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SYSTEMS BIFURCATION OF LIMIT CYCLES FOR A FAMILY OF DISCONTINUOUS PIECEWISE ISOCHRONOUS DIFFERENTIAL SYSTEMS SEPARATED BY IRREGULAR LINE

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Abstract. The study of Hilbert's 16th problem for piecewise linear differential systems has received significant attention from many researchers. It was shown that the upper bound for the maximum number of limit cycles can vary according to the configuration of the discontinuous curve.

The family of discontinuous piecewise differential systems formed by linear isochronous centers or four families of quadratic isochronous centers separated by a straight line have been studied, and the authors have found at most two limit cycles.

In this paper, we study the same family but instead of a straight line, we consider an irregular line separation, and we prove that there are at most five crossing limit cycles intersecting the separation curve at two points.

Keywords: Limit cycles, quadratic isochronous centers, linear differential center, irregular line.

AMS Subject Classification: Primary 34C29, 34C25, 47H11

1. Introduction

In 1920s, Andronov, Vitt and Khaikin [1] started investigating the behavior of piecewise differential systems because of their widespread usage in modeling biological processes and certain mechanical and electrical applications (e.g., in the book [9] and the survey [23]).

The problem of cyclicity, which refers to discovering the limit cycles of piecewise differential systems, is considered one of the most crucial and challenging topics in this field of theory. A limit cycle refers to a periodic orbit of a differential system in $\mathbb{R}^2$ isolated from all other periodic orbits of the same system. In the qualitative study, one of the main problems of planar differential systems is determining the existence and the maximum number of limit cycles. The importance of this issue comes from the main role of limit cycles in understanding the behavior of a given differential system, such as the limit

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cycle of Van der Pol equations [28, 29]. Poincaré introduced this concept in the late 19th century, as documented in his works [25, 26]. Several articles, such as [4, 14, 27], have shown the correlation between various occurrences and limit cycles.

David Hilbert [12, 13] provided a list of 23 problems. Up to now, at least three problems are still open, including the Riemann Conjecture and the sixteenth problem.

Recently, researchers have made significant progress in solving the second part of Hilbert's 16th problem for certain special classes of smooth or piecewise smooth polynomial differential systems. This problem seeks to determine the maximum number and possible configurations of limit cycles within a given class of differential systems. Consequently, one of the key challenges for piecewise differential systems involves controlling the existence and maximum number of crossing limit cycles within these systems. Many authors have focused on studying the simplest class of planar discontinuous piecewise differential systems, which are linear and separated by a straight line. For instance, refer to [2, 6, 7, 16, 17, 20] and the references therein. However, while these papers provide examples with a maximum of three limit cycles, no one has yet proven that three is the absolute maximum number for this class of systems.

In 2021, Esteban *et al.* [10] solved Hilbert's 16th problem for discontinuous piecewise isochronous centers of degree one or two separated by a straight line. In the same year, Benterki and Llibre [5] extended the same problem to some classes of discontinuous piecewise isochronous centers of degree one or three.

In this paper we are interested in studying the existence and the maximum number of limit cycles for four classes of discontinuous piecewise isochronous centers, separated by the irregular line $\Sigma = \Sigma_1 \cup \Sigma_2$ such that $\Sigma_1 = \{(x, y) : x = 0 \ \ and \ \ y \geq 0\}$, and $\Sigma_2 = \{(x, y) : x \geq 0 \ \ and \ \ y = 0\}$. This irregular line divides the plane into two regions, where the first one is $\Sigma^+ = \{(x, y) : x > 0, y > 0\}$ and the second one is $\Sigma^- = \{(x, y) : x \geq 0, y < 0\} \cup \{(x, y) : x < 0\}$, and we consider in one region an arbitrary linear differential center and in the other region we consider one of the four classes of quadratic isochronous differential systems given in Lemma 1.2.

Because of their applications, the piecewise differential systems with irregular lines separation are frequently observed in diverse fields such as biology, mechanical engineering, and control systems, etc, see for examples hybrid systems [15] and predator-prey model [11].

Now, we will give the isochronous differential centers, starting with the linear differential systems having a center given in the following lemma.

Lemma 1.1. *Through a linear change of variables and a rescaling of the independent variable, every center in* $\mathbb{R}^2$ *can be written as*

$$\dot{x} = B - Ax - (A^2 + \omega^2)y, \qquad \dot{y} = x + Ay + C, \tag{C_1}$$

with $\omega > 0$, A, B, $C \in \mathbb{R}$ *and* $A \neq 0$.
The first integral of this system is

$$H_1(x, y) = (Ay + x)^2 + 2(Cx - By) + y^2\omega^2. \tag{1}$$

Or we can define the linear differential center as follows

$$\dot{x} = -Ax - (A^2 + \omega^2)y, \qquad \dot{y} = x + Ay, \tag{C_2}$$

with $\omega > 0$, $A \in \mathbb{R} - \{0\}$ *and its corresponding first integral is*

$$H(x, y) = (Ay + x)^2 + y^2\omega^2. \tag{2}$$

For the proof of Lemma 1.1 see [18].

Quadratic polynomial differential systems with an isochronous center have been classified into four types by Loud [22].

Lemma 1.2. *Any Quadratic polynomial differential system with an isochronous center can be classified by one of the four classes*

(I) *The first class is given by*
$\dot{x} = \left(x^2 - y^2\right) - y, \quad \dot{y} = x(2y+1)$ *has the first integral* $H_2(x,y) = \frac{x^2+y^2}{2y+1}$.

(II) *The second class is written as*
$\dot{x} = x^2 - y, \quad \dot{y} = x(1+y)$ *exhibits the first integral* $H_3(x,y) = \frac{x^2+y^2}{(y+1)^2}$.

(III) *The third class is*
$\dot{x} = \frac{x^2}{4} - y, \quad \dot{y} = x(y+1)$ *has the first integral* $H_4(x,y) = \frac{\left(x^2+4y+8\right)^2}{y+1}$.

(IV) *The fourth class is*
$\dot{x} = 2x^2 - \frac{y^2}{2} - y, \quad \dot{y} = x(y+1)$ *has the first integral* $H_5(x,y) = \frac{4x^2-2(y+1)^2+1}{(y+1)^4}$.

In this work, we will study the limit cycles intersect the separation line Σ at two points, where we examine two possible configurations of limit cycles. We will denote by **Conf 1** the limit cycles that intersect Σ_1 at two points, i.e., intersecting Σ_1 at $(0, y_1)$ and $(0, y_2)$ such that $y_1 \neq y_2$. The case when the limit cycles intersect Σ at two points, where the first point on Σ_1 and the second one on Σ_2, i.e., the first point $(x_1, 0) \in \Sigma_2$ and the second point $(0, y_2) \in \Sigma_1$, we called it the second configuration and we denote it by **Conf 2**. We notice that the combination of the two configurations **Conf 1** and **Conf 2** gives a third configuration witch is denoted by **Conf 3**.

In the following theorems, we will give our main results concerning the three possible configurations of limit cycles mentioned previously.

Theorem 1.1. *The maximum number of crossing limit cycles with* **Conf 1** *of the discontinuous piecewise differential system separated by* Σ *and formed by*

(i) *the quadratic isochronous system (I) after an arbitrary affine change of variables and the linear differential center* (C_1) *is at most one. There are systems of this type with exactly one limit cycle; see Fig 1(a);*
(ii) *the quadratic isochronous system (II) after an arbitrary affine change of variables and the linear differential center* (C_1) *is at most one. There are systems of this type with exactly one limit cycle; see Fig 1(b);*
(iii) *the quadratic isochronous system (III) after an arbitrary affine change of variables and the linear differential center* (C_1) *is at most two. There are systems of this type with exactly two limit cycles, see Fig 2(a);*
(iv) *the quadratic isochronous system (IV) after an arbitrary affine change of variables and the linear differential center* (C_1) *is at most two. There are systems of this type with exactly two limit cycles; see Fig 2(b).*

Theorem 1.1 is proved in Section 3.

Theorem 1.2. *The maximum number of crossing limit cycles with* **Conf 2** *of the discontinuous piecewise differential system separated by* Σ *and formed by*

(i) *the quadratic isochronous system (I) after an arbitrary affine change of variables and the linear differential center* (C_2) *is at most one. There are systems of this type with one limit cycle; see Fig 3(a);*

(ii) *the quadratic isochronous system (II) after an arbitrary affine change of variables and the linear differential center* (C_2) *is at most two. There are systems of this type with two limit cycles; see Fig 3(b);*
(iii) *the quadratic isochronous system (III) after an arbitrary affine change of variables and the linear differential center* (C_2) *is at most two. There are systems of this type with two limit cycles, see Fig 4(a);*
(iv) *the quadratic isochronous system (IV) after an arbitrary affine change of variables and the linear differential center* (C_2) *is at most three. There are systems of this type with three limit cycles; see Fig 4(b).*

Theorem 1.2 is proved in Section 4.

Theorem 1.3. *The maximum number of crossing limit cycles with* **Conf 3** *of the discontinuous piecewise differential system separated by the non-regular line* Σ *and formed by*

(i) *the quadratic isochronous system (I) after an arbitrary affine change of variables and the linear differential center* (C_1) *is two, see Fig 5(a);*
(ii) *the quadratic isochronous system (II) after an arbitrary affine change of variables and the linear differential center* (C_1) *is three, see Fig 5(b);*
(iii) *the quadratic isochronous system (III) after an arbitrary affine change of variables and the linear differential center* (C_1) *is four, see Fig 6(a);*
(iv) *the quadratic isochronous system (IV) after an arbitrary affine change of variables and the linear differential center* (C_1) *is five, see Fig 6(b).*

Theorem 1.3 is proved in Section 5.

2. The quadratic isochronous differential centers after an affine change of variables

In this section, we present the expressions of the whole classes of the quadratic isochronous centers (I), (II), (III) and (IV) after the change of variables $(x, y) \to (ax + by + d, \alpha x + \beta y + \gamma)$, with $b\alpha - a\beta \neq 0$. This affine transformation is used to generalize all the classes, enabling a broader analysis. Thus, after this affine change of variables, system (I) becomes

$$\begin{aligned} \dot{x} = &\ \tfrac{1}{\alpha b - a\beta}(\beta(-a^2x^2 - 2adx - d^2 + (\gamma + \alpha x + \beta y)(\gamma + \alpha x + \beta y + 1)) \\ &+ b(ax + d)(2\gamma + 2\alpha x + 1) + b^2y(2\gamma + 2\alpha x + \beta y + 1)), \\ \dot{y} = &\ -\tfrac{1}{\alpha b - a\beta}(a^2x(2\gamma + \alpha x + 2\beta y + 1) + a(by + d)(2\gamma + 2\beta y + 1) \\ &+ \alpha(-b^2y^2 - 2bdy - d^2 + (\gamma + \alpha x + \beta y)(\gamma + \alpha x + \beta y + 1))), \end{aligned} \tag{3}$$

with its first integral

$$H_2(x, y) = \frac{(ax+by+d)^2 + (\gamma+\alpha x+\beta y)^2}{2(\gamma+\alpha x+\beta y)+1}. \tag{4}$$

The differential system (II) is given by

$$\begin{aligned} \dot{x} = &\ \tfrac{1}{\alpha b - a\beta}(b(ax + d)(\gamma + \alpha x - \beta y + 1) + \beta(-(ax + d)^2 + \gamma + \alpha x + \beta y) \\ &+ b^2y(\gamma + \alpha x + 1)), \\ \dot{y} = &\ \tfrac{1}{\alpha b - a\beta}(a^2(-x)(\gamma + \beta y + 1) - a(by + d)(\gamma + \alpha(-x) + \beta y + 1) \\ &+ \alpha((by + d)^2 - \gamma - \alpha x - \beta y)), \end{aligned} \tag{5}$$

where its first integral is

$$H_3(x, y) = \frac{(ax+by+d)^2 + (\gamma+\alpha x+\beta y)^2}{(\gamma+\alpha x+\beta y+1)^2}. \tag{6}$$

The differential system (III) written as

$$\begin{aligned} \dot{x} = & \ \tfrac{1}{4\alpha b-4a\beta}(2b(ax+d)(2\gamma+2\alpha x+\beta y+2)+\beta(-(ax+d)^2+4\alpha x \\ & +4(\gamma+\beta y))+b^2y(4\gamma+4\alpha x+3\beta y+4)), \\ \dot{y} = & \ \tfrac{1}{4\alpha b-4a\beta}(a^2(-x)(3\alpha x+4(\gamma+\beta y+1))-2a(by+d)(\alpha x+2(\gamma+\beta y \\ & +1))+\alpha((by+d)^2-4(\gamma+\alpha x)-4\beta y)), \end{aligned} \tag{7}$$

and its corresponding first integral is

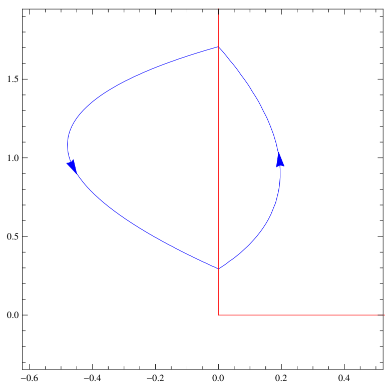
$$H_4(x,y) = \frac{\left((ax+by+d)^2+4(\gamma+\alpha x+\beta y)+8\right)^2}{\gamma+\alpha x+\beta y+1}. \tag{8}$$

Finally, the differential system (IV) becomes

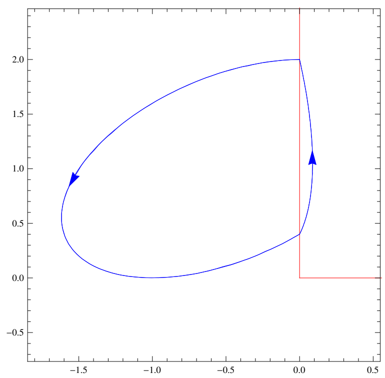
$$\begin{aligned} \dot{x} = & \ \tfrac{1}{2\alpha b-2a\beta}(\beta(-4a^2x^2-8adx-4d^2+(\gamma+\alpha x+\beta y)(\gamma+\alpha x+\beta y \\ & +2))+2b(ax+d)(\gamma+\alpha-3\beta y+1)+2b^2y(\gamma+\alpha x-\beta y+1)), \\ \dot{y} = & \ \tfrac{1}{2\alpha b-2a\beta}(2a^2x(-\gamma+\alpha x-\beta y-1)-2a(by+d)(\gamma-3\alpha x+\beta y+1) \\ & -\alpha(-4b^2y^2-8bdy-4d^2+(\gamma+\alpha x+\beta y)(\gamma+\alpha x+\beta y+2))), \end{aligned} \tag{9}$$

where its first integral is

$$H_5(x,y) = \frac{4(ax+by+d)^2-2(\gamma+\alpha x+\beta y+1)^2+1}{(\gamma+\alpha x+\beta y+1)^4}. \tag{10}$$



(a)



(b)

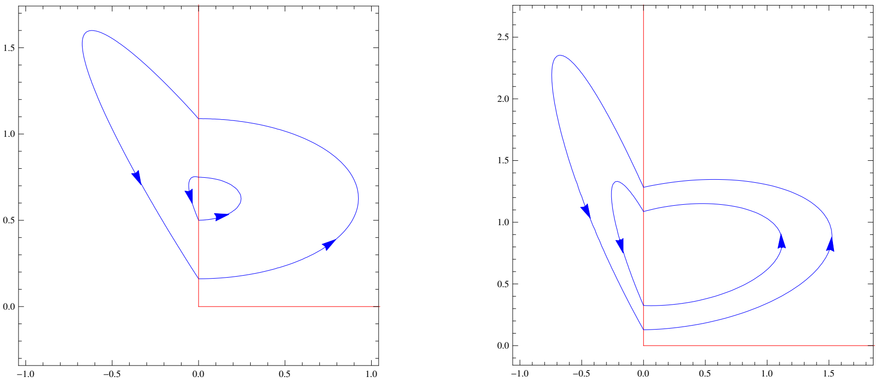
FIGURE 1. The unique limit cycle of the discontinuous piecewise differential system with **Conf 1**, (a) for (12)–(13), (b) for (14)–(15).

3. Proof of Theorem 1.1

In the region Σ^+ we consider the linear differential center (C_1) with its first integral $H_1(x,y)$ given by (1). In the region Σ^-, we consider one of the four quadratic isochronous systems with their first integrals $H_k(x,y)$ with $k=2,\ldots,5$, after an affine change of variable.

The crossing limit cycles of the discontinuous piecewise differential system (C_1)–(3), (C_1)–(5), (C_1)–(7) and (C_1)–(9) intersect Σ_1 at two different points $(0,y_1)$ and $(0,y_2)$, such that these two points must satisfy the system of equations

$$\begin{aligned} e_1 = H_1(0,y_1)-H_1(0,y_2) = & \quad (y_1-y_2)(-2B+(y_1+y_2)(A^2+\omega^2)) = 0 \\ e_2 = H_k(0,y_1)-H_k(0,y_2) = & \quad P_k(y_1,y_2) = 0, \end{aligned} \tag{11}$$



(a) (b)

FIGURE 2. The two limit cycles of the discontinuous piecewise differential system with **Conf 1**, (a) for (16)–(17), (b) for (18)–(19).

where $H_k(x, y)$ for $k = 2 \dots 5$ are the first integrals given by (4), (6), (8), (10). Since $y_1 \neq y_2$ we consider $0 < y_1 < y_2$, and by solving $e_1 = 0$, we get $y_1 = f(y_2) = -y_2 + D$ where $D = \frac{2B}{A^2+\omega^2}$. Substituting y_1 into $P_k(y_1, y_2) = 0$ we obtain an equation in the variable y_2.

Proof of statement (i) *of Theorem 1.1.* For $k = 2$ the first integral in this case is $H_2(x, y)$ given in (4), so the solutions y_2 satisfying $P_2(f(y_2), y_2) = 0$ is equivalent to the solutions y_2 of the quadratic equation $F_1(y_2) = 0$ such that

$$F_1(y_2) = \; b^2(2\gamma D + D) + 2\beta D y_2(b^2 + \beta^2) - 2\beta(b^2 + \beta^2)y_2^2 + 2b(2\gamma d + d) + \beta(2\gamma(\gamma +1) - 2d^2 + D(2\beta\gamma + \beta)).$$

This equation has at most two real solutions y_{21} and y_{22}. Consequently, system (11) has at most two real solutions namely (y_{11}, y_{21}) and (y_{12}, y_{22}). The two solutions are symmetric in the sense that $(y_{11}, y_{21}) = (y_{22}, y_{12})$. Then both solutions provide the unique limit cycle for the discontinuous piecewise differential systems (C_1)–(3).

Now we prove that the result of statement (i) is reached by giving an example with exactly one limit cycle.

In the region Σ^+, we consider the linear differential center

$$\dot{x} = 2 - x - 2y, \quad *\dot{y} = \tfrac{3}{2} + x + y, \tag{12}$$

with its first integral $H_1(x, y) = 2\left(\frac{3}{2}x - 2y\right) + (x + y)^2 + y^2$.

In the region Σ^-, we consider the quadratic isochronous center

$$\begin{aligned} \dot{x} &= \tfrac{14x^2}{5} + x\left(\tfrac{106y}{5} + \tfrac{754373}{100000}\right) + \tfrac{106y^2}{5} - \tfrac{67173y}{25000} - \tfrac{197827}{25000}, \\ \dot{y} &= -\tfrac{2x^2}{5} + x\left(\tfrac{101393}{1000000} - \tfrac{8y}{5}\right) + \tfrac{17y^2}{5} + \tfrac{845627y}{100000} + \tfrac{344957}{100000}, \end{aligned} \tag{13}$$

where its first integral is $H_2(x, y) = \dfrac{\left(x + 2y - \frac{398733}{10000}\right)^2 + \left(x + 7y + 4\right)^2}{2x + 4y - \frac{393733}{5000}}$.

The unique real solution of system (11) for these systems is $\left(\frac{292893}{1000000}, \frac{170711}{100000}\right)$. Then the discontinuous piecewise differential system (12)–(13) has one limit cycle shown in Fig 1(a).

$\square$

Proof of statement (ii) *of Theorem 1.1.* For $k=3$ the first integral in this case is $H_3(x,y)$ given in (6), so to study the solutions y_2 satisfying $P_3(f(y_2),y_2)=0$ we must study the solutions y_2 of the quadratic equation $F_2(y_2)=0$ such that

$$\begin{aligned} F_2(y_2) = {} & 2\beta D y_2(b^2(\gamma+1)+\beta^2-b\beta d)-2\beta y_2^2(b^2(\gamma+1)+\beta^2-b\beta d)+b^2(\gamma+1)^2D \\ & +2b(\gamma+1)^2 d+\beta(2\gamma+d^2(-(2\gamma+\beta D+2))+2\gamma(\gamma+\beta D)+\beta D). \end{aligned}$$

This equation has at most two real solutions again symmetric in the sense of the proof of statement (i). Therefore, system (11) has at most one limit cycle.

Now we prove that the result of statement (ii) is reached by giving an example with exactly one limit cycle. In the region Σ^+, we consider the linear differential center

$$\dot{x}=\tfrac{3}{2}+\tfrac{1}{2}x-\tfrac{5}{4}y, \quad \dot{y}=1+x-\tfrac{1}{2}y, \tag{14}$$

with its first integral $H_1(x,y)=\big(x-\frac{y}{2}\big)^2+2\big(x-\frac{3y}{2}\big)+y^2$.

In the region Σ^- we consider the quadratic isochronous center

$$\begin{aligned} \dot{x} = {} & \tfrac{3062x^2}{785}+x\big(\tfrac{366}{157}-\tfrac{102y}{157}\big)-\tfrac{680y^2}{157}+\tfrac{170y}{157}+\tfrac{50}{157}, \\ \dot{y} = {} & \tfrac{4827x^2}{15700}+x\big(\tfrac{6436y}{785}-\tfrac{1609}{1570}\big)+\tfrac{365y^2}{157}+\tfrac{262y}{157}-\tfrac{43}{157}, \end{aligned} \tag{15}$$

and its corresponding first integral is

$$H_3(x,y)=\frac{(4x+y+2)^2+\big(\frac{3x}{10}+4y-1)^2\big)}{\frac{3x}{5}+8y-1}.$$

The unique real solution of system (11) is $\big(\frac{2}{5},2\big)$, that provides the unique crossing limit cycle of the discontinuous piecewise differential system (14)–(15) shown in Fig 1(b). □

Proof of statement (iii) *of Theorem 1.1.* For $k=4$ the first integral in this case is $H_4(x,y)$ given in (8), so to study the solutions y_2 satisfying $P_4(f(y_2),y_2)=0$ is equivalent to study the solutions y_2 of the quartic equation $F_3(y_2)=0$, and due to the big expression of this equation we omit it. By solving $F_3(y_2)=0$, we know that it has at most four real solutions. Due to the symmetry of these solutions, we conclude that the discontinuous piecewise differential system (C_1)–(7) has at most two limit cycles.

Now to prove that the result of statement (iii) we consider the following example.

In the region Σ^+, we consider the linear differential center

$$\dot{x}=\tfrac{5}{2}+\tfrac{1}{100}x-\tfrac{40001}{10000}y, \qquad \dot{y}=\tfrac{1}{100}+x-\tfrac{1}{100}y, \tag{16}$$

with its first integral

$$H_1(x,y)=\big(x-\tfrac{y}{100}\big)^2+2\big(\tfrac{x}{100}-\tfrac{5y}{2}\big)+4y^2.$$

In Σ^-, we consider the quadratic isochronous center

$$\begin{aligned} \dot{x} = {} & 2x^2+x\big(\tfrac{3y}{2}-\tfrac{427081}{100000}\big)-\tfrac{5y}{2}+\tfrac{78123}{50000}, \\ \dot{y} = {} & -2x^2+x\big(\tfrac{566943}{100000}-y\big)+\tfrac{3y^2}{8}+\tfrac{14323y}{5000}-\tfrac{113507}{50000}, \end{aligned} \tag{17}$$

with the first integral

$$H_4(x,y)=\frac{\big((2x+\frac{3y}{2}-\frac{37500}{40001}\big)^2+4\big(-\frac{9x}{10}+\frac{1}{2}\big)+8\big)^2}{-\frac{9x}{10}y+\frac{3}{2}}.$$

The discontinuous piecewise differential system formed by the differential centers (16)–(17) has exactly two crossing limit cycles, because the system of equations (11) has exactly two different real solutions $\big(\frac{80323}{500000},\frac{27233}{25000}\big)$ and $\big(\frac{20001}{40000},\frac{93743}{125000}\big)$. See Fig 2(a). □

Proof of statement (iv) of Theorem 1.1. For $k = 5$ the corresponding isochronous quadratic system is (9) with its first integral $H_5(x, y)$ given in (10). The solutions y_2 satisfying $P_4(f(y_2), y_2) = 0$ are equivalent to ones of the quartic equation $F_4(y_2) = 0$, for the same reason as in the previous case we will not give the large expression of $F_4(y_2)$.

Therefor system (11) has at most four real solutions. Consequently the discontinuous piecewise differential system (C_1)–(9) has at most two limit cycles.

Now we complete the proof of this statement by giving an example of the discontinuous piecewise differential system formed by linear and isochronous centers separated by Σ, with exactly two limit cycles.

In the region Σ^+, we consider the linear differential center

$$\begin{aligned} \dot{x} &= 3 + \tfrac{1}{2}x - \tfrac{17}{4}y, \\ \dot{y} &= \tfrac{1}{10} + x - \tfrac{1}{2}y, \end{aligned} \tag{18}$$

with its first integral

$$H_1(x, y) = \left(x - \tfrac{y}{2}\right)^2 + 2\left(\tfrac{x}{10} - 3y\right) + 4y^2.$$

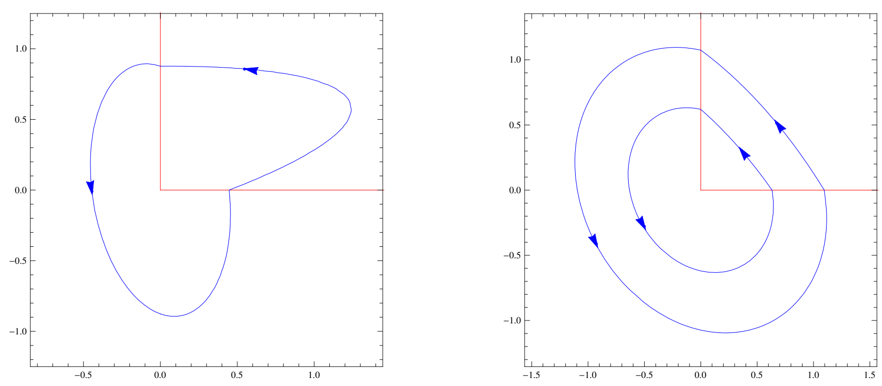
In the region Σ^-, we consider the quadratic isochronous differential center

$$\begin{aligned} \dot{x} &= 2x^2 + x\left(y - \tfrac{381699}{100000}\right) - \tfrac{38889y}{25000} + \tfrac{27451}{25000}, \\ \dot{y} &= \tfrac{719x^2}{200} + x\left(6y + \tfrac{324693}{100000}\right) + 2y^2 + \tfrac{143791y}{500000} - \tfrac{83977}{50000}, \end{aligned} \tag{19}$$

whose first integral is

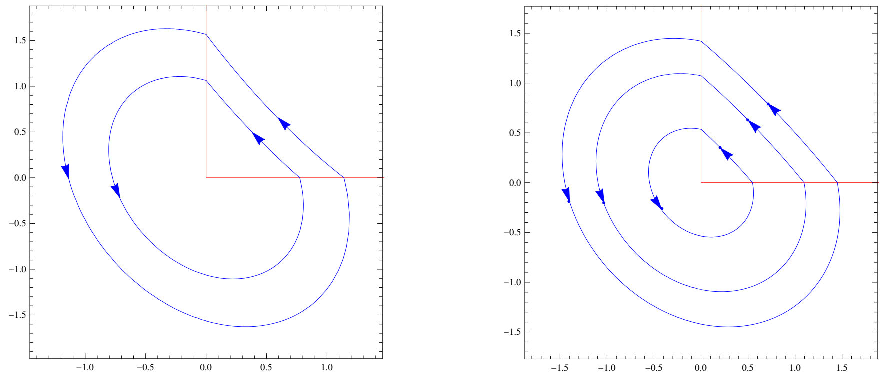
$$H_5(x, y) = \frac{-2\left(-\tfrac{9x}{10} + \tfrac{7}{5}\right)^2 + 4\left(2x + y - \tfrac{12}{17}\right)^2 + 1}{\left(-\tfrac{9x}{10} + \tfrac{7}{5}\right)^4}.$$

The two real solutions of system (11) are $\left(\tfrac{32083}{250000}, \tfrac{128343}{100000}\right)$ and $\left(\tfrac{162331}{500000}, \tfrac{10871}{10000}\right)$. Then the discontinuous piecewise differential system (18)-(19) has two limit cycles shown in Fig 2(b). This completes the proof of Theorem 1.1. □



(*a*) (*b*)

FIGURE 3. (*a*) The unique limit cycle of the discontinuous piecewise differential system (21)–(22) with **Conf 2**, (*b*) the two limit cycles of the discontinuous piecewise differential system (23)–(24) with **Conf 2**.



(a) (b)

FIGURE 4. (a) The two limit cycles of the discontinuous piecewise differential system (25)–(26) with **Conf 2**, (b) the three limit cycles of the discontinuous piecewise differential system (27)–(28) with **Conf 2**.

4. Proof of Theorem 1.2

In one region we consider the linear differential center (C_2) with its first integral $H(x,y)$ given by (2). In the other region, we consider one of the four quadratic isochronous systems with its first integral $H_k(x,y)$ with $k=2,\ldots,5$, after an affine change of variable. If the discontinuous piecewise differential system (C_2)–(3), (C_2)–(5), (C_2)–(7) and (C_2)–(9) has limit cycles, intersect the separation line Σ at two distinct points $(x_1,0)\in\Sigma_2$ and $(0,y_2)\in\Sigma_1$ where $x_1y_2\neq 0$, then these points must satisfy the system of equations

$$\begin{aligned} e_1 &= H(x_1,0)-H(0,y_2)=Q_1(x_1,y_2)=0,\\ e_2 &= H_i(x_1,0)-H_i(0,y_2)=P_i(x_1,y_2)=0, \end{aligned} \tag{20}$$

with $i=2\ldots5$. Solving $Q_1(x_1,y_2)=0$, we get $y_2=g(x_1)=Mx_1$, with $M=1/\sqrt{A^2+\omega^2}$. Substituting y_2 into $P_i(x_1,y_2)=0$ we obtain an equation in the variable x_1.

Proof of statement (i) *of Theorem 1.2.* For $i=2$ the corresponding isochronous quadratic system is (3) with its first integral $H_2(x,y)$ given in (4). The solutions x_1 satisfying $P_2(x_1,g(x_1))=0$ are the ones x_1 of the cubic equation $G_1(x_1)=0$ such that

$$\begin{aligned} G_1(x_1)= &-x_1^2a^2(2\gamma+1)-4a\beta dM+b^2(2\gamma+1)M^2+4\alpha bdM-(2\gamma+1)(\alpha-\beta M)(\alpha\\ &+\beta M))+2Mx_1^3(\alpha b^2M-\beta(a^2+\alpha(\alpha-\beta M)))-2x_1(a(2\gamma d+d)-b(2\gamma\\ &+1)dM-(d^2-\gamma(\gamma+1))(\alpha-\beta M). \end{aligned}$$

The equation $G_1(x_1)=0$ has at most two real solutions. Therefore, the discontinuous piecewise differential system (C_2)–(3) has at most one limit cycle because system (20) has at most one real solution.

Now we will prove that the result of statement (i) is reached.

In the region Σ^+, we consider the quadratic isochronous differential center

$$\begin{aligned} \dot{x} &= \tfrac{98221x^2}{500000}+x\big(\tfrac{42719}{20000}-\tfrac{21697y}{20000}\big)-\tfrac{542423}{10000000}y^2-\tfrac{488181y}{50000}+\tfrac{43043}{12500},\\ \dot{y} &= \tfrac{355829}{10000000}x^2+x\big(\tfrac{711658}{100000000}y+\tfrac{160123}{250000}\big)-\tfrac{35107y^2}{62500}-\tfrac{26719y}{20000}+\tfrac{31313}{20000}, \end{aligned} \tag{21}$$

with the first integral

$$H_2(x,y)=\frac{\big(\frac{x}{10}+\frac{y}{100}+\frac{2}{5}\big)^2+\big(\frac{x}{10}-\frac{138017}{250000}y+\frac{1}{5}\big)^2}{2\big(\frac{x}{10}+\frac{y}{100}+\frac{2}{5}\big)+1}.$$

In the region Σ^-, we consider the linear differential center

$$\dot{x} = -\tfrac{1}{10}x - \tfrac{13}{50}y, \qquad \dot{y} = x + \tfrac{1}{10}y, \tag{22}$$

which has the first integral

$$H(x,y) = \big(x + \tfrac{y}{10}\big)^2 + \tfrac{y^2}{4}.$$

The discontinuous piecewise differential system (21)–(22) has one limit cycle because the unique solution of system (20) is $\big(\frac{223607}{500000}, \frac{438529}{500000}\big)$, see Fig 3(a). □

Proof of statement (ii) of Theorem 1.2. For $i = 3$ the isochronous quadratic system is defined by (5), which has the first integral $H_3(x,y)$ given in (6). Hence, the solution x_1 of $P_3(x_1, g(x_1)) = 0$ is equivalently the ones of the quartic equation $G_3(x_1) = 0$ such that

$$\begin{aligned} G_3(x_1) = \ & x_1^2(-a^2(\gamma+1)^2 - 4a\beta(\gamma+1)dM + b^2(\gamma+1)^2M^2 + 4\alpha b(\gamma+1)dM + (-2\gamma \\ & +d^2-1)(\alpha+\beta M)(\alpha-\beta M)) - 2Mx_1^3(\beta(a^2(\gamma+1) + a\beta dM + \alpha(\alpha-\beta M)) \\ & +\alpha b^2(\gamma+1)(-M) - \alpha^2 bd) - 2(\gamma+1)x_1(a(\gamma+1)d - b(\gamma+1)dM + (d^2 \\ & -\gamma)(-(\alpha-\beta M))) + M^2x_1^4(a\beta+\alpha b)(\alpha b - a\beta). \end{aligned}$$

The discontinuous piecewise differential system (C_2)–(5) has at most two limit cycles because system (20) has at most four real solutions.

To confirm the result statement (ii) of Theorem 1.2 we consider the following example. In the region Σ^+, we define the quadratic isochronous differential center

$$\begin{aligned} \dot{x} = \ & x\big(-\tfrac{640971}{10000000}y - \tfrac{352959}{1000000}\big) - \tfrac{47817y}{62500} - \tfrac{130743}{50000}, \\ \dot{y} = \ & \tfrac{156013x}{100000} - \tfrac{640971}{10000000}y^2 - \tfrac{118939y}{500000} + \tfrac{251513}{100000}, \end{aligned} \tag{23}$$

with its first integral

$$H_3(x,y) = \frac{\big(\frac{x}{10} + \frac{y}{100} + \frac{1}{5}\big)^2 + \big(-\frac{640971}{10000000}y - \frac{98473}{500000}\big)^2}{\big(\frac{x}{10} + \frac{y}{100} + \frac{6}{5}\big)^2}.$$

In the region Σ^-, we define the linear differential center

$$\dot{x} = -\tfrac{1}{5}x - \tfrac{26}{25}y, \qquad \dot{y} = x + \tfrac{1}{5}y, \tag{24}$$

and its first integral

$$H(x,y) = \big(x + \tfrac{y}{5}\big)^2 + y^2.$$

The discontinuous piecewise differential system (23)–(24) has two limit cycles, because system (20) has the two real solutions $\big(\frac{21909}{20000}, \frac{107417}{100000}\big)$ and $\big(\frac{79057}{125000}, \frac{310087}{500000}\big)$. These limit cycles are shown in Fig 3(b). □

Proof of statement (iii) of Theorem 1.2. For $i = 4$ the isochronous quadratic system is (7) where its first integral is $H_4(x,y)$ given in (8), so the solutions x_1 satisfying $P_4(x_1, g(x_1)) = 0$ are the ones of the quintic equation $G_4(x_1) = 0$, and again we will not give its large expression.

Equation $G_4(x_1) = 0$ has at most five real solutions. Therefore, the discontinuous piecewise differential system (C_2)–(7) has at most two limit cycles.

To confirm the result statement (iii) of Theorem 1.2 we consider the following example. In the region Σ^+, we consider the quadratic isochronous differential center

$$\begin{aligned} \dot{x} = \ & -\tfrac{227309}{2500000}x^2 + x\big(\tfrac{115143}{100000} - \tfrac{111463y}{1250000}\big) + \tfrac{107733}{10000000}y^2 + \tfrac{16301y}{25000} - \tfrac{240997}{50000}, \\ \dot{y} = \ & \tfrac{121019x^2}{2000000} + x\big(\tfrac{39809y}{1250000} - \tfrac{15703}{6250}\big) - \tfrac{185627}{5000000}y^2 + \tfrac{910233}{10000000}y + \tfrac{163031}{25000}, \end{aligned} \tag{25}$$

where its corresponding first integral is

$$H_4(x,y)=\frac{\Big(\big(-\frac{x}{10}-\frac{27237y}{250000}+\frac{51769}{62500}\big)^2+4\big(-\frac{x}{5}+\frac{3y}{100}+\frac{2}{5}\big)+8\Big)^2}{-\frac{x}{5}+\frac{3y}{100}+\frac{7}{5}}.$$

In the region Σ^-, we define the linear differential center

$$\dot{x}=-\tfrac{1}{5}x-\tfrac{53}{100}y,\qquad \dot{y}=x+\tfrac{1}{5}y, \tag{26}$$

which has the first integral

$$H(x,y)=\big(x+\tfrac{y}{5}\big)^2+\tfrac{49}{100}y^2.$$

In this case system (20) has the two real solutions $\big(\frac{57009}{50000},\frac{31323}{20000}\big)$ and $\big(\frac{774597}{1000000},\frac{106399}{100000}\big)$, which provide two limit cycles for the discontinuous piecewise differential system (25)–(26) shown in Fig 4(a). □

Proof of statement (iv) of Theorem 1.2. For $i=5$ the isochronous quadratic system is (9) where its first integral is $H_5(x,y)$ given in (10). To obtain the number of real solutions of system (20) we must solve the equation $P_5(x_1,g(x_1))=0$ which has the same solutions as the equation of degree six $G_5(x_1)=0$. Consequently the discontinuous piecewise differential system (C_2)–(9) has at most three limit cycles.

To confirm the result of statement (*iv*) of Theorem 1.2 we consider the following example.

In the region Σ^+, we have the quadratic isochronous differential center

$$\begin{aligned}\dot{x}=&\ -\tfrac{3x^2}{40}+x\big(-\tfrac{662873}{1000000}y-\tfrac{8}{5}\big)-\tfrac{91y^2}{500}-\tfrac{203961y}{100000}-\tfrac{11}{10},\\ \dot{y}=&\ \tfrac{85801x^2}{500000}+x\big(\tfrac{13y}{20}+\tfrac{49029}{25000}\big)+\tfrac{764853}{10000000}y^2+\tfrac{8y}{5}+\tfrac{13483}{12500},\end{aligned} \tag{27}$$

with the first integral

$$H_5(x,y)=\frac{4\big(\frac{x}{10}-\frac{5099y}{50000}\big)^2-2\big(\frac{3x}{10}+\frac{305941}{1000000}y+\frac{6}{5}\big)^2+1}{\big(\frac{3x}{10}+\frac{305941}{1000000}y+\frac{6}{5}\big)^4}.$$

In the region Σ^-, we have the linear differential center

$$\dot{x}=-\tfrac{1}{5}x-\tfrac{26}{25}y,\qquad \dot{y}=x+\tfrac{1}{5}y, \tag{28}$$

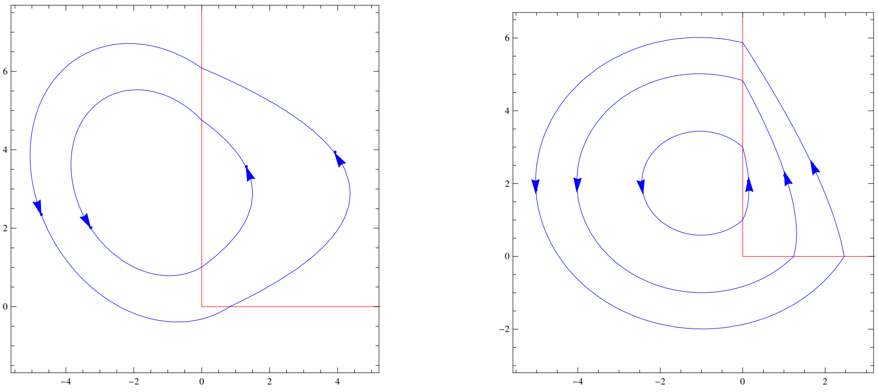
and its first integral is

$$H(x,y)=\big(x+\tfrac{1}{5}y\big)^2+y^2.$$

The three solutions of system (20) for these systems are $\big(\frac{72457}{50000},\frac{1421}{1000}\big)$, $\big(\frac{21909}{20000},\frac{107417}{100000}\big)$ and $\big(\frac{547723}{1000000},\frac{268543}{500000}\big)$. Then the three crossing limit cycles of system (27)–(28) are shown in Fig 4(b). □

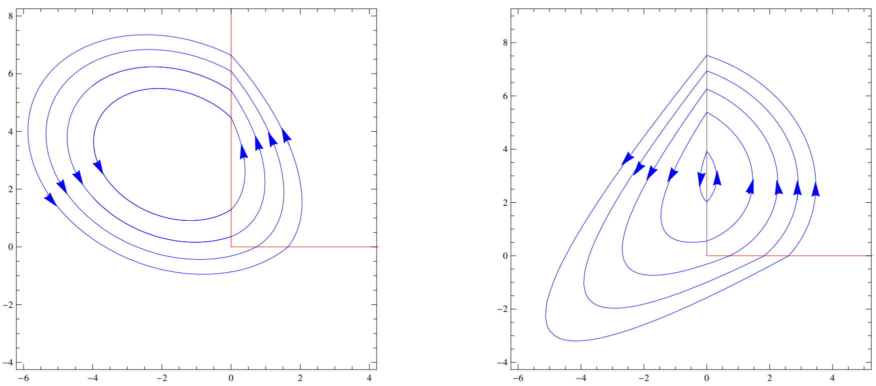
5. Proof of Theorem 1.3

In order to build limit cycles with **Conf 1** and **Conf 2** simultaneously, the limit cycles with **Conf 1** which intersect the discontinuity line Σ_1 at two points must satisfy system (11), in the other hand the intersection points of limit cycles with **Conf 2** with the curve Σ must satisfy system (20). In Theorem 1.1 and Theorem 1.2 we have already provided the maximum number of limit cycles with **Conf 1** and **Conf 2**, respectively. Then we have the following results.



(a) (b)

FIGURE 5. (a) The two limit cycles with **Conf 3** of system (29)–(30), (b) the three limit cycles with **Conf 3** of system (31)–(32).



(a) (b)

FIGURE 6. (a) The four limit cycles with **Conf 3** of system (33)–(34), (b) the five limit cycles with **Conf 3** of system (35)–(36).

Proof of statement (*i*) *of Theorem 1.3.* In Theorem 1.1 and 1.2 we proved that the maximum number of limit cycles with **Conf 1** and **Conf 2** is at most one. Then we know that the upper bound for the maximum number of limit cycles with configuration **Conf 3** is at most two.

Example with two limit cycles for the class formed by linear and the quadratic isochronous center (3) **with Conf 3.** In the region Σ^+ we consider the quadratic isochronous center

$$\begin{aligned} \dot{x} &= \frac{892x^2}{1953125} + x\Big(\frac{27899}{5000} - \frac{100209y}{100000}\Big) + \frac{57207y^2}{50000} - \frac{59021y}{10} + \frac{85099}{5}, \\ \dot{y} &= \frac{x^2}{5000} + x\Big(\frac{29449}{12500} - \frac{1784y}{1953125}\Big) - \frac{498957}{1000000}y^2 - \frac{25899y}{5000} + \frac{138749}{20}, \end{aligned} \tag{29}$$

with the first integral

$$H_2(x,y) = \frac{\Big(-\frac{x}{100} + \frac{1784y}{78125} - \frac{593981}{10000}\Big)^2 + \Big(\frac{1}{10} - \frac{y}{2}\Big)^2}{2\Big(-\frac{x}{100} + \frac{1784y}{78125} - \frac{593981}{10000}\Big) + 1}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\tfrac{1}{5}x - \tfrac{26}{25}y + 3, \qquad \dot{y} = x + \tfrac{1}{5}y + \tfrac{4}{5}, \tag{30}$$

which has the first integral

$$H_1(x,y) = \big(x + \tfrac{1}{5}y\big)^2 + 2\big(\tfrac{4x}{5} - 3y\big) + y^2.$$

These systems have the unique real solution $\big(\frac{103101}{125000}, \frac{24341}{4000}\big)$ for system (11) and they have the unique real solution $\big(\frac{101023}{100000}, \frac{4759}{1000}\big)$ for system (20). These two real solutions provide the two limit cycles drawn in Fig 5(a). This completes the proof of statement (i). □

Proof of statement (ii) *of Theorem 1.3.* In Theorem 1.1 and 1.2 we proved that the maximum number of limit cycles with **Conf 1** and **Conf 2** is at most one and two, respectively. We know that the upper bound for the maximum number of limit cycles with **Conf 3** is at most three.

Example with three limit cycles for the class formed by linear and the quadratic isochronous center (5) **with Conf 3.** In the region Σ^+ we consider the quadratic isochronous center

$$\begin{aligned}\dot{x} =& \ -\tfrac{x^2}{10} + x\big(-\tfrac{859609}{10000000}y - \tfrac{666483}{100000}\big) - \tfrac{18669y}{3125} + \tfrac{424187}{100000},\\ \dot{y} =& \ x\big(\tfrac{717167}{100000} - \tfrac{y}{10}\big) - \tfrac{859609}{10000000}y^2 + \tfrac{606483y}{100000} + \tfrac{1278}{125},\end{aligned} \tag{31}$$

and its first integral is

$$H_3(x,y) = \frac{\big(-\frac{x}{100} - \frac{416854}{10000000}y + \frac{26861}{20000}\big)^2 + \big(-\frac{x}{10} - \frac{859609}{10000000}y - \frac{1}{5}\big)^2}{2\big(-\frac{x}{100} - \frac{416854}{10000000}y + \frac{26861}{20000}\big) + 1}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\tfrac{x}{100} - \tfrac{10001}{10000}y + 2, \qquad \dot{y} = x + \tfrac{y}{100} + 1, \tag{32}$$

which has the first integral

$$H_1(x,y) = \big(x + \tfrac{y}{100}\big)^2 + 2(x - 2y) + y^2.$$

For the discontinuous piecewise differential system (31)–(32), system (11) has the unique solution $\big(\frac{24641}{10000}, \frac{293627}{50000}\big)$, and system (20) has the two solutions $\big(\frac{123607}{100000}, \frac{241401}{50000}\big)$ and $\big(\frac{20001}{20000}, \frac{59991}{20000}\big)$. These three limit cycles are drawn in Fig 5(b). This completes the proof of statement (ii). □

Proof of statement (iii) *of Theorem 1.3.* In Theorem 1.1 and 1.2 we proved that the maximum number of limit cycles with **Conf 1** and **Conf 2** is at most two for the two configurations, then we know that the upper bound for the maximum number of limit cycles with **Conf 3** is at most four.

Example with four limit cycles for the class formed by linear and the quadratic isochronous center (7) **with Conf 3.** In the region Σ^+ we define the quadratic isochronous center

$$\begin{aligned}\dot{x} =& \ -\tfrac{1}{10}x^2 + x\big(-\tfrac{76721}{500000}y - \tfrac{168473}{500000}\big) - \tfrac{59809y}{50000} + \tfrac{345053}{100000},\\ \dot{y} =& \ \tfrac{30549x^2}{625000} + x\big(\tfrac{y}{20} + \tfrac{6669}{4000}\big) - \tfrac{76721y^2}{2000000} + \tfrac{12511y}{12500} + \tfrac{107599}{100000},\end{aligned} \tag{33}$$

its first integral is given by

$$H_4(x,y) = \frac{\big(\big(-\frac{x}{10} - \frac{76721}{500000}y + \frac{442621}{1000000}\big)^2 + 4\big(\frac{x}{5} + \frac{559133}{1000000}\big) + 8\big)^2}{\frac{x}{5} + \frac{155913}{100000}}.$$

In the region Σ^- we define the linear differential center

$$\dot{x} = -\frac{x}{5} - \frac{26}{25}y + 3, \quad \dot{y} = x + \frac{y}{5} + 1, \tag{34}$$

with the first integral

$$H_1(x,y) = \left(x + \frac{y}{5}\right)^2 + 2(x - 3y) + y^2.$$

For the discontinuous piecewise differential system (33)–(34), system (11) has the two solutions $\left(\frac{6583}{4000}, \frac{663831}{100000}\right)$, $\left(\frac{732051}{1000000}, \frac{24341}{4000}\right)$, and system (20) has the two solutions $\left(\frac{355203}{1000000}, \frac{541403}{100000}\right)$ and $\left(\frac{128719}{100000}, \frac{112051}{25000}\right)$. The four limit cycles for this example are drawn in Fig 6(a). Then statement (iii) holds. □

Proof of statement (iv) *of Theorem 1.3.* In Theorem 1.1 and 1.2 we proved that the maximum number of limit cycles with **Conf 1** and **Conf 2** is at most two and three, respectively. Then the upper bound for the maximum number of limit cycles with both configurations is five.

Example with five limit cycles for the class formed by linear and the quadratic isochronous center (9) **with Conf 3.** In the region Σ^+ we have the linear differential center

$$\dot{x} = -\frac{x}{10} - \frac{101}{100}y + 3, \qquad \dot{y} = 1 + x + \frac{1}{10}y, \tag{35}$$

with the first integral

$$H_1(x,y) = 2\,(x - 3y) + \left(x + \frac{1}{10}y\right)^2 + y^2.$$

In the region Σ^- we have the quadratic isochronous center

$$\begin{aligned}\dot{x} =&\ \frac{61741x^2}{20000} + x\left(\frac{805467}{10000} - \frac{642523y}{100000}\right) + \frac{334311y^2}{100000} - \frac{804797y}{10000} + \frac{209557}{1000},\\ \dot{y} =&\ \frac{7311x^2}{2500} + x\left(\frac{737729}{10000} - \frac{608643y}{100000}\right) + \frac{316669y^2}{100000} - \frac{183913y}{2500} + \frac{347051}{5000},\end{aligned} \tag{36}$$

with the first integral

$$\begin{aligned}H_5(x,y) =&\ -2\left(\frac{204283}{10000000}x - \frac{10721y}{500000} + 2\right)^2 + 4\left(\frac{175341}{10000000}x - \frac{22963y}{1250000} + \frac{13963}{10000}\right)^2\\ &+\frac{1}{\left(\frac{204283}{10000000}x - \frac{10721y}{500000} + 2\right)^4}.\end{aligned}$$

For the discontinuous piecewise differential system (35)–(36), system (11) has the two solutions $\left(\frac{52111}{20000}, \frac{150409}{20000}\right)$, $\left(\frac{182843}{100000}, \frac{138787}{20000}\right)$, and system (20) has the three solutions $big\left(\frac{732051}{1000000}, \frac{625707}{100000}\right)$, $\left(\frac{55113}{100000}, \frac{269473}{50000}\right)$ and $\left(\frac{202063}{100000}, \frac{391997}{100000}\right)$. The five limit cycles are drawn in Fig 6(b). This completes the proof of Theorem 1.3. □

Conflict of interest

The authors declare no potential conflict of interest.

References

[1] Andronov, A., Vitt, A. and Khaikin, S., (1966), Theory of Oscillations. Pergamon Press, Oxford.
[2] Artés, J. C., Llibre, J., Medrado, J. C. and Teixeira, M. A., (2013), Piecewise linear differential systems with two real saddles, Math. Comput. Simul., 95, pp. 13–22.
[3] Baymout, L., Benterki, R. and Llibre, J., (2024), Limit Cycles of the Discontinuous Piecewise Differential Systems Separated by a Nonregular Line and Formed by a Linear Center and a Quadratic One, Int. J. Bifurc. Chaos., 34, 2450058.
[4] Belousov, B. P., (1958), Periodically acting reaction and its mechanism, Collection of abstracts on radiation medicine, Moscow, pp. 145-147.
[5] Benterki, R. and Llibre, J., (2020), The solution of the second part of the 16th Hilbert problem for nine families of discontinuous piecewise differential systems, Nonlinear Dyn., 102, pp. 2453–2466.

[6] Braga, D. C. and Mello, L. F., (2013), Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane, Nonlinear Dyn., 73, pp. 1283–1288.
[7] Castillo, J., Llibre, J. and Verduzco, F., (2017), The pseudo-Hopf bifurcation for planar discontinuous piecewise linear differential systems, Nonlinear Dyn., 90, pp. 1829–1840.
[8] Damene, L. and Benterki, R., (2021), Limit cycles of discontinuous piecewise linear differential systems formed by centers or Hamiltonian without equilibria separated by irreducible cubics, Moroccan Journal of Pure and Applied Analysis., 7(2), pp. 248-276.
[9] di Bernardo, M., Budd, C. J., Champneys, A. R. and Kowalczyk, P., (2008), Piecewise-Smooth Dynamical Systems, Theory and Applications, Appl. Math. Sci. Series 163, Springer-Verlag, London.
[10] Esteban, M., Llibre., J. and Valls, C., (2021), The 16th Hilbert problem for discontinuous piecewise isochronous centers of degree one or two separated by a straight line, Chaos., 31, 043112.
[11] Georgescu, P. and Hsieh, W. H., (2007), Global Dynamics of a Predator-Prey Model with Stage Structure for the Predator, SIAM Journal on Applied Mathematics., 67(5), pp. 1379-1395.
[12] Hilbert, D., (1900), Probleme, Mathematische, Lecture, Second Internat. Congr. Math. (Paris), Nachr. Ges. Wiss. Göttingen Math. Phys. KL, pp. 253–297.
[13] Hilbert, D., (2000), Problems in Mathematics, Bull. (New Series) Amer. Math. Soc., 37, pp. 407-436.
[14] Liénard, A. M., (1982), Etude des oscillations entrenues, Revue G´enérale del Electricité., 23, pp. 901-912.
[15] Liberzon, D. and Morse, A. S., (1999), Basic problem in stability and design of switched systems, IEEE Control Systems Magazine., 19. 59-70.
[16] Llibre, J., Teixeira, M. A. and Torregrosa, J., (2013), Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation, Int. J. Bifurc. Chaos., 23, 1350066.
[17] Llibre, J. and Teixeira, M. A., (2017), Piecewise linear differential systems without equilibria produce limit cycles?, Nonlinear Dyn., 88, pp. 157–164.
[18] Llibre, J. and Teixeira, M. A., (2018), Piecewise linear differential systems with only centers can create limit cycles?, Nonlinear Dyn., 91, pp. 249-255.
[19] Llibre, J, and Teixeira, M. A., (2018), Limit cycles in Filippov systems having a circle as switching manifold, preprint.
[20] Llibre, J. and Zhang, X., (2018), Limit cycles for discontinuous planar piecewise linear differential systems separated by one straight line and having a center, J. Math. Anal. Appl., 467, pp. 537–549.
[21] Llibre, J. and Zhang, X., (2019), Limit cycles for discontinuous planar piecewise linear differential systems separated by an algebraic curve, Int. J. Bifurc. Chaos., 29(2), 1950017.
[22] Loud, W. S., (1964), Behaviour of the period of solutions of certain plane autonomous systems near centers, Contrib. Differ. Equations., 3, pp. 21-36.
[23] Makarenkov, O. and Lamb, J. S. W., (2012), Dynamics and bifurcations of nonsmooth systems, a survey.Phys., D 241, pp. 1826-1844.
[24] Mardesic, P., Rousseau, C. and Toni, B., (1995), Linearization of isochronous centers, J.Differential Equations., 121, pp. 67-108.
[25] Poincaré, M., (1881), Mémoire sur les courbes définies par une équations differentielle, I. J. Math, Pures Appl, Sér., 3(7), pp. 375-422.
[26] Poincaré, M., (1886), Mémoire sur les courbes définies par une équations differentielle, IV. J. Math, Pures Appl, Sér., 4(2), pp. 155-217.
[27] Simpson, D. J. W., (2010), Bifurcations in Piecewise-Smooth Continuous Systems, World Scientific Series on Nonlinear Science A. 69, World Scientific, Singapore.
[28] Van Der Pol, B., (1920), A theory of the amplitude of free and forced triode vibrations. Radio Rev, (Later Wirel. World)., 1, pp. 701–710.
[29] Van Der Pol, B., (1926), On relaxation-oscillations. Lond. Edinb. Dublin Ppil, Mag. J. Sci., 2, pp. 978–992.

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