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PAIRED DOMINATION INTEGRITY OF DERIVED GRAPHS OF CYCLES

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Abstract. The study of the vulnerability of real-life networks helps network designers construct networks such that their stability is maintained even under the disruption of a few nodes or links connecting the nodes. In this paper, we study the vulnerability of larger networks through a vulnerability parameter called paired domination integrity. The paired domination integrity of a graph G is defined as the minimum value of the sum of the cardinality of a paired dominating set S of G and the order of the largest component in $< V(G) - S >$. The minimum is taken over all possible paired dominating sets. The above-mentioned large networks are modelled by some derived graphs of C_n, such as the Middle, Total, Central, and Mycielskian graphs.

Keywords: Domination integrity, Middle graph, Total graph, Central graph, Mycielskian graph.

AMS Subject Classification: 05C38, 05C40, 05C69, 05C76

1. Introduction

Let $G = (V, E)$ represent an undirected, simple, and finite graph with vertex set V and edge set E. Any extensive network can be modeled into a graph, and various graph parameters can be used to investigate distinct properties of the network. Here, we study the vulnerability of a network through the graph vulnerability parameter called paired domination integrity. A paired dominating set is a total dominating set S of G such that the subgraph induced by S contains a perfect matching. The minimum cardinality of a paired dominating set in G is the paired domination number denoted by γ_{pd} [10]. Any paired dominating set in G of cardinality equal to γ_{pd} is said to be a γ_{pd}-set of G. The concept of paired domination in graphs was introduced by T. W. Haynes and P. J. Slater [8]. Later, in the year 2010, R. Sundareswaran and V. Swaminathan [13] initiated the study of the domination integrity of graphs. The paired domination integrity [1] of G with no isolated vertices, denoted by $PDI(G)$, is defined as $PDI(G) = min\{|S| + m(G - S) :$

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$S \subseteq V(G)\}$, where S is a paired dominating set (PDS) of G and $m(G-S)$ is the total number of vertices in the largest connected component in the induced subgraph of $G-S$. A paired dominating set $S \subseteq V(G)$ with $PDI(G) = |S|+m(G-S)$ is said to be a $PDI-$set of G. The minimum paired dominating set need not be a paired domination integrity set. Terminologies related to graph theory that are not covered here can be found in [7, 15]. Several variations of domination integrity have been introduced in [2, 3, 5]

A detailed study of this parameter helps improve the paired domination integrity of a graph, thereby enhancing the network's resilience to disruptions and failures. This can be done by identifying and eliminating unnecessary nodes that do not impact the connectivity or dominance relationships within the network.

The concept of vulnerability of a network can be used in several real-life networks [11, 12]. One such network is the supply chain network, where the nodes represent the distinct individuals or companies involved in creating the product and delivering it to the customer. Studying the vulnerability parameter helps in identifying critical nodes that are essential in maintaining the flow of products. By focusing on these nodes, supply chain managers can ensure that disruptions at these points are minimized. The larger the value of paired domination integrity, the greater the resilience of the network to disruptions. Therefore, the idea of vulnerability of a network helps in analyzing the network. The analysis of the vulnerability of any network helps network designers construct a more stable and efficient network.

In this paper, the authors study the paired domination integrity for larger graphs such as the Middle, Total, Central and Mycielskian graphs of cycles. The following results are used to prove some of the results in our study.

Proposition 1.1. [10] *Let C_n be a cycle having n vertices, then $\gamma_{pd}(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$ for $n \geq 3$.*

Proposition 1.2. [1] *If C_n is a cycle of order $n \geq 10$, then $PDI(C_n) = 2\left\lceil \frac{n+4}{4} \right\rceil$.*

2. PDI of Middle graph of C_n

Definition 2.1. [6] *The middle graph of a graph $G = (V, E)$, denoted by $M(G)$, is the graph with vertex set $V \cup E$ and two vertices in $M(G)$ are adjacent if they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.*

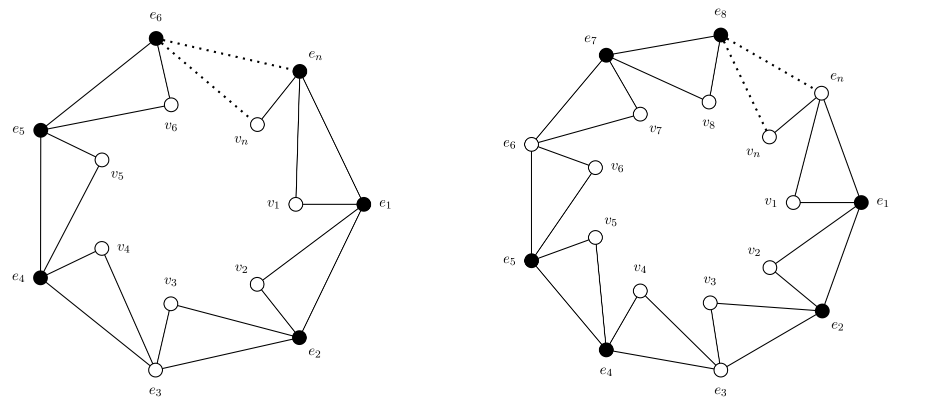
Theorem 2.1. *If $M(C_n)$ is the middle graph of C_n, then $\gamma_{pd}(M(C_n)) = 2\left\lceil \frac{n}{3} \right\rceil$ for $n \geq 3$.*

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{e_1, e_2, \dots, e_n\}$. Then $V \cup E$ is the vertex set of $M(C_n)$. The order and size of $M(C_n)$ are $2n$ and $3n$ respectively. In $M(C_n)$, each v_i, where $1 \leq i \leq n$, has neighbors in $\{e_1, e_2, \dots, e_n\}$ only whereas each e_j, where $1 \leq j \leq n$, has neighbors in both $\{v_1, v_2, \dots, v_n\}$ and $\{e_1, e_2, \dots, e_n\}$. Let $e_1, e_2, \dots, e_{n-1}, e_n$ be the vertices that subdivide $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ respectively in $M(C_n)$ (see Fig.1). The degree of v_i and e_j in $M(C_n)$ are two and four respectively. Let S be any PDS of $M(C_n)$. Clearly, $S = \{v_1, v_2, \dots, v_n\}$ is not a PDS since it is an independent set and hence does not induce a perfect matching. Therefore, if v_i belongs to any paired dominating set, it must be paired with an $e_j \in N(v_i)$, where $1 \leq i, j \leq n$.

Now, we consider distinct paired dominating sets. Let $S = \{e_1, e_2, \dots, e_n\}$. For even order n, $S = \{e_1, e_2, \dots, e_n\}$ is a paired dominating set since every vertex in $M(C_n)$ is adjacent to a vertex in S and $< S >$ contains a perfect matching. However, for odd order n, one vertex in S is left without a pair. The unpaired vertex can be paired with any one of its neighbors $v_i \notin S$. Therefore, $S = \{e_1, e_2, \ldots, e_n, v_i\}$ is a paired dominating set for an odd order of $M(C_n)$. Here, $|S| = n$ when n is even and $|S| = n + 1$ when n is odd.

Let S contain vertices such that the end vertices of every edge in the matching induced by S are v_i and e_j. We choose the least possible number of vertices in S such that its paired vertices are of the form v_i, e_j. Let us consider $S = \{v_{1+2t}, e_{1+2t}\}$ and $S = \{v_{1+2t}, e_{1+2t}, v_n, e_n\}$ for even and odd orders of $M(C_n)$ respectively, where $0 \le t \le \left\lfloor \frac{n-1}{2} \right\rfloor$. This is a paired dominating set with each v_{1+2t} paired with e_{1+2t} in S such that every vertex in $M(C_n)$ is adjacent to a vertex in S and $< S >$ contains a perfect matching. We observe that $|S| = 2\left\lceil \frac{n}{2} \right\rceil$. In a similar way, $S = \{v_{2+2t}, e_{2+2t}\}$ and $S = \{v_{2+2t}, e_{2+2t}, v_n, e_n\}$ for even and odd orders respectively, where $0 \le t \le \left\lfloor \frac{n-2}{2} \right\rfloor$, is also a paired dominating set with $|S| = 2\left\lceil \frac{n}{2} \right\rceil$.

In the above-mentioned paired dominating set, we can also pair each e_{1+2t} with v_{2+2t} to get another paired dominating set. That is, $S = \{e_{1+2t}, v_{2+2t}\}$. Here, S is a paired dominating set since every vertex in $M(C_n)$ has a neighbor in S, and the $< S >$ is a disjoint union of paths of order two, which clearly contains a perfect matching. The cardinality of S is found to be $2\left\lceil \frac{n}{2} \right\rceil$.



(A) $M(C_{4+3t})$ (B) $M(C_n)$, where $n \neq 4 + 3t$

FIGURE 1. PDI-set and γ_{pd}-set of Middle graph of cycles.

Now, we choose $S = \{e_{1+3t}, e_{2+3t}\}$, where $0 \le t \le \left\lfloor \frac{n-1}{3} \right\rfloor$ (see Fig. 1(B)). Here, S is a paired dominating set as each e_{1+3t} is paired with e_{2+3t} in S, thereby dominating the entire vertex set of $M(C_n)$ and $< S >$ is a disjoint union of paths of order two which contains a perfect matching. However, this is not true for all $M(C_n)$. We observe that for $M(C_{4+3t})$, the paired dominating set $S = \{e_{1+3t}, e_{2+3t}, e_{n-1}, e_n\}$, where $0 \le t \le \frac{n-4}{3}$ (see Fig. 1(A)) and each e_{1+3t} is paired with e_{2+3t} and e_{n-1} is paired with e_n in S. Therefore, $|S| = 2\left\lceil \frac{n}{3} \right\rceil$.

When the minimum is taken over all S, we can conclude that the γ_{pd}-set is $S = 2\left\lceil \frac{n}{3} \right\rceil$. That is, $\gamma_{pd}(M(C_n)) = 2\left\lceil \frac{n}{3} \right\rceil$. □

Theorem 2.2. *If $M(C_n)$ is the middle graph of C_n, then*

$$PDI(M(C_n)) = \begin{cases} 5 & ; \quad n = 3, 4 \\ 2\left\lceil \frac{n}{3} \right\rceil + 3; & n \geq 5 \end{cases}$$

Proof. Let $S \subset V(M(C_n))$ be any PDS of $M(C_n)$. In Theorem 2.1, we have mentioned the possible paired dominating sets of $M(C_n)$. We find the sum $|S| + m(M(C_n) - S)$ for all possible S and identify the set that gives the minimum sum.

If $S = \{e_1, e_2, \ldots, e_n\}$ (for even n) and $S = \{e_1, e_2, \ldots, e_n, v_i\}$ (for odd n), where $v_i \notin S$ is paired with the unpaired vertex in S, then $|S| + m(M(C_n) - S)$ is $n + 1$ and $n + 2$ respectively.

Let S contain the least possible number of vertices such that its paired vertices are of the form v_i, e_j, where $1 \leq i, j \leq n$ and $v_i \in N(e_j)$. We consider the set found in Theorem 2.1. In this case, we observe that $< M(C_n) - S >$ is the disjoint union of paths of order two. As a result, the order of the largest component in $< M(C_n) - S >$ is at least two. Therefore, $|S| + m(M(C_n) - S) \geq 2\left\lceil \frac{n}{2} \right\rceil + 2$.

Now, let S be the γ_{pd}-set found in the previous theorem. We consider the following two cases.

Case(i): $n = 3, 4$
When $n = 3$, we choose $S = \{e_1, e_2\}$ and hence $m(M(C_n) - S) = 3$. The sum $|S| + m(M(C_n) - S)$ is the same when $S = \{e_1, e_3\}$ or $S = \{e_2, e_3\}$. We can also have $S = \{e_1, e_2, e_3\} \cup v_i$, which would give the same sum since $m(M(C_n) - S) = 1$. When $n = 4$, $S = \{e_1, e_2, e_3, e_4\}$ is the only PDS that results in a minimum sum. Here, $m(M(C_n) - S) = 1$. Therefore, $|S| + m(M(C_n) - S) = 5$ when $n = 3$ and $n = 4$.

Case(ii): $n \geq 5$
For all $n \not\equiv 1 \pmod 3$, we have $S = \{e_{1+3t}, e_{2+3t}\}$, where t is any integer such that $0 \leq t \leq \left\lfloor \frac{n-1}{3} \right\rfloor$. For $n \equiv 1 \pmod 3$, $S = \{e_{1+3t}, e_{2+3t}, e_{n-1}, e_n\}$, where $0 \leq t \leq \frac{n-4}{3}$. The cardinality of S is found to be $2\left\lceil \frac{n}{3} \right\rceil$ as mentioned in Theorem 2.1. Since $e_j \in V(M(C_n)) - S$ is adjacent to only v_j and v_{j+1}, the components of $< M(C_n) - S >$ are either isolated vertices or paths of order three resulting in $< M(C_n) - S >$ being three. Therefore, $|S| + m(M(C_n) - S) = 2\left\lceil \frac{n}{3} \right\rceil + 3$.

Hence, the paired domination integrity is given by

$$\begin{aligned} PDI(M(C_n)) &= min\{|S| + m(M(C_n) - S) : S \text{ is a paired dominating set of } M(C_n)\} \\ &= 2\left\lceil \frac{n}{3} \right\rceil + 3. \end{aligned}$$

□

3. PDI of Total graph of C_n

Definition 3.1. [4] *The total graph of $G = (V, E)$, denoted by $T(G)$, is the graph with vertex set $V \cup E$ and two vertices in $T(G)$ are adjacent if they are either adjacent or incident in G.*

Theorem 3.1. *If $T(C_n)$ is the total graph of C_n, then $\gamma_{pd}(T(C_n)) = 2\left\lceil \frac{n}{3} \right\rceil$ for $n \geq 3$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$. Then, $V \cup E$ is the vertex set of $T(C_n)$. Since C_n is a 2-regular graph , $T(C_n)$ is a 4-regular graph. Let S be any PDS of $T(C_n)$. We observe that $N(v_i) = \{v_{i-1}, v_{i+1}, e_{i-1}, e_i\}$ for $2 \leq i \leq n-1$. However, $N(v_1) = \{v_n, v_2, e_1, e_n\}$ and $N(v_n) = \{v_{n-1}, v_1, e_{n-1}, e_n\}$.

We examine all possible paired dominating sets to find the minimum among them. The sets $\{e_1, e_2, \ldots, e_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ are paired dominating sets of $T(C_n)$ for even n since every vertex in $T(C_n)$ has a neighbor in S and $< S >$ is a path graph of order n that contains a perfect matching. We can either remove one vertex from, or add one vertex to, the above sets for odd orders to yield new paired dominating sets.

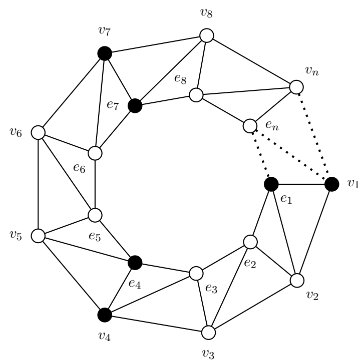


FIGURE 2. γ_{pd}-set of $T(C_n)$.

Let us consider $S = \{v_{1+2t}, e_{1+2t}\}$. Here, S is a PDS since every vertex of $T(C_n)$ is adjacent to a vertex in S and $< S >$ is the disjoint union of paths of order two, which forms a perfect matching. The cardinality of S is n and $n+1$ for even and odd orders of $T(C_n)$, respectively.

The set $S = \{e_{1+3t}, e_{2+3t}\}$ also forms a paired dominating set since every vertex of $T(C_n)$ has a neighbor in S and $< S >$ is a connected graph containing vertices $e_i \notin S$ that is adjacent to v_i and v_{i+1} and a cycle of order n formed by the vertices $\{v_1, v_2, \ldots, v_n\}$. The cardinality of S is found to be $2\left\lceil \frac{n}{3} \right\rceil$. However, for $T(C_{4+3t})$, $S = \{e_{1+3t}, e_{2+3t}, e_n, e_{n-1}\}$.

It is observed that $T(C_n)$ contains two cycles of order n as a subgraph, the outer and the inner cycle, as can be seen in Fig. 2. We consider $S = \{v_{1+3t}, e_{1+3t}\}$ for $n \geq 5$, where $0 \leq t \leq \left\lfloor \frac{n-1}{3} \right\rfloor$. The set S is a PDS since each v_{1+3t} is paired with e_{1+3t} in S, and $< S >$ is either the disjoint union of diamond graphs or consists of all diamond graphs and one P_2 as components. Either way, $< S >$ contains a perfect matching. That is, at least one out of every three vertices of each C_n belongs to S. The vertices e_{1+3t} may be paired with any one of its neighbors such that every e_{1+3t} is paired with the same kind of neighbor. For example, each e_{1+3t} can also be paired with v_{2+3t} which also yields a PDS. Here, $|S|$ is found to be the same as the previous PDS. That is, $|S| = 2\left\lceil \frac{n}{3} \right\rceil$. On taking the minimum over all the paired dominating sets discussed above, we get $\gamma_{pd}(T(C_n)) = 2\left\lceil \frac{n}{3} \right\rceil$ for $n \geq 3$. □

Theorem 3.2. *If $T(C_n)$ is the total graph of C_n, then*

$$PDI(T(C_n)) = \begin{cases} n+2 & ; \quad n = 3, 4 \\ 2\left\lceil \frac{n+6}{3} \right\rceil & ; \quad n \geq 5 \end{cases}$$

Proof. Let $V(T(C_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n\}$. The adjacencies of $T(C_n)$ are discussed in Theorem 3.1 which states that $\gamma_{pd}(T(C_n)) = 2\left\lceil \frac{n}{3} \right\rceil$ for $n \geq 3$. Let S be any PDS of $T(C_n)$. We consider all the paired dominating sets discussed in the previous theorem and find the one that gives the minimum value for the sum $|S| + m(T(C_n) - S)$.

If S contains vertices from either $\{v_i\}$ only or $\{e_i\}$ only, then $< S >$ need not be connected but $< V - S >$ is always connected. In this case, $|S| + m(T(C_n) - S) = 2n = V(T(C_n))$. When $S = \{v_{1+2t}, e_{1+2t}\}$, $|S| + m(T(C_n) - S)$ is $n + 2$ and $n + 3$ for even and odd orders respectively. Let us consider $S = \{e_{1+3t}, e_{2+3t}\}$ for $T(C_n)$ except for $n = 4 + 3t$ and $S = \{e_{1+3t}, e_{2+3t}, e_n, e_{n-1}\}$ for $T(C_{4+3t})$. Here, we observe that $< V - S >$ is a connected graph and hence $|S| + m(T(C_n) - S) = 2n = V(T(C_n))$.

If $S = \{v_{1+3t}, e_{1+3t}\}$ for $0 \leq t \leq \left\lfloor \frac{n-1}{3} \right\rfloor$, where $n \geq 5$, such that each v_{1+3t} is paired with e_{1+3t} in S, then the order of any component in $< V - S >$ is atmost four since the components in $< V - S >$ are either diamond graphs or P_2. Therefore, $|S| + m(T(C_n) - S) = 2\left\lceil \frac{n}{3} \right\rceil + 4 = 2\left\lceil \frac{n+6}{3} \right\rceil$.

Since $PDI(T(C_n)) = min\{|S| + m(T(C_n) - S) : S$ is a paired dominating set of $T(C_n)\}$, we choose $S = \{v_{1+3t}, e_{1+3t}\}$ for $0 \leq t \leq \left\lfloor \frac{n-1}{3} \right\rfloor$, where $n \geq 5$. However, S is not the same when $n = 3$ or $n = 4$. We consider the following cases.

Case(i): $n = 3, 4$
For $n = 3$, if $|S| = \gamma_{pd} = 2$, then $|S| + m(T(C_n) - S) = 2 - (2n - 2) = 2n$ since $m(T(C_n - S))$ is connected. However, if S contains four vertices such that $< V - S >$ contains two isolated vertices, then $|S| + m(T(C_n) - S) = 4 + 1 = 5$. When $n = 4$, we choose $S = \{v_{1+2t}, e_{1+2t}\}$ and $S = \{v_{2+2t}, e_{2+2t}\}$, where $0 \leq t \leq 1$. Here, $< V - S >$ contains two components, each of which is a P_2. Then $|S| + m(T(C_n) - S) = 4 + 2 = 6$. Therefore, $PDI(T(C_n) = n + 2$ when $n = 3$ or $n = 4$.

Case(ii): $n \geq 5$
For $n \geq 5$, we choose $S = \{v_{1+3t}, e_{1+3t}\}$ for $0 \leq t \leq \left\lfloor \frac{n-1}{3} \right\rfloor$ such that every v_{1+3t} is paired with e_{1+3t} in S so that $< S >$ contains a perfect matching. The set $S = \{v_1, e_n, v_{1+3t}, e_{3t}\}$ for $1 \leq t \leq \left\lceil \frac{n}{4} \right\rceil$ also gives a minimum value of $|S| + m(T(C_n) - S)$. The cardinality of both the paired dominating sets equals the paired domination number, which is found in Theorem 3.1. Then, $|V - S| = 2n - 2\left\lceil \frac{n}{3} \right\rceil$. The number of components in $< V - S >$ is found

to be $\left\lceil \frac{n-1}{3} \right\rceil$. Therefore, the cardinality of any component in $< V - S >$ is at most four. For example, for $T(C_9)$ (see Fig. 2), $|V - S| = 2n - 2\left\lceil \frac{n}{3} \right\rceil = 12$. Then, $m(T(C_9) - S) = \frac{|V-S|}{\left\lceil \frac{n-1}{3} \right\rceil} = \frac{12}{3} = 4$. Therefore, $|S| + m(T(C_n) - S) = 2\left\lceil \frac{n}{3} \right\rceil + 4 = 2\left\lceil \frac{n+6}{3} \right\rceil$. □

4. PDI of Central graph of C_n

Definition 4.1. [14] *The central graph of G, denoted by C(G), is the graph obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices in G.*

Theorem 4.1. *If $C(C_n)$ is the central graph of C_n, then*

$$\gamma_{pd}(C(C_n)) = \begin{cases} 4 & ; \quad n = 3, 4, 5 \\ 2\left\lceil \frac{n}{4} \right\rceil ; & n \geq 6 \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$. Then $V(C(C_n)) = \{v_1, v_2, \ldots, v_{n-1}, v_n, u_1, u_2, \ldots, u_{n-1}, u_n\}$, where $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ be the vertices of $C(C_n)$ that subdivide the edges $v_nv_1, v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_n$ respectively. Let S be any PDS of $C(C_n)$. Further, we note that $\{u_1, u_2, \ldots, u_n\}$ is an independent set in $C(C_n)$. The following two cases are considered.

Case(i): $n = 3, 4, 5$
When $n = 3$, no v_i has a neighbor in $\{v_1, v_2, v_3\}$. We observe that $C(C_3)$ is the cycle C_6 with vertices labelled as $u_1, v_1, u_2, v_2, u_3, v_3$ in order. Therefore, the end vertices of every edge in the matching induced by a PDS are v_i and u_j, where $1 \leq i, j \leq n$. Consequently, $\gamma_{pd}(C(C_3)) = \gamma_{pd}(C_6) = 2\left\lceil \frac{n}{4} \right\rceil = 4$.

For $C(C_4)$, every $v_i \in C(C_4)$ has only one neighbor in $V(C_n)$. That is, each $v_i \in C(C_n)$ is adjacent to a $v_j \in C(C_n)$ such that $v_j \notin N(v_i)$ in C_n. We observe that at least four vertices must belong to a PDS of $C(C_4)$. Therefore, $\gamma_{pd}(C(C_4)) = 4$.

When $n = 5$, any $n - 1$ vertices from $V(C_5)$ is enough to dominate the entire vertex set of $C(C_5)$. This is because C_5 is an induced subgraph of $C(C_5)$ with vertices $\{v_i\}$, where $1 \leq i \leq 5$. Therefore, $\gamma_{pd}(C(C_5)) = 4$.

Case(ii): $n \geq 6$
In this case, we observe that any two vertices v_i, v_j, where $1 \leq i, j \leq n$ and $i \neq j$, dominate all of $V(C_n)$ in $C(C_n)$ such that v_i and v_j are adjacent in $C(C_n)$ and must have no common neighbor in C_n. Therefore, any PDS of $C(C_n)$ must contain at least two such vertices of $C(C_n)$. Further, we choose more pairs of vertices from $\{v_1, v_2, \ldots, v_n\}$ until the entire vertex set of $C(C_n)$ is dominated. The vertices u_i are of degree two and have neighbors from $\{v_1, v_2, \ldots, v_n\}$ only. If $u_i \in S$, then it can only be paired with v_j such that v_j is adjacent to u_i. As a result, S will contain more vertices, and hence it is not the minimum. Therefore, the γ_{pd}-set must contain only vertices from $\{v_1, v_2, \ldots, v_n\}$. The domination number of $C(C_n)$ is found to be $|D| = \gamma_{pd}(C(C_n)) = 2\left\lceil \frac{n}{4} \right\rceil$.

□

Theorem 4.2. *The paired domination integrity for the central graph C of C_n is given by*

$$PDI(C(C_n)) = \begin{cases} 5 \quad ; & n = 3, 4 \\ n+2; & n \geq 5 \;\; \& \;\; n \text{ is odd} \\ n+1; & n \geq 6 \;\; \& \;\; n \text{ is even} \end{cases}$$

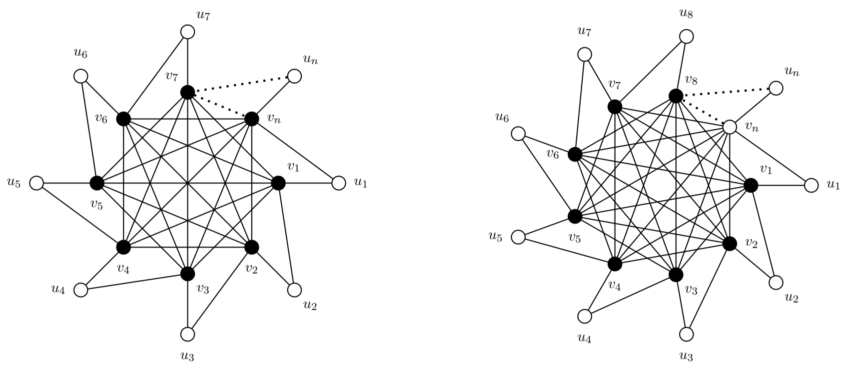
Proof. Let S be any paired dominating set.
Case(i): $n = 3, 4$
When $n = 3$, the order of $C(C_n)$ is six. In this case, both $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$ are independent sets. By Theorem 4.1, $\gamma_{pd}(C(C_3)) = 4$ and we choose γ_{pd}-set S such that $< V - S >$ is a totally disconnected graph. Therefore, $PDI(C(C_3)) = \gamma_{pd} + m(C(C_n) - S) = 4 + 1 = 5$.

When $n = 4$, the order of $C(C_n) = 8$. In this case, each v_i is adjacent to a v_j such that v_j is a nonadjacent vertex of v_i in C_4. By the previous theorem, $\gamma_{pd}(C(C_4)) = 4$ and hence the possible cardinalities of S are 4, 6 and 8. When $|S|$ is 6 or 8, the sum is clearly larger as compared to when $|S| = 4$. We choose S such that $m(C(C_4) - S) = 1$. Therefore, $PDI(C(C_4)) = \gamma_{pd} + m(C(C_4) - S) = 4 + 1 = 5$.

It is observed that for $n \geq 6$, no γ_{pd}-set of $C(C_n)$ gives a minimum value of $|S| + m(C(C_n) - S)$ since $m(C(C_n) - S)$ is very large. Moreover, we observe that if S contains at least one pair of vertices of the form $v_i \; u_j$, then such a PDS also does not give a minimum sum. This is because each u_i dominates exactly two vertices from $\{v_1, v_2, \ldots, v_n\}$.

Case(ii): $n \geq 5 \;\; \& \;\; n$ is odd
In this case, we choose S as all but one vertices of $\{v_1, v_2, \ldots, v_n\}$. Such a set gives $m(C(C_n) - S) = 3$. Therefore, $PDI(C(C_n)) = |S| + m(C(C_n) - S) = (n - 1) + 3 = n + 2$.

Case(iii): $n \geq 6 \;\; \& \;\; n$ is even
For even orders of $C(C_n)$, where $n \geq 6$, we choose S as $\{v_1, v_2, \ldots, v_n\}$. Such a set gives $m(C(C_n) - S) = 1$. Therefore, $PDI(C(C_n)) = |S| + m(C(C_n) - S) = n + 1$. □



(A) $PDI(C(C_n))$ when $n \geq 6 \;\; \&$ n is even. (B) $PDI(C(C_n))$ when $n \geq 5 \;\; \&$ n is odd

FIGURE 3. PDI of Central graph of cycles

5. PDI of Mycielskian of C_n

Definition 5.1. [9] *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Then the Mycielskian of G, denoted by $\mu(G)$, has the vertex set $V(\mu(G)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$, where $\{u_1, u_2, \ldots, u_n\}$ is a copy of V called the shadow* *vertices and w is the root vertex. The edge set is $E(\mu(G)) = E(G) \cup \{u_i w : 1 \leq i \leq n\} \cup \{u_i v_j : v_i v_j \in E(G)\}$.*

Theorem 5.1. *Let $\mu(C_n)$ denote the Mycielskian of C_n, where $n \geq 4$. Then $\gamma_{pd}(\mu(C_n)) = PDI(C_n) = \gamma_{pd}(C_n) + 2$ except for C_{5+4t}, where $t \geq 0$. For C_{5+4t}, $\gamma_{pd}(\mu(C_n)) = PDI(C_n) - 2 = \gamma_{pd}(C_n)$.*

Proof. Let $V(\mu(C_n)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$, where $\{v_1, v_2, \ldots, v_n\}$ is the vertex set of C_n, and $u_1, u_2, \ldots, u_n$ subdivides $v_n v_2, v_1 v_3, v_2 v_4, \ldots, v_{n-1} v_1$ respectively (Fig. 4). The vertex w is adjacent to every vertex in $\{u_1, u_2, \ldots, u_n\}$. Let S be any paired dominating set of $\mu(C_n)$. Clearly, every paired dominating set of $\mu(C_n)$ must contain vertices from $\{v_1, v_2, \ldots, v_n\}$ as $\{u_1, u_2, \ldots, u_n\}$ is an independent set and w is not adjacent to any v_i. We observe that w is the vertex with the maximum degree that dominates the whole of u_i and hence may belong to S. If $w \in S$, then it is paired with one of its neighbors u_i. To dominate $\{v_1, v_2, \ldots, v_n\}$, we can either choose paired vertices of the form v_i, v_k or u_i, v_k, where $1 \leq i, k \leq n$. We consider γ_{pd}-set of C_n found in [1], which are paired vertices of the form v_i, v_k. This set dominates all v_i, where $1 \leq i \leq n$. As a result, $\{w, u_i\} \cup \{\gamma_{pd} - \text{set of } C_n\}$ forms the γ_{pd}-set of $\mu(C_n)$. Therefore, $\gamma_{pd}(\mu(C_n)) = 2 + \gamma_{pd}(C_n) = 2 + 2\left\lceil \frac{n}{4} \right\rceil = PDI(C_n)$. However, the γ_{pd}-set is not the same for $\mu(C_{5+4t})$, where $t \geq 0$. For the γ_{pd}-set of $\mu(C_{5+4t})$, we choose the γ_{pd}-set of C_n and replace the last pair of vertices v_{n-1}, v_n by u_{n-1}, v_n so that w is dominated. Therefore, for $n = 5 + 4t$, $\gamma_{pd}(\mu(C_n)) = \gamma_{pd}(C_n) = 2\left\lceil \frac{n}{4} \right\rceil = PDI(C_n) - 2$.

As mentioned earlier, another γ_{pd}-set of $\mu(C_n)$ is the set that contains the minimum possible number of paired vertices of the form u_i, v_k that dominates $\{v_1, v_2, \ldots, v_n\}$ along with w paired with one of its neighbors u_i that will dominate all of $\{v_i\}$. However, if S contains only vertices of the form u_i, v_k, then the PDS is found to be of larger cardinality since $w \notin S$. □

Theorem 5.2. *For $n \geq 3$, the PDI of the Mycielskian of a cycle $\mu(C_n)$ is given by,*

$$PDI(\mu(C_n)) = \begin{cases} n+3 & ; \quad 3 \leq n \leq 8 \\ 2\left\lceil \frac{n}{3} \right\rceil + 5 & ; \quad 9 \leq n \leq 15 \\ 2\left\lceil \frac{n}{4} \right\rceil + 7 & ; \quad n \equiv 1 \pmod 4 \\ 2\left\lceil \frac{n-3}{3} \right\rceil + 6; & 19 \leq n \leq 27. \end{cases}$$

Further, $PDI(\mu(C_n)) \leq 2\left\lceil \frac{n}{3} \right\rceil + 5$ otherwise.

Proof. Let S be any paired dominating set of $\mu(C_n)$. The vertices of $\mu(C_n)$ are labeled as seen in Fig. 4. Clearly, w is the vertex with maximum degree, as mentioned in Theorem 5.1. We observe that w need not belong to a $\gamma_p d-$ set of $\mu(C_n)$. However, to find the $PDI(\mu(C_n))$, we choose a paired dominating set S such that $w \in S$. This is because if $w \notin S$, then $< V - S >$ is either connected or will contain larger connected components. We consider all possible paired dominating sets that contain w and identify the one that gives the minimum value for $|S| + m(\mu(C_n) - S)$. We choose paired vertices from $\{v_1, v_2, \ldots, v_n\}$, such that we get a minimum value for $|S| + m(\mu(C_n) - S)$. This is obtained by choosing v_i either in pairs of the form $v_1, v_2, v_4, v_5, \ldots, v_{n-1}, v_n$ or in pairs of the form $v_1, v_2, v_5, v_6, \ldots, v_{n-1}, v_n$ depending on the order of $\mu(C_n)$. Further, we choose the vertex u_i that is more connected in $< V - S >$ and pair it with w that dominates the whole of u_i, where $1 \leq i \leq n$. We consider the following cases.

Case(i): $3 \leq n \leq 8$
In this case, we observe that the paired dominating set S of the form $\{w, u_i, v_1, v_2, v_4, v_5, \ldots, v_{n-1}, v_n\}$ gives the minimum sum $|S| + m(\mu(C_n) - S)$. The connected components of $< V - S >$ are P_1, P_2 or P_3 which implies that $m(\mu(C_n) - S)$ is atmost 3. Hence, we get $|S| + m(\mu(C_n)) = n + 3$, and the obtained sum is the minimum over all the paired dominating sets. Therefore, $PDI(\mu(C_n)) = n + 3$.

Case(ii): $9 \leq n \leq 15$
As in Case(i), the paired dominating set S of the form $\{w, u_i, v_1, v_2, v_4, v_5, \ldots, v_{n-1}, v_n\}$ is the $PDI-$set for this case. The order of the largest component of $< V(\mu(C_n)) - S >$ is exactly three as shown in Fig. 4 and the cardinality of S is found to be $2\left\lceil \frac{n}{3} \right\rceil + 2$. Therefore, $PDI(\mu(C_n)) = 2\left\lceil \frac{n}{3} \right\rceil + 2 + 3 = 2\left\lceil \frac{n}{3} \right\rceil + 5$.

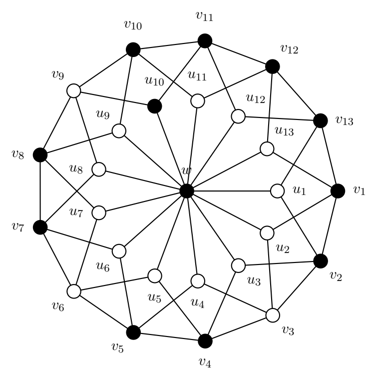


FIGURE 4. PDI-set of $\mu(C_{13})$.

Case(iii): $n \equiv 1 \pmod 4$
If $n \equiv 1 \pmod 4$, then the paired dominating set of the form $S = \{w, u_n, v_1, v_2, v_5, v_6, \ldots, v_{n-4}, v_{n-3}\}$ is a $PDI-$set. If $S = \{w, u_i, v_1, v_2, v_4, v_5, \ldots, v_{n-1}, v_n\}$, then $m(\mu(C_n) - S) = 3$. However, the cardinality of S is much larger. Hence, two out of every four vertices in the cycle formed by $\{v_1, v_2, \ldots, v_n\}$ of $\mu(C_n)$ belong to S to dominate the cycle. However, v_{n-1} remains undominated. The vertex w is paired with u_n in order to dominate v_{n-1} and the vertices of $\{u_1, u_2, \ldots, u_n\}$ that are not dominated. Thus, the cardinality of S is found to be $2\left\lceil \frac{n}{4} \right\rceil$ and $m(\mu(C_n) - S) = 7$. Therefore, $PDI(\mu(C_n)) = 2\left\lceil \frac{n}{4} \right\rceil + 7$.

Case(iv): $19 \leq n \leq 27$
For $19 \leq n \leq 27$, the paired dominating set S of the form $\{w, u_i, v_1, v_2, v_5, v_6, \ldots\}$ gives the minimum value for $|S| + m(\mu(C_n) - S)$. Here, w paired with any u_i results in $m(\mu(C_n) - S) = 6$ (see Fig. 5). Hence, $PDI(\mu(C_n)) = 2\left\lceil \frac{n-3}{3} \right\rceil + 6$.

Case(v):
In this case, we obtain an upper bound for $PDI(\mu(C_n))$ for all values of n not considered in the above four cases. The paired dominating set S is of the form $\{w, u_i, v_1, v_2, v_5, v_6, \ldots\}$. We observe that $m(\mu(C_n) - S) = 6$ for such a set. Therefore,

$$
\begin{aligned}
PDI(\mu(C_n)) &\leq \left(2\left\lceil\frac{n}{3}\right\rceil - 1\right) + 6 \\
&\leq \left(2\left\lceil\frac{n}{3}\right\rceil\right) + 5.
\end{aligned}
$$

□

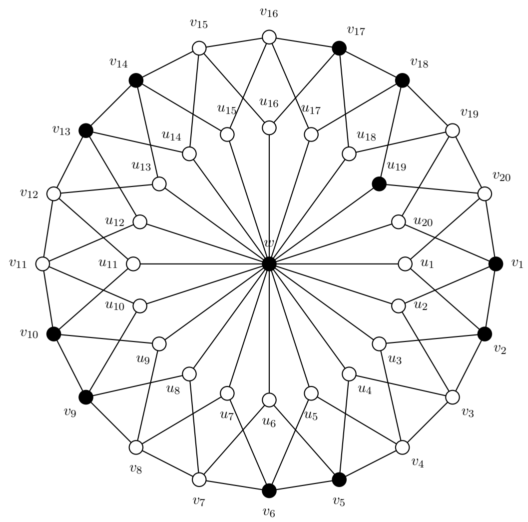


FIGURE 5. PDI-set of $\mu(C_{20})$.

Remark 5.1. *Any PDI-set of $\mu(C_n)$ must contain the vertex of degree equal to $\Delta(\mu(C_n))$. A paired dominating set S not containing the maximum degree vertex will result in $< V - S >$ being connected or $< V - S >$ containing larger components.*

The vertex of $\mu(C_n)$ with maximum degree is the root vertex w (see Definition 5.1). This root vertex is adjacent to all the shadow vertices u_i of $\mu(C_n)$ as shown in Fig. 5 and thus has a degree equal to n. If w does not belong to the paired dominating set S then the induced subgraph of $V(\mu(C_n)) - S$ is a connected graph of larger order formed by w, u_i and v_i not in S. As a result, $|S| + m(\mu(C_n) - S)$ is a larger value and thus S is not a PDI-set. Therefore, the root vertex must belong to any PDI-set of $\mu(C_n)$.

6. Conclusion

In this paper, the authors have studied the vulnerability parameter, namely, paired domination integrity for larger networks modelled by a few derived graphs of cycles such as the Middle, Total, Central and Mycielskian graphs. The paired domination integrity can be explored for other graph classes and operations. As paired domination integrity is a graph vulnerability parameter, it can be extended to various real-life networks. Understanding and using the principles of the graph vulnerability parameters is essential in a world that relies on networked systems.

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