\sigma)| + l_1|u(\sigma)| + |x(\sigma)|(|\alpha_2(\sigma)| + l_2|u(\sigma)|),$$

which simplifies to

$$|Au(\sigma)| \leq |\alpha_1(\sigma)| + l_1|u(\sigma)| + M(|\alpha_2(\sigma)| + l_2|u(\sigma)|).$$

Taking the norm on both sides, we obtain

$$\|Au\| \leq \|\alpha_1\| + l_1\|u\| + M(\|\alpha_2\| + l_2\|u\|).$$

Substituting $\|u\| \leq \rho$, we get

$$\|Au\| \leq \|\alpha_1\| + l_1\rho + M(\|\alpha_2\| + l_2\rho) = \rho.$$

Thus, the operator $A$ maps $B_\rho$ into itself.

Now, let $\rho > 0$ be given and take $u_1, u_2 \in B_\rho$, such that $\|u_2 - u_1\| \leq \delta$, then for every $x \in BC(\mathbb{R}_+)$ we have

$$\begin{aligned} |Au_2(\sigma) - Au_1(\sigma)| &= \Big| \eta_1(\sigma, u_2(\sigma)) + x(\sigma)\eta_1(\sigma, u_2(\sigma)) \\ &\quad -\eta_1(\sigma, u_1(\sigma)) - x(\sigma)\eta_2(\sigma, u_1(\sigma))\Big|, \end{aligned}$$

which implies that

$$\begin{aligned} |Au_2(\sigma) - Au_1(\sigma)| &\leq |\eta_1(\sigma, u_2(\sigma)) - \eta_1(\sigma, u_1(\sigma))| \\ &\quad +|x(\sigma)||\eta_1(\sigma, u_2(\sigma)) - \eta_1(\sigma, u_1(\sigma))|. \end{aligned} \tag{16}$$

(i) Choosing $T > 0$ and for $\sigma \geq T$, we get

$$\begin{aligned} |Au_2(\sigma) - Au_1(\sigma)| &\leq l_1\|u_2 - u_1\| + Ml_2\|u_2 - u_1\| \\ &\leq (l_1 + Ml_2)\|u_2 - u_1\| \\ &\leq (l_1 + Ml_2)\delta = \epsilon. \end{aligned}$$

(ii) For $T > 0$ and $\sigma \in [0, T]$, from equation (16) we have:

$$\begin{aligned} |Au_2(\sigma) - Au_1(\sigma)| &\leq l_1\|u_2 - u_1\| + Ml_2\|u_2 - u_1\| \\ &\leq (l_1 + Ml_2)\delta = \epsilon. \end{aligned}$$

Thus, the operator $A$ is continuous.
Now, let $u \in B_\rho$ be nonempty. Then for any $u_2, u_1 \in X$ and $T > 0$ such that $\sigma \geq T$, then from (16) we obtain

$$\begin{aligned} |Au_2(\sigma) - Au_1(\sigma)| &\leq l_1|u_2(\sigma) - u_1(\sigma)| + Ml_2|u_2(\sigma) - u_1(\sigma)| \\ &\leq l_1 sup\{|u_2(\sigma) - u_1(\sigma)|, u_1, u_2 \in X\} \\ &\quad +Ml_2 sup\{|u_2(\sigma) - u_1(\sigma)|, u_1, u_2 \in X\} \\ &\leq l_1 \; diamX(\sigma) + Ml_2 \; diamX(\sigma). \end{aligned}$$

Thus, we have:

$$\begin{aligned} diamAX(\sigma) &\leq (l_1 + Ml_2) \; diamX(\sigma) \\ \lim_{\sigma\to\infty} sup \; diamAX(\sigma) &\leq (l_1 + Ml_2) \lim_{\sigma\to\infty} sup \; diamX(\sigma). \end{aligned} \tag{17}$$

Let $T > 0$ and $\delta > 0$ be given, choose a function $u \in X$ and $\sigma \in [0, T]$ such that $|\sigma_2 - \sigma_1| < \delta$, $\sigma_1 \leq \sigma_2$ and define

$$\begin{aligned} \theta_{\eta_1}(\delta) &= \sup \{|\eta_1(\sigma_2, u) - \eta_1(\sigma_1, u)| : \sigma_1, \sigma_2 \in [0, T], \; ||u|| \leq \rho\}, \\ \theta_{\eta_2}(\delta) &= \sup \{|\eta_2(\sigma_2, u) - \eta_2(\sigma_1, u)| : \sigma_1, \sigma_2 \in [0, T], \; ||u|| \leq \rho\}. \end{aligned}$$

We then have:

$$\begin{aligned}
& |Au(\sigma_2) - Au(\sigma_1)| \\
= & \left|\eta_1(\sigma_2, u(\sigma_2)) + x(\sigma_2)\eta_2(\sigma_2, u(\sigma_2)) - \eta_1(\sigma_1, u(\sigma_1)) - x(\sigma_1)\eta_2(\sigma_1, u(\sigma_1))\right| \\
\leq & \ \theta_{\eta_1}(\delta) + l_1|u(\sigma_2) - u(\sigma_1)| + |x(\sigma_2) - x(\sigma_1)||\eta_2(\sigma_2, u(\sigma_2))| \\
+ & \ |x(\sigma_1)||\eta_2(\sigma_2, u(\sigma_2)) - \eta_2(\sigma_1, u(\sigma_2))| + |x(\sigma_1)||\eta_2(\sigma_1, u(\sigma_2)) - \eta_2(\sigma_1, u(\sigma_1))| \\
\leq & \ \theta_{\eta_1}(\delta) + l_1|u(\sigma_2) - u(\sigma_1)| + |x(\sigma_2) - x(\sigma_1)|(\|\alpha_2\| + l_2\rho) + M\theta_{\eta_2}(\delta) \\
+ & \ Ml_2|u(\sigma_2) - u(\sigma_1)|.
\end{aligned}$$

Then

$$\begin{aligned}
|Au(\sigma_2) - Au(\sigma_1)| \leq & \ \theta_{\eta_1}(\delta) + l_1 sup\{|u(\sigma_2) - u(\sigma_1)|\} + (\|\alpha_2\| + l_2\rho)\epsilon \\
+ & \ M\theta_{\eta_2}(\delta) + Ml_2 sup\{|u(\sigma_2) - u(\sigma_1)|\}.
\end{aligned}$$

Now, let $\sigma_1, \sigma_2 \in [0, T]$, $|\sigma_2 - \sigma_1| < \delta$. Then we deduce that

$$\begin{aligned}
\omega^T(AX, \epsilon) &\leq (l_1 + Ml_2)\omega^T(X, \epsilon) + \theta_{\eta_1}(\delta) + M\theta_{\eta_2}(\delta) + (\|\alpha_2\| + l_2\rho)\epsilon \\
\omega_0^T(AX) &\leq (l_1 + Ml_2)\omega_0^T(X)
\end{aligned}$$

and as $T \to \infty$

$$\omega_0(AX) \leq (l_1 + Ml_2)\omega_0(X). \tag{18}$$

Now, from (17) and (18) we obtain

$$\mu(AX) \leq (l_1 + Ml_2)\mu(X).$$

Since $l_1 + Ml_2 < 1$, $A$ is a contraction with regard to the measure of noncompactness and equation (4) has at least one solution $u \in B_\rho$.

3.2.1. *Asymptotic Stability.* We now address the asymptotic stability of the solution $u \in BC(\mathbb{R}_+)$ to the equation (1). This is based on the asymptotic behavior of the solutions $x \in BC(\mathbb{R}_+)$ of equations (5).

**Theorem 3.3.** *The solution $u \in BC(\mathbb{R}_+)$ of the equation (1) is asymptotically stable. Specifically, for any $\epsilon > 0$, there exist $T(\epsilon) > 0$ and $\rho > 0$ such that for any two solutions $u_1, u_2 \in B_\rho$, we have $|u_2(\sigma) - u_1(\sigma)| \leq \epsilon$ for $\sigma \geq T(\epsilon)$.*

**Proof.** Let $u_1, u_2 \in B_\rho$ be two solutions of equation (5). We have

$$\begin{aligned}
& |u_2(\sigma) - u_1(\sigma)| \\
= & \left|\eta_1(\sigma, u_2(\sigma)) + x_2(\sigma)\eta_2(\sigma, u_2(\sigma)) - \eta_1(\sigma, u_1(\sigma)) - x_1(\sigma)\eta_2(\sigma, u_1(\sigma))\right| \\
\leq & \ |\eta_1(\sigma, u_2(\sigma)) - \eta_1(\sigma, u_1(\sigma))| + |x_2(\sigma)||\eta_2(\sigma, u_2(\sigma)) - \eta_2(\sigma, u_1(\sigma))| \\
+ & \ |x_2(\sigma) - x_1(\sigma)||\eta_2(\sigma, u_1(\sigma))| \\
\leq & \ l_1|u_2(\sigma) - u_1(\sigma)| + Ml_2|u_2(\sigma) - u_1(\sigma)| + |x_2(\sigma) - x_1(\sigma)|(\|\alpha_2\| + l_2 r_2),
\end{aligned}$$

from Theorems 3.1 we have

$$|x_2(\sigma) - x_1(\sigma)| \leq \epsilon^*, \ t \geq T(\epsilon^*),$$

then

$$\|u_2 - u_1\| \leq l_1\|u_2 - u_1\| + Ml_2\|u_2 - u_1\| + (\|\alpha_2\| + l_2 r_2)\epsilon^*.$$

Hence

$$\|u_2 - u_1\| \quad \leq \quad \frac{(\|\alpha_2\| + l_2 r_2)\epsilon^*}{1 - (l_1 + M l_2)} = \epsilon,$$

that is

$$|u_2(\sigma) - u_1(\sigma)| \quad \leq \quad \|u_2 - u_1\| \ \leq \ \epsilon.$$

Then $u \in BC(\mathbb{R}_+)$ is asymptotically stable of equation (4).

3.2.2. *Example.* Consider the following multidimensional functional equation:

$$\begin{aligned} x(\sigma) \quad &= \quad \frac{\sigma}{1+\sigma^2}\arctan(\sigma + x(\sigma)) \\ &. \quad \sum_{i=1}^{k} \frac{\sigma}{1+2\sigma}\sin\left(\sigma + x_i(\sigma)\int_0^\sigma \left(\frac{2\sigma - \tau}{1+\sigma^4} + \frac{\sigma|x_i(\tau)|}{2\pi(\sigma^2+1)(\tau+1)}\right)d\tau\right), \end{aligned}$$

under the feedback control differential equation:

$$\frac{dv_x(\sigma)}{d\sigma} = -4v_x(\sigma) + \sigma e^{-\sigma^2} + \frac{0.1|x(\sigma)| + 0.1|v_x(\sigma)|}{1+\sigma}, \quad v_x(0) = 0.02.$$

We now examine the solvability of this functional equation in the space $BC(\mathbb{R}_+)$.

Take into account that this multidimensional functional equation is a specific case of equation (5), where

$$x(\sigma) = \frac{\mu(\sigma) - \frac{2e^{-\sigma}}{7}\frac{|\cos\mu(\sigma)|}{1+|\cos\mu(\sigma)|}}{\frac{e^{-\pi\sigma}}{1+\sigma} + \frac{e^{-\ln(\sigma+1)\mu^2(\sigma)}}{1+\mu^2(\sigma)}}.$$

Thus,

$$\eta_1(\sigma, \mu(\sigma)) = \frac{2e^{-\sigma}}{7}\frac{|\cos\mu(\sigma)|}{1+|\cos\mu(\sigma)|},$$

$$\eta_2(\sigma, \mu(\sigma)) = \frac{e^{-\pi\sigma}}{1+\sigma} + \frac{e^{-\ln(\sigma+1)}\mu^2(\sigma)}{1+\mu^2(\sigma)},$$

with $l_1(\sigma) = \frac{1}{\sigma^2+1}$ and $l_2(\sigma) = \frac{\sqrt{\sigma}}{e^{\sigma+1}}$. We define the following functions:

$$\begin{aligned} g_1(\sigma, v_x, x(\sigma)) \quad &= \quad \frac{\sigma^2}{1+\sigma^2}\arctan(\sigma + v_x(\sigma) + x(\sigma)), \\ g_2(\sigma, x_1(\sigma), ..., x_k(\sigma)) \quad &= \quad \sum_{i=1}^{k}\frac{\sigma}{1+2\sigma}\sin(\sigma + x_i(\sigma)), \\ h_i(\sigma, \tau, x(\tau)) \quad &= \quad \frac{2\sigma - \tau}{1+\sigma^4} + \frac{\sigma|x_i(\tau)|}{2\pi(\sigma^2+1)(\tau+1)} \\ f(\sigma, v_x(\sigma), x(\sigma)) \quad &= \quad \sigma e^{-\sigma^2} + \frac{0.1|x(\sigma)| + 0.1|v_x(\sigma)|}{1+\sigma} \end{aligned}$$

It is evident that $g_1$ is a continuous function. Moreover, for any $x, y \in \mathbb{R}$ and $\sigma \in [0, 1]$, we have:

$$|g_1(\sigma, x(\sigma)) - g_1(\sigma, y(\sigma))| \quad \leq \quad \frac{1}{2}[|v_x(\sigma) - v_y(\sigma)| + |x(\sigma) - y(\sigma)|].$$

This implies that condition (iv) is satisfied, where $G_1 = \frac{\pi}{4}$ and $\eta = \frac{1}{2}$, with $g_1(\sigma, 0, 0) = \frac{\sigma^2}{1+\sigma^2} \arctan(\sigma)$. Next, we observe:

$$\begin{aligned} |g_2(\sigma, x_1(\sigma), ..., x_k(\sigma)) - g_2(\sigma, y_1(\sigma), ..., y_k(\sigma))| &\leq \sum_{i=1}^{k} \frac{\sigma}{1+2\sigma} |x_i(\sigma) - y_i(\sigma)| \\ &\leq \sum_{i=1}^{k} \frac{|x_i(\sigma) - y_i(\sigma)|}{2}, \end{aligned}$$

where $l = \frac{1}{2}$ and $g_2(\sigma, 0, ..., 0) = \frac{\sigma}{1+2\sigma} \sin(\sigma)$. Therefore, $G_2 = \frac{1}{2}$ holds. Also for function $f$

$$\begin{aligned} |f(\sigma, v_x(\sigma), x(\sigma))| &\leq |\sigma e^{-\sigma^2}| + |\frac{0.1|x(\sigma)| + 0.1|v_x(\sigma)|}{1+\sigma}| \\ &\leq \frac{1}{2e} + 0.1(|x| + |v_x|), \end{aligned}$$

where $\aleph = 0.02$, $\aleph_1 = \frac{0.01}{4}$, $\aleph_2 = \frac{0.3}{4}$, $\aleph_3 = \frac{1}{8e}$. Moreover, we notice that $h_i(\sigma, \tau, x)$ satisfies assumption (v), with:

$$|h_i(\sigma, \tau, x(\tau))| \leq \frac{2\sigma - \tau}{1+\sigma^4} + \frac{\sigma |x(\tau)|}{2\pi(\sigma^2+1)(\tau+1)}.$$

Thus, we can define:

$$\alpha_i(\sigma, \tau) = \frac{\sigma(2\sigma - \tau)}{2(1+\sigma^4)}, \quad \beta_i(\sigma, \tau) = \frac{\sigma}{2\pi(\sigma^2+1)(\tau+1)}.$$

To verify assumption (v), we compute:

$$\lim_{\sigma\to\infty} \int_0^{\sigma} \alpha_i(\sigma, \tau) d\tau = \lim_{\sigma\to\infty} \frac{3\sigma^3}{4\sigma^4 + 4} = 0,$$

and

$$\lim_{\sigma\to\infty} \int_0^{\sigma} \beta_i(\sigma, \tau) d\tau = \lim_{\sigma\to\infty} \frac{\sigma \ln(\sigma+1)}{2\pi(\sigma^2+1)} = 0.$$

Moreover, we calculate $\alpha_i \approx 0.14246919$ and $\beta_i \approx 0.0906987$.

Finally, the roots of the cubic equation from Theorem 3.1 are approximated:

$$0.02927r^3 + 0.22475r^2 - 0.68125r + 0.409 = 0$$

The roots of the cubic equation are approximately:

$$r_1 \approx 0.722, \; r_2 \approx -0.912, \; r_3 \approx -0.092,$$

and the root $r \simeq 0.722$ satisfies the inequality:

$$\eta\ (\Delta+1)\big[\ G_2 + l\ k\ r|[\alpha + \beta\ r]\big] + l\ k\ [G_1 + \eta\ (\aleph + \aleph_1 + \aleph_2\ r + \aleph_3 + r)][\alpha + \beta\ r] \simeq 0.390 < 1.$$

Hence, all the conditions of Theorem 3.1 are satisfied, and we conclude that equation (1) accompanied with feedback control (2) has at least one solution in the space $BC(\mathbb{R}_+)$.

## Conclusion

The unforeseen disturbances can continuously affect physical systems in many practical scenarios. These disturbances, often caused by variations in system parameters, are referred to as perturbations. In control theory, perturbations can be treated as control variables, particularly in integral equations. Many real-world scenarios, such as population dynamics, control systems, and signal processing play a crucial role in modeling interactions over time.

The feedback control mechanism can be applied to harvesting or other biological control strategies. For further literature on feedback control problems, we refer the reader to [10, 11, 12].

In this work, we have established the existence and asymptotic stability of solutions for a nonlinear cubic functional integral inclusion with feedback control on the real half-line, using a measure of noncompactness technique. Our study is situated in the space of bounded continuous functions, $BC(\mathbb{R}_+)$. Furthermore, by selecting a suitable noncompactness measure, we have shown that these solutions are asymptotically stable in the Banach space $BC(\mathbb{R}_+)$.

The example presented illustrates the stability and existence results derived in our study.

## References

[1] Banaś, J., Chlebowicz, A., Woś, W., (2020), On measures of noncompactness in the space of functions defined on the half-axis with values in a Banach space, J. Math. Anal. Appl., 489, 124187.

[2] Banaś, J., Chlebowicz, A., (2019), On solutions of an infinite system of nonlinear integral equations on the real half-axis, Banach J. Math. Anal., 13, pp. 944–968.

[3] Banaś, J., O'Regan, D., (2008), On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order, Journal of Mathematical Analysis and Applications, 345, pp. 573–582.

[4] Banaś, J., Rzepka, B., (2003), On existence and asymptotic stability of solutions of nonlinear integral equation, Math. Anal. Appl., 284, pp. 165–173.

[5] Banaś, J., Rzepka, B., (2003), An application of a Measure of Noncompactness in the study of asymptotic stability, Applied Mathematics Letters, 16, pp. 1–6.

[6] Banaś, J., (1984), On measures of noncompactness in Banach spaces and their applications to integral equations, Dissertationes Mathematicae, 234, pp. 1–58.

[7] El-Sayed, A. M. A., Hashem, H. H. G., Al-Issa, S. M., (2023), New Aspects on the Solvability of a Multidimensional Functional Integral Equation with Multivalued Feedback Control, Axioms, 12 (653). https://doi.org/10.3390/axioms12070653.

[8] Hashem, H. H. G., El-Sayed, A. M. A., Al-Issa, S. M., (2023), Investigating Asymptotic Stability for Hybrid Cubic Integral Inclusion with Fractal Feedback Control on the Real Half-Axis, Fractal Fract., 7 (449). https://doi.org/10.3390/fractalfract7060449.

[9] Darbo, G., (1955), Punti in transformazationi a condominio non compatto, Rend. Sem. Math. Univ. Padova, 4, pp. 84–92.

[10] Chen, F., (2006), The permanence and global attractivity of Lotka–Volterra competition system with feedback controls, Nonlinear Analysis: Real World Applications, 7, pp. 133–143.

[11] Nasertayoob, P., (2018), Solvability and asymptotic stability of a class of nonlinear functional-integral equations with feedback control, Commun. Nonlinear Anal., 5, pp. 19–27.

[12] Nasertayoob, P., Mansour Vaezpour, S., (2013), Positive periodic solution for a nonlinear neutral delay population equation with feedback control, J. Nonlinear Sci. Appl., 6, pp. 152–161.

[13] Alsaadi, A., Cichon, M., Metwali, M. M. A., (2022), Integrable solutions for Gripenberg-type equations with m-product of fractional operators and applications to initial value problems, Mathematics, 10 (1172).

[14] Hashem, H. H. G., Zaki, M. S., (2013), Cartheodory theorem for quadratic integral equations of Erdélyi–Kober type, Journal of Fractional Calculus and Applications, 4, pp. 1–8.

[15] Metwali, M. M. A., Mishra, V. N., (2023), On the measure of noncompactness in $L^p(\mathbb{R}^+)$ and applications to a product of n-integral equations, Turkish Journal of Mathematics, 47, pp. 372–386.

[16] Brestovanska, E., Medveď, M., (2012), Fixed point theorems of the Banach and Krasnosel'skii type for mappings on m-tuple Cartesian product of Banach algebras and systems of generalized Gripenberg's equations, Acta Universitatis Palackianae Olomucensis Facultas Rerum Naturalium Mathematica, 51, pp. 27–39.

[17] Darbo, G., (1955), Punti in transformazationi a condominio non compatto, Rend. Sem. Math. Univ. Padova, 4, pp. 84–92.

[18] Y. Ma, L. Yang, Z. Li, C. Cheng, B. Lev, A. Hajiyev, (2023), Routing Optimization for Origin-to-Destination Orders with Crowdsourced Delivery: An Application for Real Orders from the Wenzhou Power Grid. Appl. Comput. Math., V.22, N.4, pp.400-425

**A. M. A. El-Sayed** a full-time professor in the Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Egypt. He obtained the Medal of Science and Arts (first class) in basic sciences to obtain the State Award for Excellence in Basic Sciences 2015. He has more than 400 publications in the areas of Fractional Calculus, Applied Functional Analysis, Functional Differential and Functional Integral Equations, and Dynamical Systems.

**Sh. M. Al-Issa** received her B.Sc. in Pure Mathematics from the Lebanese University, Lebanon, in 2003. She received M.Sc. and Ph.D. in Pure Mathematics from Beirut Arab University, Lebanon, in 2007 and 2011, respectively. She served as an Assistant Professor in the Faculty of Mathematics at King Saud University (KSU). Since October 2018, she has held the position of Assistant Professor in the Faculty of Arts and Sciences at the Lebanese International University, Lebanon. Her research interests span applied mathematics, nonlinear functional analysis and fractional calculus.

**H. H. G. Hashem** graduated from Alexandria University, Egypt. H. received M.Sc. and Ph.D. degrees in Mathematics from the Department of Mathematics and Computer Science, Faculty of Science, Alexandria University. H. does research in Analysis. Their most recent publications are Quadratic Integral Equations, Coupled Systems of Integral Equations, Block Matrix Operators and Differential inclusions.

**I. H. Kaddoura** is currently working as an associate professor in the department of mathematics and physics at the Lebanese International University in Saida, and the International University of Beirut, Lebanon. He received his Ph.D. degree from Baghdad University in 2000. He is interested in various research domains, including recurrences, difference equations, matrix theory, number theory, dynamical systems, and differential topology.

**A. A. Najdi** earned his Bachelor of Science degree in Pure Mathematics from the Lebanese University in 2022. He subsequently pursued advanced studies by enrolling in the Master's program at the Lebanese International University in 2023. His academic interests are centered around Functional Analysis, where he explores complex mathematical structures and contributes to understanding their theoretical frameworks and applications.