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EXISTENCE, UNIQUENESS, AND STABILITY OF SOLUTIONS FOR NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS AND FRACTIONAL DERIVATIVES

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Abstract. This study investigates a nonlinear fractional integro-differential equation defined by Riemann-Liouville fractional derivatives, focusing on the existence, uniqueness, and stability of its solutions. Using advanced fixed-point theorems, specifically the Banach and Krasnoselskii's fixed-point theorems, we derive precise conditions for the existence and uniqueness of solutions. We also conduct a stability analysis, establishing criteria to ensure the robustness of solutions under minor perturbations. The theoretical results extend existing frameworks in fractional differential equations and provide novel insights into fractional dynamic systems. To validate our theoretical findings and demonstrate their practical applicability, we present a numerical example that illustrates the solution behavior under varying fractional orders, nonlinearities, and boundary conditions. This example highlights the effectiveness of the proposed methods and lays the foundation for future research on fractional integro-differential equations in real-world applications.

Keywords: Fractional calculus, Riemann-Liouville fractional derivatives, Nonlinear integro-differential equations, Existence and uniqueness, Stability analysis, Numerical example.

AMS Subject Classification: Primary: 26A33, 34A08, 45J05; Secondary: 34D20, 34G20, 65L05.

1. Introduction

Fractional calculus extends the traditional framework of calculus by allowing differentiation and integration to be performed at non-integer orders. This generalization has significantly impacted various fields by enabling the modeling of complex phenomena that

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cannot be captured by classical integer-order differential equations [26], [27]. The ability of fractional calculus to incorporate memory and hereditary effects into mathematical models makes it particularly useful in domains such as viscoelasticity [3], fluid mechanics [14], and control theory [25], among others.

The fractional derivatives, including the Riemann-Liouville, Caputo, and Grunwald-Letnikov derivatives, offer a versatile framework for describing systems with intricate dynamics. The Riemann-Liouville derivative, in particular, has gained prominence in practical applications due to its compatibility with initial conditions expressed in integer-order terms [6], [18]. This compatibility is crucial in scenarios where the initial state of a system is known, allowing for accurate modeling of physical and engineering processes where initial conditions significantly influence future behavior [7], [32].

In recent years, the study of nonlinear fractional integro-differential equations (NLFDEs), particularly those characterized by Riemann fractional derivatives, has emerged as a significant area of research. These equations extend classical models by incorporating fractional orders, thereby providing a robust framework for describing processes with long-range dependencies and non-local interactions [3], [4], [10]. The inclusion of nonlinearities further enhances the model's capability to capture complex, often chaotic behaviors observed in natural systems [34], [31]. Notably, such equations have become indispensable in explaining phenomena in various scientific and engineering fields, such as anomalous diffusion [25], complex fluid flows [14], and control systems [8], [9].

Recent advancements have led to the development of new techniques for addressing the existence, uniqueness, and stability of solutions for nonlinear fractional differential equations with nonlocal boundary conditions. These studies have expanded our understanding of fractional systems and their behavior under different conditions. Notably, the work on boundary value problems for fractional differential equations has seen significant progress [1], [15], [28], highlighting the importance of fractional derivatives in modeling systems with memory effects. Stability and well-posedness studies in fractional calculus have also gained traction, providing a foundation for ensuring the stability of solutions in the presence of perturbations [11], [23].

However, while much research has been devoted to these topics, the study of nonlinear fractional integro-differential equations (NLFDEs) with nonlocal boundary conditions, particularly those involving the Riemann-Liouville fractional derivative, remains an area that requires further exploration. In particular, solutions to such equations, which involve both fractional differentiation and integral terms, are essential for understanding the complex dynamics of real-world systems. The majority of existing work focuses on integer-order or standard fractional models without fully exploiting the potential of more generalized fractional systems, including nonlocal boundary conditions [31], [12], [13].

In this context, our focus is on analyzing a specific class of nonlinear fractional integro-differential equations characterized by Riemann-Liouville fractional derivatives. We aim to investigate the existence, uniqueness, and stability of solutions, providing insights into the practical applications and implications of these theoretical results. The equation under consideration is given by:

$$D^{\alpha}_{0^+}\left(\chi(\xi) - g(\xi,\chi(\xi))\right) = f(\xi,\chi(\xi), D^{\alpha-1}_{0^+}\chi(\xi), \int_{0^+}^{\xi} \varphi(\xi,\tau) D^{\alpha-1}_{0^+}\chi(\tau), d\tau), \tag{1}$$

subject to the nonlocal boundary conditions:

$$\chi(0) + D^{\alpha-1}_{0^+}\chi(0) = \sigma_1 \chi(\eta_1), \quad 0 < \eta_1 < 1, \tag{2}$$

$$\chi(1) + D_{0^+}^{\alpha-1}\chi(1) = \sigma_2\chi(\eta_2), \quad 0 < \eta_2 < 1. \tag{3}$$

Here, $\alpha \in (1,2]$ is a real number, $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and the kernel $\varphi : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and for $i = 1,2$, we have $\eta_i \in (0,1)$ and $\sigma_i \in \mathbb{R}$. The equation captures the interplay between fractional differentiation and nonlinear dynamics, with the nonlocal boundary conditions reflecting the global influence of boundary values on the entire solution domain.

The main aim of this study is to examine the existence, uniqueness, and stability of solutions for the nonlinear fractional integro-differential equation presented. Our contributions are threefold: First, we provide a rigorous proof of existence by applying fixed-point theorems specifically designed for fractional calculus. Second, we investigate the uniqueness of solutions through an analysis of the relevant functional spaces and the properties of the nonlinear components. Third, we address the stability of solutions, outlining the conditions that ensure stability in the presence of small perturbations. Our findings expand upon and generalize current theories in fractional differential equations, offering fresh insights into the dynamics of systems described by fractional models. Additionally, to validate our theoretical results and demonstrate their practical application, we present a numerical example. This example is strategically chosen to illustrate the solution behavior under various conditions, including different fractional orders, types of nonlinearities, and boundary conditions. By juxtaposing analytical results with numerical simulations, we affirm the accuracy and robustness of our methods. This example not only underscores the practical importance of our theoretical contributions but also provides a foundation for future research in applying fractional integro-differential equations to real-world scenarios.

By addressing these aspects, we contribute to the growing body of literature on fractional integro-differential equations, particularly those involving Riemann derivatives and nonlocal boundary conditions. Our findings not only advance the theoretical understanding of such equations but also have potential applications in various fields where fractional dynamics play a critical role.

2. Main Results

In the following, we present essential notations, definitions, lemmas, and theorems that serve as foundational elements for our research.

Definition 2.1. [26] *Let $\alpha > 0$, and consider an interval $[a,b]$ where $-\infty < a < \xi < b < +\infty$, with $\chi \in L_1([a,b],\mathbb{R})$. Then,*

(1) *The Riemann-Liouville fractional integral of a function $\chi(\xi)$ of order α is given by:*

$$I_{a^+}^{\alpha}\chi(\xi) = \frac{1}{\Gamma(\alpha)}\int_a^{\xi}(\xi-\tau)^{\alpha-1}\chi(\tau)\,ds, \tag{4}$$

where $\Gamma(\alpha)$ denotes the Euler gamma function, defined by $\Gamma(\alpha) = \int_0^{+\infty}\xi^{\alpha-1}e^{-\xi}\,dt$.

(2) *The Riemann-Liouville fractional derivative of order α, with $n = [\alpha]+1$ where $[\alpha]$ is the integer part of α, is defined as:*

$$D_{a^+}^{\alpha}\chi(\xi) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^{\xi}(\xi-\tau)^{n-\alpha-1}\chi(\tau)\,ds. \tag{5}$$

To examine the existence of solutions for a fractional differential equation, it is crucial to transform it into an equivalent integral equation, leveraging the key properties of $I_{a^+}^{\alpha}$ and $D_{a^+}^{\alpha}$.

Lemma 2.1. [3] *Let $\alpha, \beta \in \mathbb{R}^+$, and $f(\xi) \in L_1[a,b]$. Then, for all $\xi \in [a,b]$:*

(1) *The fractional integral $I_{a^+}^{\alpha}\chi(\xi)$ exists almost everywhere.*
(2) $I_{a^+}^{\alpha} I_{a^+}^{\beta} f(\xi) = I_{a^+}^{\beta} I_{a^+}^{\alpha} f(\xi) = I_{a^+}^{\alpha+\beta} f(\xi)$.
(3) $(I_{a^+}^{\alpha})^n f(\xi) = I_{a^+}^{n\alpha} f(\xi)$, *where $n \in \mathbb{N}$.*
(4) $D_{a^+}^{\alpha} I_{a^+}^{\alpha}\chi(\xi) = \chi(\xi)$ *for all ξ.*
(5) $D_{0^+}^{\beta} I_{0^+}^{\alpha}\chi(\xi) = I_{0^+}^{\alpha-\beta}\chi(\xi)$ *for $\beta \in [0, \alpha)$.*
(6) $D_{0^+}^{\alpha}\chi(\xi) = I_{0^+}^{-\alpha}\chi(\xi)$ *for $\alpha < 0$ and $\xi \geq 0$.*

Lemma 2.2. *For $\beta > -1$, $\beta \neq \alpha-1, \alpha-2, \dots, \alpha_1 - n$, then for $\xi \geq 0$:*

$$D_{0^+}^{\alpha}\xi^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\xi^{\beta-\alpha}, \tag{6}$$

and $D_{0^+}^{\alpha}\xi^{\alpha-i} = 0$ for all $i = 1, 2, 3, \dots, n$.

Lemma 2.3. [26] *If $\alpha > 0$, then the general solution to the homogeneous equation $D_{0^+}^{\alpha}u(\xi) = 0$ in $C(I,\mathbb{R}) \cap L_1(I,\mathbb{R})$ is given by:*

$$u(\xi) = c_1\xi^{\alpha-1} + c_2\xi^{\alpha-2} + c_3\xi^{\alpha-3} + \cdots + c_n\xi^{\alpha-n}, \quad \xi \in I. \tag{7}$$

Definition 2.2. *A function $x \in C([0,1],\mathbb{R})$ is said to be a solution to the fractional differential equation (1) if it satisfies the following condition:*

$$D_{0^+}^{\alpha}\left(\chi(\xi) - g(\xi,\chi(\xi))\right) = f(\xi, \chi(\xi), D_{0^+}^{\alpha-1}\chi(\xi), \int_{0^+}^{\xi}\varphi(\xi,\tau)D_{0^+}^{\alpha-1}\chi(\tau)\,d\tau),$$

along with the associated nonlocal boundary conditions:

$$\chi(0) + D_{0^+}^{\alpha-1}\chi(0) = \sigma_1\chi(\eta_1), \quad 0 < \eta_1 < 1,$$

and

$$\chi(1) + D_{0^+}^{\alpha-1}\chi(1) = \sigma_2\chi(\eta_2), \quad 0 < \eta_2 < 1.$$

Now, to facilitate the understanding of the solution to the nonlinear implicit fractional differential equation (NLFDE), we present the following lemma. It provides an equivalent fractional integral form of the solution, which will be essential for further analysis of existence and uniqueness.

Lemma 2.4. *Let $1 < \alpha \leq 2$, and suppose $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. A function $\chi(\xi)$, defined on the interval $[0,1]$, is said to be a solution of the nonlinear implicit fractional differential equation NLFDE (1-3) if and only if it satisfies the following fractional integral equation:*

$$\chi(\xi) = g(\xi,\chi(\xi)) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\tau)^{\alpha-1}u(\tau)d\tau \tag{8}$$

$$+\xi^{\alpha-1}\left(\begin{array}{c} -\frac{p}{\Delta}g(0,\chi(0)) + \frac{p\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{n}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right) \end{array}\right)$$

$$+\xi^{\alpha-2}\left(\begin{array}{c}\frac{z}{\Delta}g(0,\chi(0))-\frac{z\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1))-\frac{z\sigma_1}{\Gamma(\alpha)\Delta}\int\limits_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau\\ +\frac{m}{\Delta}\left(\begin{array}{c}\sigma_2 g(\eta_2,\chi(\eta_2))-g(1,\chi(1))+\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau\\ -\frac{1}{\Gamma(\alpha)}\int\limits_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau-\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int\limits_0^1 u(\tau)d\tau\end{array}\right)\end{array}\right).$$

Proof. Let $\chi(\xi)$ be a solution to the nonlinear fractional integro-differential equation (NLFDE) (1): We define the continuous function $u(\xi)$ as follows:

$$u(\xi)=f(\xi,\chi(\xi),D_{0^+}^{\alpha-1}\chi(\xi),\int_{0^+}^{\xi}\varphi(\xi,\tau)D_{0^+}^{\alpha-1}\chi(\tau)\,d\tau). \tag{9}$$

Consequently, the NLFDE (1)-(3) can be rewritten in the equivalent form:

$$D_{0^+}^{\alpha}\left[\chi(\xi)-g(\xi,\chi(\xi))\right]=u(\xi). \tag{10}$$

By employing Lemma 2.3 and applying the Riemann-Liouville fractional integral, the solution can be expressed as:

$$\chi(\xi)=g(\xi,\chi(\xi))+c_0\xi^{\alpha-1}+c_1\xi^{\alpha-2}+I_{0^+}^{\alpha}u(\xi). \tag{11}$$

Incorporating the boundary conditions (2)-(3), the following equations are obtained:

$$c_0(\Gamma(\alpha)-\sigma_1\eta_1^{\alpha-1})-c_1\ \sigma_1\eta_1^{\alpha-2}=\ \sigma_1 g(\eta_1,\chi(\eta_1))+\sigma_1 I^{\alpha}u(\eta_1)-g(0,\chi(0)) \tag{12}$$

and

$$\begin{aligned}&c_0(1+\Gamma(\alpha)-\sigma_2\eta_2^{\alpha-1})+c_1(1-\sigma_2\eta_2^{\alpha-2}) \qquad (13)\\ &=\sigma_2 g(\eta_2,\chi(\eta_2))-g(1,\chi(1))+\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau\\ &-\frac{1}{\Gamma(\alpha)}\int\limits_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau-\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int\limits_0^1 u(\tau)d\tau\end{aligned}$$

For the sake of simplicity, let us consider

$$m=\Gamma(\alpha)-\sigma_1\eta_1^{\alpha-1},\ n=\sigma_1\eta_1^{\alpha-2},\ z=1+\Gamma(\alpha)-\sigma_2\eta_2^{\alpha-1},\ \text{and } p=1-\sigma_2\eta_2^{\alpha-2}.$$

By setting $nz+pm=\Delta$ and solving equations (12) and (13), we derive the following:

$$\begin{aligned}c_0=&-\frac{p}{\Delta}g(0,\chi(0))+\frac{p\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1))+\frac{p\sigma_1}{\Gamma(\alpha)\Delta}\int\limits_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau\\ &+\frac{n}{\Delta}\left(\begin{array}{c}\sigma_2 g(\eta_2,\chi(\eta_2))-g(1,\chi(1))+\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau\\ -\frac{1}{\Gamma(\alpha)}\int\limits_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau-\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int\limits_0^1 u(\tau)d\tau\end{array}\right)\end{aligned} \tag{14}$$

and

$$\begin{aligned}c_1=&\frac{z}{\Delta}g(0,\chi(0))-\frac{z\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1))-\frac{z\sigma_1}{\Gamma(\alpha)\Delta}\int\limits_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau\\ &+\frac{m}{\Delta}\left(\begin{array}{c}\sigma_2 g(\eta_2,\chi(\eta_2))-g(1,\chi(1))\\ +\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau-\frac{1}{\Gamma(\alpha)}\int\limits_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau\\ -\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int\limits_0^1 u(\tau)d\tau\end{array}\right)\end{aligned} \tag{15}$$

Substituting c_0 and c_1 in these into (11), we obtain that:

$$\chi(\xi) = g(\xi,\chi(\xi)) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\tau)^{\alpha-1}u(\tau)d\tau \tag{16}$$

$$+\xi^{\alpha-1}\left(\begin{array}{c} -\frac{p}{\Delta}g(0,\chi(0)) + \frac{p\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{n}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right) \end{array}\right)$$

$$+\xi^{\alpha-2}\left(\begin{array}{c} \frac{z}{\Delta}g(0,\chi(0)) - \frac{z\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) - \frac{z\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{m}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right) \end{array}\right).$$

□

To establish the necessary conditions for the nonlinear fractional differential equation (NLFDE), we present the following lemma, which outlines key assumptions regarding the continuity and boundedness of the functions involved.

Lemma 2.5. *Consider the nonlinear fractional differential equation (NLFDE) given by (1), subject to the following assumptions:*

(H_1) : *The nonlinear function* $f : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ *is continuous. Additionally, there exists a continuous function* $\lambda \in C([0,1],\mathbb{R}^+)$ *with a norm* $\|\lambda\|$, *such that for all* $\xi \in [0,1]$ *and for any* $u_i, v_i \in \mathbb{R}$ $(i = 1,2,3)$, *the following inequality holds:*

$$|f(\xi,u_1,u_2,u_3) - f(\xi,v_1,v_2,v_3)| \le \lambda(\xi)\left(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|\right).$$

(H_2) : *The kernel function* $\varphi : [0,1]\times[0,1] \to \mathbb{R}$ *is continuous for all* $(\tau,v) \in [0,1]\times[0,1]$. *Moreover, there exists a constant* $\Phi > 0$ *such that*

$$\max_{\tau,v\in[0,1]}|\varphi(\tau,v)| = \Phi.$$

(H_3) : *The function* $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous, and there exists a continuous function* $\omega \in C([0,1],\mathbb{R}^+)$ *satisfying the inequality:*

$$|g(\xi,u) - g(\xi,v)| \le \omega(\xi)|u-v|,$$

for all $\xi \in [0,1]$ *and for any* $u, v \in \mathbb{R}$. *Additionally, there exists a constant* $G > 0$ *such that*

$$G = \max_{\xi\in[0,1]} g(\xi,u(\xi)).$$

Remark 2.1. *From the assumption* (H_1), *we deduce that*

$$|f(\tau,u_1,u_2,u_3)| - |f(\tau,0,0,0)| \le |f(\tau,u_1,u_2,u_3) - f(\tau,0,0,0)| \le \|\lambda\|\left(|u_1| + |u_2| + |u_3|\right),$$

and

$$|f(\tau,u_1,u_2,u_3)| \le \|\lambda\|\left(|u_1| + |u_2| + |u_3|\right) + F,$$

where $F = \max_{\tau\in[0,1]}|f(\tau,0,0,0)|$.

Definition 2.3. *Define the operator* $\Theta : C([0,1],\mathbb{R}^+) \to C([0,1],\mathbb{R}^+)$ *as follows:*

$$\Theta(\xi) = g(\xi,\chi(\xi)) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\tau)^{\alpha-1}u(\tau)d\tau$$

$$+\xi^{\alpha-1}\left(\begin{array}{c} -\frac{p}{\Delta}g(0,\chi(0)) + \frac{p\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{n}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right)\end{array}\right)$$

$$+\xi^{\alpha-2}\left(\begin{array}{c} \frac{z}{\Delta}g(0,\chi(0)) - \frac{z\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) - \frac{z\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{m}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right)\end{array}\right).$$

3. Existence of solutions

In this section, we establish the existence of solutions to the nonlinear fractional differential equation (NLFDE) as defined by equations (1)-(3). To achieve this, we apply Krasnoselskii's fixed point theorem. The following theorem outlines the conditions under which a solution to the NLFDE exists.

Theorem 3.1. *Assume that the conditions* (H_1) *through* (H_3) *are satisfied. If the following inequalities hold:*

$$r \geq \frac{a_0}{1-b_0 d_0}, \tag{17}$$

and

$$\left(\begin{array}{c} |\frac{p}{\Delta}|\,\|\omega\| + |\frac{p\sigma_1}{\Delta}|\,\|\omega\| + b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}| \\ +|\frac{n}{\Delta}|\left(\begin{array}{c} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0 + \sigma_2\,\|\omega\| + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{array}\right) + |\frac{z}{\Delta}|\,\|\omega\| + |\frac{z\sigma_1}{\Delta}|\,\|\omega\| \\ +b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}| + |\frac{m}{\Delta}|\left(\begin{array}{c} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0 + \sigma_2\,\|\omega\| + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{array}\right) \end{array}\right) < 1, \tag{18}$$

then the NLFDE (1) has at least one solution in $C[0,1]$.

Proof. Consider the operator $\Theta(\chi(\xi)) = \Theta_1(\chi(\xi)) + \Theta_2(\chi(\xi)); \xi \in [0,1]$, where

$$\Theta_1(\chi(\xi)) = \xi^{\alpha-1}\left(\begin{array}{c} -\frac{p}{\Delta}g(0,\chi(0)) + \frac{p\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{n}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) \\ +\frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right)\end{array}\right) \tag{19}$$

$$+\xi^{\alpha-2}\left(\begin{array}{c}\frac{z}{\Delta}g(0,\chi(0))-\frac{z\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1))-\frac{z\sigma_1}{\Gamma(\alpha)\Delta}\int\limits_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau\\ +\frac{m}{\Delta}\left(\begin{array}{c}\sigma_2 g(\eta_2,\chi(\eta_2))-g(1,\chi(1))\\ +\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau\\ -\frac{1}{\Gamma(\alpha)}\int\limits_0^{1}(1-\tau)^{\alpha-1}u(\tau)d\tau\\ -\frac{1}{\Gamma(1-\alpha)}\int\limits_0^{1}(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int\limits_0^{1}u(\tau)d\tau\end{array}\right)\end{array}\right),$$

and

$$\Theta_2(\chi(\xi))=g(\xi,\chi(\xi))+\frac{1}{\Gamma(\alpha)}\int\limits_0^{\xi}(\xi-\tau)^{\alpha-1}u(\tau)d\tau.$$

Consider the closed and convex subset $B_r=\{\chi\in C([0,1],\mathbb{R});\parallel\chi\parallel\leq r\}$ of $C([0,1]$, where r is a positive real number such that:

$$r\geq\frac{a_0}{1-b_0d_0},$$

where

$$a_0=\left(\begin{array}{c}|\frac{p}{\Delta}|G\left(1+\sigma_1\right)+F\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{n}{\Delta}|\left[G+\frac{F}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+F+\sigma_2G+\frac{\sigma_2F\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]\\ +|\frac{z}{\Delta}|G\left(1+\sigma_1\right)+F\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+G+\frac{F}{\Gamma(\alpha+1)}\\ +|\frac{m}{\Delta}|\left[G+\frac{F}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+F+\sigma_2G+\frac{\sigma_2F\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]\end{array}\right) \tag{20}$$

and

$$b_0=\|\lambda\|\left(1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right) \tag{21}$$

and

$$\begin{aligned}d_0=\eta_1^{\alpha}\left|\frac{p\sigma_1}{\Gamma(\alpha+1)\Delta}\right|+\left|\frac{n}{\Delta}\right|\left[\frac{1}{\Gamma(\alpha+1)}+1+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]\\ +\eta_1^{\alpha}\left|\frac{z\sigma_1}{\Gamma(\alpha_1+1)\Delta}\right|+\left|\frac{m}{\Delta}\right|\left[\frac{1}{\Gamma(\alpha+1)}+1+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{1}{\Gamma(\alpha+1)}.\end{aligned} \tag{22}$$

The proof will be decomposed into three steps:

Step 1: In this step, we have to prove that for every $\chi_1,\ \chi_2\in B_r$, the operator $\Theta_1\chi_1+\Theta_2\chi_2\in B_r$.

$$\begin{aligned}|\Theta_1\chi_1+\Theta_2\chi_2|&\leq|\Theta_1\chi_1|+|\Theta_2\chi_2|\\ &\leq\left|\frac{p}{\Delta}\right|G+\left|\frac{p\sigma_1}{\Delta}\right|G+|\frac{p\sigma_1}{\Gamma(\alpha)\Delta}|\times\int\limits_0^{\eta_1}|\eta_1-\tau|^{\alpha-1}|u(\tau)|d\tau\\ &+|\frac{n}{\Delta}|\left(\begin{array}{c}G+\frac{1}{\Gamma(\alpha)}\int\limits_0^{1}|1-\tau|^{\alpha-1}\ |u(\tau)|d\tau+\\ \frac{1}{\Gamma(1-\alpha)}G.\int\limits_0^{1}|1-\tau|^{-\alpha}d\tau+\int\limits_0^{1}|u(\tau)|d\tau\\ +\sigma_2G+\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}|\eta_2-\tau|^{\alpha-1}\ |u(\tau)|d\tau\end{array}\right)\end{aligned} \tag{23}$$

$$+\left(\left|\frac{z}{\Delta}\right|+\left|\frac{z\sigma_1}{\Delta}\right|\right)G+\left|\frac{z\sigma_1}{\Gamma(\alpha)(\Delta)}\right|\int_0^{\eta_1}|\eta_1-\tau|^{\alpha-1}|u(\tau)|d\tau$$

$$+|\frac{m}{\Delta}|\begin{pmatrix} G+\frac{1}{\Gamma(\alpha)}\int_0^1|1-\tau|^{\alpha-1}\ |u(\tau)|d\tau+\ \frac{1}{\Gamma(1-\alpha)}G\int_0^1|1-\tau|^{-\alpha}d\tau+ \\ \int_0^1|u(\tau)|d\tau+\sigma_2G+\frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}|\eta_2-\tau|^{\alpha-1}\ |u(\tau)|d\tau] \end{pmatrix}$$

$$+G+\frac{1}{\Gamma(\alpha)}\int_0^{\xi}|\xi-\tau|^{\alpha_1-1}\ |u(\tau)|d\tau,$$

where

$$\begin{aligned}|u(\tau)|&=|f(\tau,\chi(\tau),D_{0^+}^{\alpha-1}\chi(\tau),\int_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha_1-1}\chi(v)dv)|\\&\le F+\|\lambda\|\left[|\chi(\tau)|+\frac{1}{\Gamma(2-\alpha)}|\chi(\tau)|+\frac{\Phi}{\Gamma(2-\alpha)}|\chi(\tau)|\right]\\&\le F+\|\lambda\|\left[1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right]\|\chi\|\\&\le\theta,\end{aligned}$$

where $\theta=F+\|\lambda\|\,[1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}]\,\|\chi\|$. Thus, we have

$$\begin{aligned}|\Theta_1\chi_1+\Theta_2\chi_2|\le{}&|\frac{p}{\Delta}|G+|\frac{p\sigma_1}{\Delta}|G+\theta\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)\Delta}|\\&+|\frac{n}{\Delta}|\left[G+\frac{\theta}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+\theta+\sigma_2G+\frac{\sigma_2\theta\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]\\&+|\frac{z}{\Delta}|G+|\frac{z\sigma_1}{\Delta}|G+\theta\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)\Delta}|\\&+|\frac{m}{\Delta}|\left[G+\frac{\theta}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+\theta+\sigma_2G+\frac{\sigma_2\theta\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]+G+\frac{\theta}{\Gamma(\alpha+1)}\\\le{}&|\frac{p}{\Delta}|G+|\frac{p\sigma_1}{\Delta}|G+F\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}|\\&+|\frac{n}{\Delta}|\left[G+\frac{F}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+\sigma_2G+F+\frac{\sigma_2F\eta_2^{\alpha_1}}{\Gamma(\alpha+1)}\right]\\&+\|\chi_1\|\left[b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)\Delta}|+|\frac{n}{\Delta}|\left(\frac{b_0}{\Gamma(\alpha+1)}+b_0+\frac{\sigma_2b_0\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right)\right]\\&+G\left(|\frac{z}{\Delta}|+|\frac{z\sigma_1}{\Delta}|\right)+F\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)\Delta}|\\&+|\frac{m}{\Delta}|\left[G+\frac{F}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+F+\sigma_2G+\frac{\sigma_2F\eta_2^{\alpha_1}}{\Gamma(\alpha+1)}\right]\\&+G+\frac{F}{\Gamma(\alpha+1)}+\|\chi_1\|\ \left[b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{m}{\Delta}|\left(\frac{b_0}{\Gamma(\alpha+1)}+b_0+\frac{\sigma_2b_0\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right)\right]\\&+\frac{b_0}{\Gamma(\alpha+1)}\|\chi_2\|\end{aligned}\tag{24}$$

$$\leq \left(\begin{array}{c} |\frac{p}{\Delta}|G\left(1+\sigma_1\right)+F\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{n}{\Delta}|\left(G+\frac{F}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+F+\sigma_2 G+\frac{\sigma_2 F\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ +|\frac{z}{\Delta}|G\left(1+\sigma_1\right)+F\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+G+\frac{F}{\Gamma(\alpha+1)} \\ +|\frac{m}{\Delta}|\left(G+\frac{F}{\Gamma(\alpha+1)}+\frac{G}{\Gamma(2-\alpha)}+F+\sigma_2 G+\frac{\sigma_2 F\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \end{array} \right)$$

$$+\, b_0 r \left(\begin{array}{c} |\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}|\eta_1^{\alpha}+|\frac{n}{\Delta}|[\frac{1}{\Gamma(\alpha+1)}+1+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}]+\ |\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|\eta_1^{\alpha} \\ +|\frac{m}{\Delta}|\left[\frac{1}{\Gamma(\alpha+1)}+1+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{1}{\Gamma(\alpha+1)} \end{array} \right)$$

Taking supremum over all $\tau \in [0,1]$ and solving the inequality $\|\Theta_1\chi_1+\Theta_2\chi_2\| \leq r$ for r, we obtain that:

$$r \geq \frac{a_0}{1-b_0 d_0},$$

where a_0, b_0, and d_0 are given in the equations (20), (21), and (22) respectively.
Step 2: In the following step, we have to prove that the operator Θ_1 is a contraction with coefficient

$$c=\left(\begin{array}{c} |\frac{p}{\Delta}|\,\|\omega\|+|\frac{p\sigma_1}{\Delta}|\,\|\omega\|+b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}| \\ +|\frac{n}{\Delta}|\left(\begin{array}{c} \|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha_{11}}}{\Gamma(\alpha+1)}b_0 \end{array} \right)+|\frac{z}{\Delta}|\,\|\omega\|+|\frac{z\sigma_1}{\Delta}|\,\|\omega\| \\ +b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{m}{\Delta}|\left(\begin{array}{c} \|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{array} \right) \end{array} \right)<1.$$

Consider the following inequality:

$$\begin{aligned} |\Theta_1(\chi)-\Theta_1(\vartheta)| \leq\ & |\frac{p}{\Delta}|\,\|\omega\|\,\|\chi-\vartheta\|+|\frac{p\sigma_1}{\Delta}|\,\|\omega\|\,\|\chi-\vartheta\|+b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}|\,\|\chi-\vartheta\| \\ & +|\frac{n}{\Delta}|\left(\begin{array}{c} \|\omega\|\,\|\chi-\vartheta\|+\frac{b_0}{\Gamma(\alpha+1)}\|\chi-\vartheta\|+\frac{\|\omega\|}{\Gamma(2-\alpha)}\|\chi-\vartheta\| \\ +b_0\|\chi-\vartheta\|+\sigma_2\|\omega\|\,\|\chi-\vartheta\| \\ +\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0\|\chi-\vartheta\| \end{array} \right) \\ & +|\frac{z}{\Delta}|\,\|\omega\|\,\|\chi-\vartheta\|+|\frac{z\sigma_1}{\Delta}|\,\|\omega\|\,\|\chi-\vartheta\|+b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|\,\|\chi-\vartheta\| \\ & +|\frac{m}{\Delta}|\left(\begin{array}{c} \|\omega\|\,\|\chi-\vartheta\|+\frac{b_0}{\Gamma(\alpha+1)}\|\chi-\vartheta\|+\frac{\|\omega\|}{\Gamma(2-\alpha)}\|\chi-\vartheta\| \\ +b_0\|\chi-\vartheta\|+\sigma_2\|\omega\|\,\|\chi-\vartheta\| \\ +\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0\|\chi-\vartheta\| \end{array} \right) \end{aligned}$$

$$\leq \left(\begin{array}{c} |\frac{p}{\Delta}|\,\|\omega\|+|\frac{p\sigma_1}{\Delta}|\,\|\omega\|+b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}| \\ +|\frac{n}{\Delta}|\left(\begin{array}{c} \|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{array} \right)+|\frac{z}{\Delta}|\,\|\omega\|+|\frac{z\sigma_1}{\Delta}|\,\|\omega\| \\ +b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{m}{\Delta}|\left(\begin{array}{c} \|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{array} \right) \end{array} \right)\|\chi-\vartheta\|.$$

Utilizing the Banach fixed-point theorem, we establish that the operator Θ_1 is a contraction mapping with a contraction coefficient given by

$$c=\begin{pmatrix} |\frac{p}{\Delta}|\,\|\omega\|+|\frac{p\sigma_1}{\Delta}|\,\|\omega\|+b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}| \\ +|\frac{n}{\Delta}|\begin{pmatrix} \|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{pmatrix}+|\frac{z}{\Delta}|\,\|\omega\|+|\frac{z\sigma_1}{\Delta}|\,\|\omega\| \\ +b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{m}{\Delta}|\begin{pmatrix} \|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0 \end{pmatrix} \end{pmatrix}. \tag{25}$$

Step 3: In this stage, our task is to demonstrate that Θ_2 is both compact and continuous. Consider a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ within the ball B_r, where χ_n converges to $\chi\in B_r$ as $n\to\infty$. We need to establish that $\|\Theta_2(\chi_n)-\Theta_2(\chi)\|$ approaches zero as n approaches infinity. To show this, observe that for every $\tau\in[0,1]$, it holds that

$$|\Theta_2(\chi_n)-\Theta_2(\chi)|\leq|g(\xi,\chi_n(\xi))-g(\xi,\chi(\xi))|+\frac{1}{\Gamma(\alpha)}\int_0^{\xi}|\xi-\tau|^{\alpha-1}|u_n(\tau)-u(\tau)|d\tau, \tag{26}$$

where

$$u_n(\tau)=f(\tau,\chi_n(\tau),D_{0^+}^{\alpha-1}\chi_n(\tau),\int_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha-1}\chi_n(v)dv), \tag{27}$$

and

$$u(\tau)=f(\tau,\chi(\tau),D_{0^+}^{\alpha-1}\chi(\tau),\int_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha-1}\chi(v)dv). \tag{28}$$

Thus,

$$\begin{aligned}|u_n(\tau)-u(\tau)|&=|f(\tau,\chi_n(\tau),D_{0^+}^{\alpha-1}\chi_n(\tau),\int_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha_1-1}\chi_n(v)dv)\\&\qquad-f(\tau,\chi(\tau),D_{0^+}^{\alpha-1}\chi(\tau),\int_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha-1}\chi(v)dv)|\\&\leq\|\lambda\|\left[1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right]\|\chi_n-\chi\|\\&\leq\|\lambda\|\left[1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right]\|\chi_n-\chi\|\\&\leq b_0\|\chi_n-\chi\|,\end{aligned}$$

with $b_0=\|\lambda\|[1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}]$. Let $\chi_n\to\chi$ as $n\to\infty$. We then have that $u_n(\tau)\to u(\tau)$ for each $\tau\in[0,1]$. Assume there exists a positive constant $\varepsilon>0$ such that for every $\tau\in[0,1]$, the inequalities $|u_n(\tau)|\leq\frac{\varepsilon}{2}$ and $|u(\tau)|\leq\frac{\varepsilon}{2}$ hold. Consequently, we obtain

$$|u_n(\tau)-u(\tau)|\leq|u_n(\tau)|+|u(\tau)|\leq\varepsilon. \tag{29}$$

By employing the Lebesgue Dominated Convergence Theorem, it follows that

$$\|\Theta_2(\chi_n)-\Theta_2(\chi)\|\to 0 \text{ as } n\to\infty, \tag{30}$$

demonstrating that Θ_2 is continuous. Furthermore, Θ_2 is also bounded since

$$\begin{aligned}|\Theta_2(\chi)| &= |g(\xi,\chi(\xi)) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\tau)^{\alpha-1}u(\tau)d\tau| \\ &\le |g(\xi,\chi(\xi))| + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}|\xi-\tau|^{\alpha-1}u(\tau)|d\tau \\ &\le G + \frac{1}{\Gamma(\alpha+1)}\xi^{\alpha}\left(F + b_0\|\chi\|\right) \\ &\le G + \frac{1}{\Gamma(\alpha+1)}\left(F + b_0 r\right).\end{aligned}$$

By taking supremum over $[0,1]$, we obtain due to the definition of r that

$$\|\Theta_2\| \le G + \frac{1}{\Gamma(\alpha+1)}\left(F + b_0 r\right) \le r. \tag{31}$$

Thus, it was prove that Θ_2 is uniformly bounded. To demonstrate that Θ_2 is equicontinuous, we proceed as follows. Consider any $\varepsilon > 0$. We aim to find a $\delta > 0$ such that for any $\tau_1, \tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $|\tau_2 - \tau_1| < \delta$, the following condition holds. Utilizing the assumption (H_3), we can derive that:

$$\begin{aligned}|\Theta_2\chi(\tau_2) - \Theta_2\chi(\tau_1)| &\le |g(\tau_2,\chi(\tau_2) + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2-v)^{\alpha-1}u(v)dv - g(\tau_1,\chi(\tau_1)) \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-v)^{\alpha-1}u(v)dv| \\ &\le |g(\tau_2,\chi(\tau_2) - g(\tau_2,\chi(\tau_1) + g(\tau_2,\chi(\tau_1) - g(\tau_1,\chi(\tau_1))| \\ &\quad + |\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2-v)^{\alpha-1}u(v)dv - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-v)^{\alpha-1}u(v)dv| \\ &\le |g(\tau_2,\chi(\tau_2) - g(\tau_2,\chi(\tau_1)| + |g(\tau_2,\chi(\tau_1) - g(\tau_1,\chi(\tau_1))| \\ &\quad + |\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_2-v)^{\alpha-1}u(v)dv + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-v)^{\alpha-1}u(v)dv \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-v)^{\alpha-1}u(v)dv|\end{aligned}$$

$$\begin{aligned}&\le \|\omega\|\,|\chi(\tau_2) - \chi(\tau_1)| + |g(\tau_2,\chi(\tau_1)) - g(\tau_1,\chi(\tau_1))| \\ &+ |\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\left|(\tau_2-v)^{\alpha-1} - (\tau_1-v)^{\alpha-1}\right|\,|u(v)|\,dv + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-v)^{\alpha-1}\,|u(v)|\,dv\end{aligned}$$

$$\leq \|\omega\| \left|\chi(\tau_2) - \chi(\tau_1)\right| + \left|g(\tau_2, \chi(\tau_1)) - g(\tau_1, \chi(\tau_1))\right|$$

$$+ \left|\frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left|{\tau_2}^{\alpha-1} - {\tau_1}^{\alpha-1}\right| \; |u(v)|\, dv + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - v)^{\alpha-1} |u(v)|\, dv\right.$$

$$\leq \|\omega\| \left|\chi(\tau_2) - \chi(\tau_1)\right| + \left|g(\tau_2, \chi(\tau_1)) - g(\tau_1, \chi(\tau_1))\right|$$

$$+ \left|\frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left|{\tau_2}^{\alpha-1} - {\tau_1}^{\alpha-1}\right| \; |u(v)|\, dv + \frac{1}{\Gamma(\alpha+1)} \left|\tau_2 - \tau_1\right|^{\alpha} |u(v)|\right. .$$

Thus,

$$\left|\Theta_2\chi(\tau_2) - \Theta_2\chi(\tau_1)\right| \leq \|\omega\| \left|\chi(\tau_2) - \chi(\tau_1)\right| + \left|g(\tau_2, \chi(\tau_1)) - g(\tau_1, \chi(\tau_1))\right|$$
$$+ \frac{\theta}{\Gamma(\alpha)} \left|{\tau_2}^{\alpha-1} - {\tau_1}^{\alpha-1}\right| \tau_1 + \frac{\theta}{\Gamma(\alpha_1+1)} \left|\tau_2 - \tau_1\right|^{\alpha} .$$

As τ_1 approaches τ_2, the right-hand side of the inequality becomes independent of χ and converges to zero. Consequently, we have

$$\left|\Theta_2\chi(\tau_2) - \Theta_2\chi(\tau_1)\right| \to 0, \quad \text{for all } \left|\tau_2 - \tau_1\right| \to 0.$$

This implies that the family $\{\Theta_2\chi\}$ is equicontinuous on B_r. By invoking the Arzela-Ascoli Theorem, it follows that Θ_2 is a compact operator. Consequently, $\Theta_2 : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ is both continuous and compact. Thus, all the conditions of Krasnoselskii's fixed point theorem are fulfilled, ensuring that the operator $\Theta = \Theta_1 + \Theta_2$ has a fixed point $\chi(\tau) \in C([0,1])$ within B_r. This fixed point $\chi(\tau)$ satisfies the nonlocal boundary conditions

$$\chi(0) + D_{0^+}^{\alpha-1}\chi(0) = \sigma_1\chi(\eta_1), \quad \text{for } \eta_1 \in (0,1),$$

and

$$\chi(1) + D_{0^+}^{\alpha-1}\chi(1) = \sigma_2\chi(\eta_2), \quad \text{for } \eta_2 \in (0,1).$$

Therefore, $\chi(\tau)$ is considered a solution to the equation (1) with the specified nonlocal boundary conditions (2) and (3). □

4. Uniqueness of Solutions

To establish the uniqueness of the solution to the nonlinear fractional differential equation NLFDE (1) , we present the following theorem:

Theorem 4.1. *If the conditions* $(H_1) - (H_3)$ *hold and if*

$$c + \|\omega\| + \frac{\|\lambda\|}{\Gamma(\alpha+1)} \left(1 + \frac{1}{\Gamma(2-\alpha)} + \frac{\Phi}{\Gamma(2-\alpha)}\right) < 1, \tag{32}$$

where

$$c = \left(\begin{array}{c} \left|\frac{p}{\Delta}\right| \|\omega\| + \left|\frac{p\sigma_1}{\Delta}\right| \|\omega\| + b_0\eta_1^{\alpha}\left|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}\right| \\ +\left|\frac{n}{\Delta}\right| \left(\begin{array}{c} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0 + \sigma_2\|\omega\| + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)} b_0 \end{array} \right) + \left|\frac{z}{\Delta}\right| \|\omega\| + \left|\frac{z\sigma_1}{\Delta}\right| \|\omega\| \\ +b_0\eta_1^{\alpha}\left|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}\right| + \left|\frac{m}{\Delta}\right| \left(\begin{array}{c} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0 + \sigma_2\|\omega\| + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)} b_0 \end{array} \right) \end{array} \right),$$

then the NLFDE (1) has a unique solution.

Proof. We have

$$\begin{aligned}|\Theta(\chi)-\Theta(\vartheta)| &= |\Theta_1(\chi)+\Theta_2(\chi)-\Theta_1(\vartheta)-\Theta_2(\vartheta)| \\ &\leq |\Theta_1(\chi)-\Theta_1(\vartheta)|+|\Theta_2(\chi)-\Theta_2(\vartheta)| \\ &\leq c\,\|\chi-\vartheta\|+|g(\xi,\chi(\xi)-g(\xi,\vartheta(\xi)| \\ &+|\frac{1}{\Gamma(\alpha)}\int\limits_0^\xi(\xi-\tau)^{\alpha-1}u(\tau)d\tau-\frac{1}{\Gamma(\alpha)}\int\limits_0^\xi(\xi-\tau)^{\alpha-1}v(\tau)d\tau| \\ &\leq c\,\|\chi-\vartheta\|+\|\omega\|\,\|\chi-\vartheta\|+\frac{1}{\Gamma(\alpha)}\int\limits_0^\xi(\xi-\tau)^{\alpha-1}|u(\tau)-v(\tau)|d\tau\end{aligned}$$

But,

$$|u(\tau)-v(\tau)|\leq b_0\,\|\chi-\vartheta\|\,,$$

where Then, $b_0=\|\lambda\|\left(1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right)$. Thus,

$$\begin{aligned}|\Theta(\chi)-\Theta(\vartheta)| &\leq c\,\|\chi-\vartheta\|+\|\omega\|\,\|\chi-\vartheta\|+\frac{b_0}{\Gamma(\alpha+1)}\,\|\chi-\vartheta\| \\ &\leq\left(c+\|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}\right)\|\chi-\vartheta\|\end{aligned}$$

Hence, if we take $\left(c+\|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}\right)<1$, then by the contarction thereom (Banach fixed point thorem), we obtain that the NLFDE (1) has a unique solution. □

5. Stability results via Ulam-Hyers type

In the following, we consider the Ulam stability for NLFDE (1). Consider the real numbers $1<\alpha\leq 2$; $\varepsilon>0$, and let $\Theta:[0,1]\to\mathbb{R}^+$ be a continuous function, and consider the following inequalities:

$$|D^\alpha_{0^+}\left(\chi(\xi)-g(\xi,\chi(\xi))-f(\xi,\chi(\xi),D^{\alpha-1}_{0^+}\chi(\xi),\int\limits_{0^+}^\xi\varphi(\xi,\tau)D^{\alpha-1}_{0^+}\chi(\tau)d\tau)\right|\leq\varepsilon(\xi),\xi\in[0,1], \tag{33}$$

$$|D^\alpha_{0^+}\left(\chi(\xi)-g(\xi,\chi(\xi))-f(\xi,\chi(\xi),D^{\alpha-1}_{0^+}\chi(\xi),\int\limits_{0^+}^\xi\varphi(\xi,\tau)D^{\alpha-1}_{0^+}\chi(\tau)d\tau)\right|\leq\Theta(\xi),\xi\in[0,1], \tag{34}$$

and

$$|D^\alpha_{0^+}\left(\chi(\xi)-g(\xi,\chi(\xi))-f(\xi,\chi(\xi),D^{\alpha-1}_{0^+}\chi(\xi),\int\limits_{0^+}^\xi\varphi(\xi,\tau)D^{\alpha-1}_{0^+}\chi(\tau)d\tau)\right|\leq\varepsilon\Phi(\xi),\xi\in[0,1]. \tag{35}$$

Definition 5.1. [35] *The nonlinear integro-differential equation (NLFDE) (1) is said to exhibit Ulam-Hyers stability if there exists a positive real constant $\kappa_f>0$ such that for any solution $z\in C([0,1],\mathbb{R})$ of the inequality (33), there is a solution $\chi\in C([0,1],\mathbb{R})$ to the NIFDE (1) satisfying*

$$|z(\xi)-\chi(\xi)|\leq\varepsilon\kappa_f\quad\forall\,\xi\in[0,1].$$

Definition 5.2. [35] *The NLFDE (1) is generalized Ulam-Hyers stable if there exists a function* $\kappa_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\kappa_f(0) = 0$ *such that for any* $\varepsilon > 0$ *and any solution* $z \in C([0,1], \mathbb{R})$ *of the inequality (34), there is a solution* $\chi \in C([0,1], \mathbb{R})$ *of the NIFDE (1) satisfying*

$$|z(\xi) - \chi(\xi)| \leq \kappa_f(\varepsilon) \quad \forall \xi \in [0,1].$$

Definition 5.3. [33] *The NLFDE (1) is Ulam-Hyers-Rassias stable with respect to a function* Θ *if there exists a positive real constant* $\kappa_{f\ \Theta} > 0$ *such that for any* $\varepsilon > 0$ *and any solution* $z \in C([0,1], \mathbb{R})$ *of the inequality (35), there is a solution* $\chi \in C([0,1], \mathbb{R})$ *of the NIFDE (1) satisfying*

$$|z(\xi) - \chi(\xi)| \leq \varepsilon \kappa_{f\ \Theta} \Theta(\xi) \quad \forall \xi \in [0,1].$$

Definition 5.4. [33] *The NLFDE (1) is generalized Ulam-Hyers-Rassias stable with respect to a function* Θ *if there exists a positive real constant* $\kappa_{f\ \Theta} > 0$ *such that for each solution* $z \in C([0,1], \mathbb{R})$ *of the inequality (35), there is a solution* $\chi \in C([0,1], \mathbb{R})$ *of the NIFDE (1) satisfying*

$$|z(\xi) - \chi(\xi)| \leq \kappa_{f\ \Theta} \Theta(\xi) \quad \forall \xi \in [0,1].$$

5.1. **Ulam-Hyers stability.** To establish the stability of the solution to the nonlinear implicit fractional differential equation NLFDE (1), we present the following theorem, which demonstrates the Ulam-Hyers stability under the conditions outlined in Theorem (4.1).

Theorem 5.1. *Suppose that the assumptions of Theorem (4.1) are satisfied. Then, the NIFDE (1) is Ulam-Hyers stable.*

Proof. Assume that $\chi(\xi)$ is a solution of of the NIFDE (1), then

$$|z(\xi) - \chi(\xi)| = |z(\xi) - g(\xi, \chi(\xi)) - \frac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi - \tau)^{\alpha-1} u(\tau) d\tau \tag{36}$$

$$-\xi^{\alpha-1} \left(\begin{array}{c} -\frac{p}{\Delta} g(0, \chi(0)) + \frac{p\sigma_1}{\Delta} g(\eta_1, \chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta} \int_0^{\eta_1} (\eta_1 - \tau)^{\alpha-1} u(\tau) d\tau \\ + \frac{n}{\Delta} \left(\begin{array}{c} \sigma_2 g(\eta_2, \chi(\eta_2)) - g(1, \chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - \tau)^{\alpha-1} u(\tau) d\tau \\ -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} u(\tau) d\tau - \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} g(\tau, \chi(\tau)) d\tau - \int_0^1 u(\tau) d\tau \end{array} \right) \end{array} \right)$$

$$-\xi^{\alpha-2} \left(\begin{array}{c} \frac{z}{\Delta} g(0, \chi(0)) - \frac{z\sigma_1}{\Delta} g(\eta_1, \chi(\eta_1)) - \frac{z\sigma_1}{\Gamma(\alpha)\Delta} \int_0^{\eta_1} (\eta_1 - \tau)^{\alpha-1} u(\tau) d\tau \\ + \frac{m}{\Delta} \left(\begin{array}{c} \sigma_2 g(\eta_2, \chi(\eta_2)) - g(1, \chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - \tau)^{\alpha-1} u(\tau) d\tau \\ -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} u(\tau) d\tau - \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} g(\tau, \chi(\tau)) d\tau - \int_0^1 u(\tau) d\tau \end{array} \right) \end{array} \right)$$

$$= |z(\xi) + g(\xi, z(\xi)) - g(\xi, \chi(\xi)) - g(\xi, z(\xi)) + \tfrac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi - \tau)^{\alpha-1} v(\tau) d\tau$$

$$- \tfrac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi - \tau)^{\alpha-1} u(\tau) d\tau - \tfrac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi - \tau)^{\alpha-1} v(\tau) d\tau$$

$$-\xi^{\alpha-1}\begin{pmatrix} -\frac{p}{\Delta}\left[g(0,\chi(0))-g(0,z(0))+g(0,z(0))\right] \\ +\frac{p\sigma_1}{\Delta}\left[g(\eta_1,\chi(\eta_1))-g(\eta_1,z(\eta_1))+g(\eta_1,z(\eta_1))\right] \end{pmatrix}$$

$$-\xi^{\alpha-1}\frac{p\sigma_1}{\Gamma(\alpha)\Delta}\left(\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau-\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}v(\tau)d\tau+\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}v(\tau)d\tau\right)$$

$$-\xi^{\alpha-1}\frac{n}{\Delta}\begin{pmatrix} (\sigma_2 g(\eta_2,\chi(\eta_2))-\sigma_2 g(\eta_2,z(\eta_2))+\sigma_2 g(\eta_2,z(\eta_2))) \\ -(g(1,\chi(1))-g(1,z(1))+g(1,z(1))) \\ +\frac{\sigma_2}{\Gamma(\alpha)}\left(\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau-\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}v(\tau)d\tau+\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}v(\tau)d\tau\right) \\ -\frac{1}{\Gamma(\alpha)}\left(\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau-\int_0^1(1-\tau)^{\alpha-1}v(\tau)d\tau+\int_0^1(1-\tau)^{\alpha-1}v(\tau)d\tau\right) \\ -\frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} \int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int_0^1(1-\tau)^{-\alpha}g(\tau,z(\tau))d\tau \\ +\int_0^1(1-\tau)^{-\alpha}g(\tau,z(\tau))d\tau \end{pmatrix} \\ -\left(\int_0^1 u(\tau)d\tau-\int_0^1 v(\tau)d\tau+\int_0^1 v(\tau)d\tau\right) \end{pmatrix}$$

$$-\xi^{\alpha-2}\begin{pmatrix} \frac{z}{\Delta}\left[g(0,\chi(0))-g(0,z(0))+g(0,z(0))\right] \\ -\frac{z\sigma_1}{\Delta}\left[g(\eta_1,\chi(\eta_1))-g(\eta_1,z(\eta_1))+g(\eta_1,z(\eta_1))\right] \end{pmatrix}$$

$$+\xi^{\alpha-2}\frac{z\sigma_1}{\Gamma(\alpha)\Delta}\begin{pmatrix} \int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau-\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}v(\tau)d\tau \\ +\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}v(\tau)d\tau \end{pmatrix}$$

$$-\xi^{\alpha-2}\cdot\frac{m}{\Delta}\begin{pmatrix} \begin{pmatrix} [\sigma_2 g(\eta_2,\chi(\eta_2))-g(\eta_2,z(\eta_2))+g(\eta_2,z(\eta_2))] \\ -[g(1,\chi(1))-g(1,z(1))+g(1,z(1))] \end{pmatrix} \\ +\frac{\sigma_2}{\Gamma(\alpha)}\left(\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau-\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}v(\tau)d\tau+\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}v(\tau)d\tau\right) \\ -\frac{1}{\Gamma(\alpha)}\left(\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau-\int_0^1(1-\tau)^{\alpha-1}v(\tau)d\tau+\int_0^1(1-\tau)^{\alpha-1}v(\tau)d\tau\right) \\ -\frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} \int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau-\int_0^1(1-\tau)^{-\alpha}g(\tau,z(\tau))d\tau \\ +\int_0^1(1-\tau)^{-\alpha}g(\tau,z(\tau))d\tau \end{pmatrix} \\ -\int_0^1 u(\tau)d\tau-\int_0^1 v(\tau)d\tau+\int_0^1 v(\tau)d\tau \end{pmatrix}.$$

This implies that

$$|z(\xi)-\chi(\xi)|\leq\begin{pmatrix} |z(\xi)-g(\xi,z(\xi))-I^\alpha v(\xi)|+|g(\xi,\chi(\xi))-g(\xi,z(\xi))| \\ +\frac{1}{\Gamma(\alpha)}\int_0^\xi|\xi-\tau|^{\alpha-1}|v(\tau)-u(\tau)|d\tau+|\frac{p}{\Delta}|\;|g(0,\chi(0))-g(0,z(0))| \\ +|\frac{p\sigma_1}{\Delta}||g(\eta_1,\chi(\eta_1))-g(\eta_1,z(\eta_1))|+|\frac{p\sigma_1}{\Gamma(\alpha)\Delta}|\int_0^{\eta_1}|\eta_1-\tau|^{\alpha-1}|u(\tau)-v(\tau)|d\tau \end{pmatrix}$$

$$+\left|\frac{n}{\Delta}\right|\left(\begin{array}{c}\sigma_2|g(\eta_2,\chi(\eta_2))-\sigma_2 g(\eta_2,z(\eta_2))|+|g(1,\chi(1))-g(1,z(1))|\\ +\frac{1}{\Gamma(\alpha)}\int\limits_0^1|1-\tau|^{\alpha-1}|u(\tau)-v(\tau)|d\tau+\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1|1-\tau|^{-\alpha}|g(\tau,\chi(\tau))-g(\tau,z(\tau))|d\tau\\ +\int\limits_0^1|u(\tau)-v(\tau)|d\tau+\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}|\eta_2-\tau|^{\alpha-1}.|u(\tau)-v(\tau)|d\tau\end{array}\right)$$

$$+\left|\frac{z}{\Delta}\right|\;|g(0,\chi(0))-g(0,z(0))|+\left|\frac{z\sigma_1}{\Delta}\right|\;|g(\eta_1,\chi(\eta_1))-g(\eta_1,z(\eta_1))|$$
$$+\left|\frac{z\sigma_1}{\Gamma(\alpha)\Delta}\right|\int\limits_0^{\eta_1}|\eta_1-\tau|^{\alpha-1}|u(\tau)-v(\tau)|d\tau$$

$$+\left|\frac{m}{\Delta}\right|\left(\begin{array}{c}|g(1,\chi(1))-g(1,z(1))|+\frac{\sigma_2}{\Gamma(\alpha)}|g(\eta_2,\chi(\eta_2))-g(\eta_2,z(\eta_2))|\\ +\frac{1}{\Gamma(\alpha)}\int\limits_0^1|1-\tau|^{\alpha-1}|u(\tau)-v(\tau)|d\tau+\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1|1-\tau|^{-\alpha}|g(\tau,\chi(\tau))-g(\tau,z(\tau))|d\tau\\ +\int\limits_0^1|u(\tau)-v(\tau)|d\tau+\frac{\sigma_2}{\Gamma(\alpha)}\int\limits_0^{\eta_2}|\eta_2-\tau|^{\alpha-1}|u(\tau)-v(\tau)|d\tau\end{array}\right)$$

But

$$|u(\tau)-v(\tau)|=|f(\xi,\chi(\xi),D_{0^+}^{\alpha-1}\chi(\xi),\int\limits_{o^+}^{\xi}\varphi(\xi,\tau)D_{0^+}^{\alpha-1}\chi(\tau)d\tau)$$
$$-f(\xi,z(\xi),D_{0^+}^{\alpha-1}z(\xi),\int\limits_{o^+}^{\xi}\varphi(\xi,\tau)D_{0^+}^{\alpha-1}z(\tau)d\tau)|$$
$$\leq|\lambda(\tau)|\left[\begin{array}{c}|\chi(\tau)-z(\tau)|+|D_{0^+}^{\alpha-1}\chi(\tau)-D_{0^+}^{\alpha-1}z(\tau)|\\ +|\int\limits_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha-1}\chi(v)dv-\int\limits_{0^+}^{\tau}\varphi(\tau,v)D_{0^+}^{\alpha-1}z(v)dv)|\end{array}\right]$$
$$\leq\|\lambda\|\left[\|\chi-z\|+\frac{\|\chi-z\|}{\Gamma(2-\alpha)}+\Phi\frac{\|\chi-z\|}{\Gamma(2-\alpha)}\right].$$

Taking supremum over all $\tau\in[0,1]$, we get

$$\|u-v\|\leq\|\lambda\|\left[1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right]\|\chi-z\|$$
$$\leq b_0\,\|\chi-z\|\;\; with\; b_0=\|\lambda\|.\left(1+\frac{1}{\Gamma(2-\alpha)}+\frac{\Phi}{\Gamma(2-\alpha)}\right).$$

Then,

$$|z(\xi)-\chi(\xi)| \leq I_{0^+}^{\alpha}\varepsilon + \|\chi - z\| \begin{pmatrix} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \left|\frac{p}{\Delta}\right| \|\omega\| \\ + \left|\frac{p\sigma_1}{\Delta}\right| \|\omega\| + \left|\frac{p\sigma_1}{\Delta}\right| \frac{b_0\eta_1^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix}$$

$$+ |\frac{n}{\Delta}|.\, \|\chi - z\| \begin{pmatrix} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0 + \sigma_2 \|\omega\| + \frac{\sigma_2 b_0 \eta_2^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix}$$

$$+ |\frac{z}{\Delta}|.\, \|\chi - z\| \left(\|\omega\| + \sigma_1 \|\omega\| + \frac{\sigma_1 b_0 \eta_1^{\alpha}}{\Gamma(\alpha+1)} \right)$$

$$+ |\frac{m}{\Delta}|.\, \|\chi - z\| \begin{pmatrix} \|\omega\| + \frac{b_0}{\Gamma(\alpha+1)} + \frac{\|\omega\|}{\Gamma(2-\alpha)} \\ +b_0 + \sigma_2 \|\omega\| + \frac{\sigma_2 b_0 \eta_2^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix}$$

Taking supremum over all $\tau \in [0,1]$, we get

$$\|\chi - z\| \leq \left[\frac{\varepsilon t^{\alpha}}{\Gamma(\alpha+1)}\right]_0^1 + \|\omega\| \begin{pmatrix} 1 + \left|\frac{p}{\Delta}\right| + \left|\frac{p\sigma_1}{\Delta}\right| \\ +|\frac{n}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \\ +|\frac{z}{\Delta}| + |\frac{z\sigma_1}{\Delta}| + |\frac{m}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \end{pmatrix} \|\chi - z\|$$

$$+ b_0 \begin{pmatrix} \frac{1}{\Gamma(\alpha+1)} + \left|\frac{p\sigma_1}{\Delta}\right| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{n}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ +|\frac{z\sigma_1}{\Delta}| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{m}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \end{pmatrix} \|\chi - z\|$$

$$\|\chi - z\| \leq \frac{\varepsilon}{\Gamma(\alpha+1)} + \|\omega\| \begin{pmatrix} 1 + \left|\frac{p}{\Delta}\right| + \left|\frac{p\sigma_1}{\Delta}\right| \\ +|\frac{n}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \\ +|\frac{z}{\Delta}| + |\frac{z\sigma_1}{\Delta}| + |\frac{m}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \end{pmatrix} \|\chi - z\|$$

$$+ b_0 \begin{pmatrix} \frac{1}{\Gamma(\alpha+1)} + \left|\frac{p\sigma_1}{\Delta}\right| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{n}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ +|\frac{z\sigma_1}{\Delta}| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{m}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \end{pmatrix} \|\chi - z\|$$

Simplifying for $\|\chi - z\|$, we get

$$\|\chi - z\| \begin{pmatrix} 1 - \|\omega\| \begin{pmatrix} 1 + \left|\frac{p}{\Delta}\right| + \left|\frac{p\sigma_1}{\Delta}\right| \\ +|\frac{n}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \\ +|\frac{z}{\Delta}| + |\frac{z\sigma_1}{\Delta}| + |\frac{m}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \end{pmatrix} \\ -b_0 \begin{pmatrix} \frac{1}{\Gamma(\alpha+1)} + \left|\frac{p\sigma_1}{\Delta}\right| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{n}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ +|\frac{z\sigma_1}{\Delta}| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{m}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \end{pmatrix} \end{pmatrix} \leq \frac{\varepsilon}{\Gamma(\alpha+1)}.$$

This implies that

$$\begin{aligned} \|\chi - z\| &\leq \varepsilon . \frac{1}{\Gamma(\alpha+1)} \begin{pmatrix} 1 - \|\omega\| \begin{pmatrix} 1 + \left|\frac{p}{\Delta}\right| + \left|\frac{p\sigma_1}{\Delta}\right| \\ +|\frac{n}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \\ +|\frac{z}{\Delta}| + |\frac{z\sigma_1}{\Delta}| + |\frac{m}{\Delta}| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \end{pmatrix} \\ -b_0 \begin{pmatrix} \frac{1}{\Gamma(\alpha+1)} + \left|\frac{p\sigma_1}{\Delta}\right| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{n}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ +|\frac{z\sigma_1}{\Delta}| \frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\frac{m}{\Delta}| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \end{pmatrix} \end{pmatrix}^{-1} \\ &\leq \varepsilon\ \kappa_f, \end{aligned}$$

where

$$\kappa_f = \frac{1}{\Gamma(\alpha+1)} \begin{pmatrix} 1 - \|\omega\| \begin{pmatrix} 1 + \left|\frac{p}{\Delta}\right| + \left|\frac{p\sigma_1}{\Delta}\right| \\ +\left|\frac{n}{\Delta}\right| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \\ +\left|\frac{z}{\Delta}\right| + \left|\frac{z\sigma_1}{\Delta}\right| + \left|\frac{m}{\Delta}\right| \left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \end{pmatrix} \\ -b_0 \begin{pmatrix} \frac{1}{\Gamma(\alpha+1)} + \left|\frac{p\sigma_1}{\Delta}\right| \frac{\eta_1^\alpha}{\Gamma(\alpha+1)} + \left|\frac{n}{\Delta}\right| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^\alpha}{\Gamma(\alpha+1)}\right) \\ +\left|\frac{z\sigma_1}{\Delta}\right| \frac{\eta_1^\alpha}{\Gamma(\alpha+1)} + \left|\frac{m}{\Delta}\right| \left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^\alpha}{\Gamma(\alpha+1)}\right) \end{pmatrix} \end{pmatrix}^{-1} \tag{37}$$

This implies that the NLFDE (1) is Ulam Hyer stable. It is revealed that the NLFDE (1) is generalised Ulam Hyer stable by setting $\Theta(\varepsilon) = \kappa_f\ \varepsilon; \Theta(0) = 0$. □

5.2. **Ulam-Hyers Rassias stability.** To establish the stability of the nonlinear fractional differential equation NLFDE (1), we present the following theorem, which ensures the Ulam-Hyers-Rassias stability under the given assumptions.

Theorem 5.2. *Assume that assumptions (H_1), (H_2), and (H_3) hold. Then,NLFDE (1)is Ulam-Hyers-Rassias stable with respect to Θ.*

Proof. Let $z \in C(I, R)$ be a solution of the inequality (35), i.e.,

$$|D_{0^+}^\alpha (z(\xi) - g(\xi, z(\xi)) - f(\xi, z(\xi), D_{0^+}^{\alpha-1} z(\xi), \int_{0^+}^{\xi} \varphi(\xi,\tau) D_{0^+}^{\alpha-1} z(\tau) d\tau)| \leq \varepsilon\Phi(\xi), \xi \in [0,1].$$

In addition, let χ be a solution of NLFDE (1), and let $u \in C(I, R)$ such that:

$$\chi(\xi) = g(\xi, \chi(\xi)) + \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi-\tau)^{\alpha-1} u(\tau) d\tau \tag{38}$$

$$+ \xi^{\alpha-1} \begin{pmatrix} -\frac{p}{\Delta} g(0, \chi(0)) + \frac{p\sigma_1}{\Delta} g(\eta_1, \chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta} \int_0^{\eta_1} (\eta_1 - \tau)^{\alpha-1} u(\tau) d\tau \\ +\frac{n}{\Delta} \begin{pmatrix} \sigma_2 g(\eta_2, \chi(\eta_2)) - g(1, \chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - \tau)^{\alpha-1} u(\tau) d\tau \\ -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} u(\tau) d\tau - \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} g(\tau, \chi(\tau)) d\tau - \int_0^1 u(\tau) d\tau \end{pmatrix} \end{pmatrix}$$

$$+ \xi^{\alpha-2} \begin{pmatrix} \frac{z}{\Delta} g(0, \chi(0)) - \frac{z\sigma_1}{\Delta} g(\eta_1, \chi(\eta_1)) - \frac{z\sigma_1}{\Gamma(\alpha)\Delta} \int_0^{\eta_1} (\eta_1 - \tau)^{\alpha-1} u(\tau) d\tau \\ +\frac{m}{\Delta} \begin{pmatrix} \sigma_2 g(\eta_2, \chi(\eta_2)) - g(1, \chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - \tau)^{\alpha-1} u(\tau) d\tau \\ -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} u(\tau) d\tau - \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} g(\tau, \chi(\tau)) d\tau - \int_0^1 u(\tau) d\tau \end{pmatrix} \end{pmatrix}$$

where

$$u(\xi) = f(\xi, \chi(\xi), D_{0^+}^{\alpha-1} \chi(\xi), \int_{0^+}^{\xi} \varphi(\xi,\tau) D_{0^+}^{\alpha-1} \chi(\tau) d\tau).$$

Operating I^{α} on both sides of inequality (35) and then integrating, we get

$$\left| z(\xi) - g(\xi,\chi(\xi)) - \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-\tau)^{\alpha-1}u(\tau)dsh(\xi) \right. \tag{39}$$

$$- \xi^{\alpha-1}\left(\begin{array}{c} -\frac{p}{\Delta}g(0,\chi(0)) + \frac{p\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) + \frac{p\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{n}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right) \end{array}\right)$$

$$\left. - \xi^{\alpha-2}\left(\begin{array}{c} \frac{z}{\Delta}g(0,\chi(0)) - \frac{z\sigma_1}{\Delta}g(\eta_1,\chi(\eta_1)) - \frac{z\sigma_1}{\Gamma(\alpha)\Delta}\int_0^{\eta_1}(\eta_1-\tau)^{\alpha-1}u(\tau)d\tau \\ +\frac{m}{\Delta}\left(\begin{array}{c} \sigma_2 g(\eta_2,\chi(\eta_2)) - g(1,\chi(1)) + \frac{\sigma_2}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-\tau)^{\alpha-1}u(\tau)d\tau \\ -\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^1(1-\tau)^{-\alpha}g(\tau,\chi(\tau))d\tau - \int_0^1 u(\tau)d\tau \end{array}\right) \end{array}\right)\right|$$

$$\leq \varepsilon.\kappa_f.\Theta(\xi)$$

Hence, for each $\xi\in I$, But, from proof of Theorem (4.1) , we have

$$\|u-v\| \leq b_0\|\chi - z\|.$$

where $v \in C(I,R)$ such that

$$v(\xi) = f(\xi, z(\xi), D_{0^+}^{\alpha-1}z(\xi), \int_{0^+}^{\xi}\varphi(\xi,\tau)D_{0^+}^{\alpha-1}z(\tau)d\tau).$$

Then, for each $\xi \in I$

$$\|\chi - z\| \leq \varepsilon.\kappa_f.\ \Theta(\xi)$$

with

$$\kappa_f = \frac{1}{\Gamma(\alpha+1)}\left(\begin{array}{c} 1 - \|\omega\|\left(\begin{array}{c} 1 + \left|\frac{p}{\Delta}\right| + \left|\frac{p\sigma_1}{\Delta}\right| \\ +\left|\frac{n}{\Delta}\right|\left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \\ +\left|\frac{z}{\Delta}\right| + \left|\frac{z\sigma_1}{\Delta}\right| + \left|\frac{m}{\Delta}\right|\left(1 + \frac{1}{\Gamma(2-\alpha)} + \sigma_2\right) \end{array}\right) \\ -b_0\left(\begin{array}{c} \frac{1}{\Gamma(\alpha+1)} + \left|\frac{p\sigma_1}{\Delta}\right|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + \left|\frac{n}{\Delta}\right|\left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ +\left|\frac{z\sigma_1}{\Delta}\right|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + \left|\frac{m}{\Delta}\right|\left(\frac{1}{\Gamma(\alpha+1)} + 1 + \frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}\right) \end{array}\right) \end{array}\right)^{-1} \tag{40}$$

Therefore, the problem $NLFDE(1)$ is Ulam-Hyers-Rassias stable with respect to Θ. □

6. Numerical Example

In this section, we provide a numerical example to illustrate the practical significance of our theoretical findings. By considering nonlinear fractional integro-differential equations (NLFDEs), we demonstrate how our criteria guarantee the existence, uniqueness, and stability of solutions. This example emphasizes the applicability of our approach in addressing complex fractional models across diverse scientific and engineering fields. Additionally, it highlights the crucial role of solution uniqueness, particularly in advanced control systems where consistent and predictable outcomes are essential.

Example 6.1. *Consider the following nonlinear fractional differential equation:*

$$D_{0^+}^{\frac{5}{4}}\left[\chi(\xi)-\frac{1}{100+\xi}.\frac{4+|\chi(\xi)|}{5+3|\chi(\xi)|}\right] \tag{41}$$

$$=\frac{e^{\sqrt{\xi+5}}}{500+e^{5t}}\left(\frac{5+|\chi(\xi)|+D_{0^+}^{\frac{1}{4}}\chi(\xi)+\int_0^{\xi}\frac{\sqrt{\xi}}{2}e^{\tau}D_{0^+}^{\frac{1}{4}}\chi(\tau)\,d\tau}{1+|\chi(\xi)|+D_{0^+}^{\frac{1}{4}}\chi(\xi)+\int_0^{\xi}\frac{\sqrt{\xi}}{2}e^{\tau}D_{0^+}^{\frac{1}{4}}\chi(\tau)\,d\tau}\right) \tag{42}$$

under the conditions:

$$\chi(0)+D_{0^+}^{\frac{1}{4}}\chi(0)=\tfrac{1}{5}\chi\left(\tfrac{1}{6}\right),$$

and

$$\chi(1)+D_{0^+}^{\frac{1}{4}}\chi(1)=\tfrac{1}{10}\chi\left(\tfrac{1}{7}\right).$$

$$|f(\xi,u,v,w)|=\frac{e^{\sqrt{\xi+5}}}{500+e^{5t}}\left(\frac{5+|u|+|v|+|w|}{1+|u|+|v|+|w|}\right),\ F=\frac{5e^{\sqrt{5}}}{501}.$$

Obviously, f is a mutually continuous function. Besides, for any $u,v,w,u_1,v_1,w_1\in R$, and $\xi\in[0,1]$ we have

$$|f(\xi,u,v,w)-f(\xi,u_1,v_1,w_1)|\leq\frac{e^{\sqrt{5}}}{501}\left(|u-u_1|+|v-v_1|+|w-w_1|\right).$$

Hence, condition (H_2) is satisfied with

$$\lambda(\xi)=\frac{e^{\sqrt{\xi+5}}}{500+e^{5t}},\ \|\lambda\|=\frac{e^{\sqrt{5}}}{501}=1.8676\times10^{-2},\ \text{and }\Phi=\frac{e}{2}.$$

In this example we take $\alpha=\frac{5}{4}$, $\eta_1=\frac{1}{6}$, $\eta_2=\frac{1}{7}$, $\sigma_1=\frac{1}{5}$, $\sigma_2=\frac{1}{10}$, $\Phi=\frac{e}{2}$, $\|\omega\|=\frac{1}{100}$, $G=\frac{1}{300}$, $b_0=0.05463$, $p=0.569648$, $m=0.77861$, $n=0.766732$, $z=1.84492$, and $\Delta=1.858$.

It clear from Theorem 3.1 that the CIFDP has at least one mild solution on $[0,1]$ since the condition

$$c=\left(\begin{array}{c}|\frac{p}{\Delta}|\,\|\omega\|+|\frac{p\sigma_1}{\Delta}|\,\|\omega\|+b_0\eta_1^{\alpha}|\frac{p\sigma_1}{\Gamma(\alpha+1)(\Delta)}|\\ +|\frac{n}{\Delta}|\left[\begin{array}{c}\|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)}\\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0\end{array}\right]+|\frac{z}{\Delta}|\,\|\omega\|+|\frac{z\sigma_1}{\Delta}|\,\|\omega\|\\ +b_0\eta_1^{\alpha}|\frac{z\sigma_1}{\Gamma(\alpha+1)(\Delta)}|+|\frac{m}{\Delta}|\left[\begin{array}{c}\|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}+\frac{\|\omega\|}{\Gamma(2-\alpha)}\\ +b_0+\sigma_2\|\omega\|+\frac{\sigma_2\eta_2^{\alpha}}{\Gamma(\alpha+1)}b_0\end{array}\right]\end{array}\right)\approx0.134715<1.$$

Moreover, the NLFDE (41) has unique solution since the followng condition holds

$$c+\|\omega\|+\frac{b_0}{\Gamma(\alpha+1)}\approx0.1893<1,$$

$$a_0=0.26941;d_0=1.08851;\frac{a_0}{1-b_0d_0}=0.2864,$$

and

$$\|\Theta_2\|\leq0.10006<0.2864.$$

Thus, the solution is also stable.

7. Conclusion

This study significantly advances the theoretical framework of nonlinear fractional integro-differential equations, particularly those involving Riemann-Liouville fractional derivatives, by addressing the existence, uniqueness, and stability of solutions through advanced fixed-point theorems. Our work establishes novel conditions for the existence and uniqueness of solutions, along with stability criteria under small perturbations, extending the current understanding of fractional calculus and its application to systems with memory and long-range dependencies. By synthesizing fractional derivatives with nonlocal boundary conditions, we offer new insights into the behavior and stability of such systems, which is a valuable extension to existing models that primarily focus on simpler cases. The findings not only advance theoretical knowledge but also have practical implications for fields such as engineering, control systems, and anomalous diffusion, where memory effects and nonlocal interactions are prevalent. Our numerical example demonstrates the applicability of the theoretical results in various scenarios with differing fractional orders, boundary conditions, and nonlinearities. Looking ahead, future research could explore more complex boundary conditions, higher-dimensional models, and computational techniques for solving intricate fractional systems, with potential applications in biological modeling, advanced materials science, and industrial systems. In conclusion, this study strengthens the theoretical foundations of nonlinear fractional integro-differential equations and provides a solid basis for future research, bridging the gap between theory and real-world applications, and offering powerful tools to tackle complex, memory-driven systems, thus advancing the field of fractional calculus and its broad range of applications.

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References

[1] Agarwal, R. P., Benchohra, M. and Hamani, S., (2009), Boundary value problems for fractional differential equations, Georgian Mathematical Journal, 16(3), pp. 401-411.

[2] Alkhezi, Y., Awad, Y., Amin, K., and Mghames, R., (2024), On the Solutions of Nonlinear Implicit ω-Caputo Fractional Order Ordinary Differential Equations with Two-Point Fractional Derivatives and Integral Boundary Conditions in Banach Algebra, Contemporary Mathematics, 5(4), pp. 4805-4835.

[3] Almeida, R., (2017), A Caputo fractional derivative of a function with respect to another function, Communications in Nonlinear Science and Numerical Simulation, 44, pp. 460-481.

[4] Almeida, R., (2018), Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, Mathematical Methods in the Applied Sciences, 41(1):336-352.

[5] Arjunan, M. M., Anbalagan, P., and Al-Mdallal, Q., (2023), Robust uniform stability criteria for fractional-order gene regulatory networks with leakage delays, Mathematical Methods in the Applied Sciences, 46(7), pp. 8372-8389.

[6] Awad, Y., Fakih, H., and Alkhezi, Y., (2023), Existence and Uniqueness of Variable-Order φ-Caputo Fractional Two-Point Nonlinear Boundary Value Problem in Banach Algebra, Axioms, 12(10):935.

[7] Awad, Y., (2023), On the Existence and Stability of Positive Solutions of Eigenvalue Problems for a Class of p-Laplacian ψ-Caputo Fractional Integro-Differential Equations, Journal of Mathematics, 2023: Article ID.

[8] Awad, Y., (2023), Well Posedness and Stability for the Nonlinear φ-Caputo Hybrid Fractional Boundary Value Problems with Two-Point Hybrid Boundary Conditions, Jordan Journal of Mathematics and Statistics (JJMS), 16(4):617-647.

[9] Awad, Y. and Alkhezi, Y., (2023), Analysis of Implicit Solutions for a Coupled System of Hybrid Fractional Order Differential Equations with Hybrid Integral Boundary Conditions in Banach Algebras, Symmetry, 15(9):1758.

[10] Awad, Y. and Fakih, H., (2024), Existence and Uniqueness Results for a Two-Point Nonlinear Boundary Value Problem of Caputo Fractional Differential Equations of Variable Order, TWMS Journal of Applied and Engineering Mathematics, 14(3), pp. 1068-1084.

[11] Awad, Y. and Kaddoura, I., (2024), On the Ulam-Hyers-Rassias stability for a boundary value problem of implicit ψ-Caputo fractional integro-differential equation, TWMS Journal of Applied and Engineering Mathematics, 14(1), pp. 617-647.

[12] Awad, Y. and Alkhezi, Y., (2024), Solutions of Second-Order Nonlinear Implicit ψ-Conformable Fractional Integro-Differential Equations with Nonlocal Fractional Integral Boundary Conditions in Banach Algebra, Symmetry, 16(9):1097.

[13] Awad, Y. and Chehade, H., (2024), Analysis of solutions for nonlinear ψ-Caputo fractional differential equations with fractional derivative boundary conditions in Banach algebra, The Eurasia Proceedings of Science Technology Engineering and Mathematics, 28, pp. 360-374.

[14] Benchohra, M., Hamani, S., and Ntouyas, S. K., (2008), Boundary value problems for differential equations with fractional order, Surveys in Mathematics and its Applications, 3, pp. 1-12.

[15] Benlabbes, A., Benbachir, M. and Lakrib, M., (2015), Boundary value problems for nonlinear fractional differential equations, Facta Universitatis-Series: Mathematics and Informatics, 30(2), pp. 157-168.

[16] Benchohra, M. and Hamani, S., (2009), Boundary value problems for differential equations with fractional order and nonlinear integral conditions, Commentationes Mathematicae, 49(2).

[17] Burton, T. A., and Kirk, C., (1998), A fixed point theorem of Krasnoselskii-Schaefer type, Mathematische Nachrichten, 189(1), pp. 23-31.

[18] Derbazi, C., and Hammouche, H., (2020), Existence and Uniqueness Results for a Class of Nonlinear Fractional Differential Equations with Nonlocal Boundary Conditions, Jordan Journal of Mathematics and Statistics (JJMS), 13(3):341-361.

[19] Derbazi, C., Al-Mdallal, Q. M., Jarad, F., and Baitiche, Z., (2021), Some qualitative properties of solutions for nonlinear fractional differential equation involving two Φ-Caputo fractional derivatives, arXiv preprint arXiv:2108.13758.

[20] Derdar, N., (2022), Nonlinear Implicit Caputo-Hadamard Fractional Differential Equation with Fractional Boundary Conditions, Jordan Journal of Mathematics and Statistics (JJMS), 15, pp. 999-1014.

[21] M. El-Shahed, (2007), Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation, Abstract and Applied Analysis, pp. 1-9.

[22] Haddouchi, F., (2018), On the existence and uniqueness of solution for fractional differential equations with nonlocal multi-point boundary conditions, arXiv preprint arXiv:1811.10706, November 26.

[23] Hamoud, A., Ghadle, K. and Pathade, P., (2019), An existence and convergence results for Caputo fractional Volterra integro-differential equations, Jordan Journal of Mathematics and Statistics (JJMS), 12(3), pp. 307-327.

[24] Hyers, D. H., (1941), On the stability of the linear functional equation, Proceedings of the National Academy of Sciences, 27, pp. 222-224.

[25] Ionescu, C., Lopes, A., Copot, D., Machado, J. T., and Bates, J. H., (2017), The role of fractional calculus in modeling biological phenomena: A review, Communications in Nonlinear Science and Numerical Simulation, 51, pp. 141-159.

[26] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., (2006), Theory and Applications of Fractional Differential Equations, Elsevier.

[27] Mainardi, F., Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London.

[28] Ntouyas, S. K., (2013), Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions, Opuscula Mathematica, 33(1), pp. 117-138.

[29] Obaidat, M. A., (2020), Existence results for fractional differential equations with multi-point boundary conditions, Journal of Inequalities and Applications, 2020;2020(1):1-14.
[30] Palanisamy, G., Kashkynbayev, A., and Rajan, R., (2023), Finite-Time Stability of Fractional-Order Discontinuous Nonlinear Systems With State-Dependent Delayed Impulses, IEEE Transactions on Systems, Man, and Cybernetics: Systems, 54(2), pp. 1312-1324.
[31] Qian, W. and Yang, X., (2008), Existence of solutions for boundary value problems of fractional differential equations, Nonlinear Analysis: Theory, Methods & Applications, 69(9), pp. 3222-3231.
[32] Rao, R., and Santos, J. A. R., (2015), Existence of solutions for boundary value problems of fractional differential equations, Mathematical Modelling and Analysis, 20(4):519-537.
[33] Rassias, Th. M., (1978), On the stability of linear mappings in Banach spaces, Proceedings of the American Mathematical Society, 72, pp. 297-300.
[34] Stojanovic, M. and Stojanovic, M. S., (2014), Existence and uniqueness results for fractional differential equations with boundary conditions, Electronic Journal of Differential Equations, 2014(68), pp. 1-13.
[35] Ulam, S. M., (1960), Problems in Modern Mathematics, Interscience Publishers.
[36] Wang, Z., and Zhang, H., (2018), Existence and uniqueness of solutions for nonlinear fractional differential equations with nonlocal boundary conditions, Mathematical Methods in the Applied Sciences, 41(8), pp. 3361-3377.

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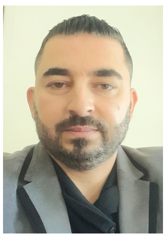

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