TWMS J. App. and Eng. Math. V.15, N.10, 2025, pp. 2505-2518

THE HYBRID FORM OF THE HYPERBOLIC HORADAM MATRIX FUNCTIONS

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Abstract. In the present paper, a special type of hybrid number is introduced whose components are the hyperbolic Horadam sine and cosine matrix functions. These hybrid numbers and their symmetrical forms are explored in detail, and their recursive relations as well as hyperbolic properties are investigated. The paper also includes the Cassini, Catalan, and De Moivre identities, along with the Pythagorean Theorem, all for the hybrid form of the hyperbolic Horadam sine and cosine matrix functions. Furthermore, the hybrid forms of the quasi-sine Horadam matrix function and three-dimensional Horadam matrix spiral related to these matrix functions are presented, offering new insights into their mathematical properties.

Keywords: Horadam number sequence, Hybrid numbers, Hyperbolic matrix functions, Matrix exponential, Trigonometric matrix functions.

AMS Subject Classification: 15A16, 15A24,11K31

1. Introduction

Matrix functions play an important role in solutions of systems of differential equations in many areas of science and engineering [26, 29, 30, 31, 42]. For instance, the solution of the differential equation system $\frac{dx}{dt} = (\ln \psi) Qx,\ x(0) = x_0$ is $x(t) = \psi^{Qt} x_0$, where $Q \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$ and $\psi = \frac{1+\sqrt{5}}{2}$. Especially, the matrix exponentials and trigonometric matrix functions play a fundamental role in the solution of the second order differential equations [1, 7, 8, 12, 27]. Sastre et al. [28] presented an algorithm based on the Taylor series to compute matrix cosine functions. The matrix exponentials to be computed by expanding in the Chebyshev, Legendre or Laguerre orthogonal polynomials are suggested by Moore [24], instead of the exact formulas or algorithms based on the Taylor series or Padé approximations. Defez and Jodar [9] gave some new methods for computing the matrix exponential, sine and cosine functions based on the Hermite matrix polynomial series. Defez et al [6] improved the method proposed by Defez and Jodar [9], for compute the hyperbolic sine and cosine matrix functions based on the Hermite matrix polynomial

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§ Manuscript received: November 21, 2024; accepted: February 24, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.10;

expansions. The method using a matrix separation of variables for the construction of exact series solutions of mixed problems for strongly coupled hyperbolic partial differential systems are given by Jodar et al [16]. Hargreaves and Higham [12] studied on an efficient algorithm for the sine and cosine matrix functions.

Integer sequences are a powerful tool that makes important contributions to science and continues to attract the attention of researchers due to this feature [10, 11, 13, 21, 22, 40]. For instance, in [13], an elementary approach was introduced to derive the explicit formula for number sequences defined by second-order linear recurrence relations, along with related identities. Furthermore, various applications of these approaches to well-known integer sequences, particularly in solving algebraic and differential equations, as well as their extensions to higher-dimensional cases, were explored. One of the most famous integer sequences is the Horadam number sequence $W_n\left(a,b;u,v\right)$, defined by the recursive relation

$$W_{n+1}=uW_n+vW_{n-1}\quad\left(n\geq 0\right),$$

where $a=W_0, b=W_1$, and u and v are nonzero real numbers [15]. The characteristic polynomial of the Horadam number sequence W_n is [15]

$$t^2-ut-v=0, \tag{1}$$

and the roots of equation (1) are

$$\psi=\frac{u+\sqrt{u^2+4v}}{2}\quad\text{and}\quad\zeta=\frac{u-\sqrt{u^2+4v}}{2}. \tag{2}$$

Horadam number sequence has the following Binet formula

$$W_n=\frac{A\psi^n-B\zeta^n}{\psi-\zeta},$$

where $A=b-a\zeta$ and $B=b-a\psi$ [15]. The Horadam number sequence W_n is reduced to some famous number sequences, such as the Fibonacci and Lucas number sequences [3]

$$F_n=W_n\left(0,1;1,1\right)=\begin{cases}\dfrac{\psi^n+\psi^{-n}}{\sqrt{5}}, & n\quad\text{is odd}\\[2ex] \dfrac{\psi^n-\psi^{-n}}{\sqrt{5}}, & n\quad\text{is even}\end{cases} \tag{3}$$

and

$$L_n=W_n\left(2,1;1,1\right)=\begin{cases}\psi^n+\psi^{-n}, & n\quad\text{is odd}\\[2ex] \psi^n-\psi^{-n}, & n\quad\text{is even.}\end{cases} \tag{4}$$

For some generalizations of the Horadam numbers we refer [5, 18, 32, 41].

The classical hyperbolic functions are defined as

$$\sinh\left(x\right)=\frac{e^x-e^{-x}}{2}\quad\text{and}\quad\cosh\left(x\right)=\frac{e^x+e^{-x}}{2}$$

for variable $x\in\mathbb{R}$. Bahşi and Solak [3] introduced hyperbolic Horadam sine and cosine functions

$$sW\left(x\right)=\frac{A\psi^{2x}-Bv^{2x}\psi^{-2x}}{\sqrt{u^2+4v}}\quad\text{and}\quad cW\left(x\right)=\frac{A\psi^{2x+1}+Bv^{2x+1}\psi^{-2x-1}}{\sqrt{u^2+4v}}$$

and their symmetrical forms

$$sWs\left(x\right)=\frac{A\psi^{x}-Bv^{x}\psi^{-x}}{\sqrt{u^{2}+4v}}\quad\text{and}\quad cWs\left(x\right)=\frac{A\psi^{x}+Bv^{x}\psi^{-x}}{\sqrt{u^{2}+4v}},$$

where ψ is as in equation (2) and u, v are nonzero real numbers such that $u^2+4v>0$. The authors gave the relationship between the Horadam numbers and symmetrical hyperbolic Horadam functions

$$W_n=\begin{cases} cWs\left(x\right) & n \quad \text{is odd} \\ sWs\left(x\right), & n \quad \text{is even}\end{cases}$$

and investigated some recursive and hyperbolic properties of symmetrical hyperbolic Horadam functions. The matrix forms of some of these functions have been derived as follows: The classical hyperbolic matrix functions are defined by Jodar et al. [16] as

$$\sinh\left(Q\right)=\frac{e^{Q}-e^{-Q}}{2}\quad\text{and}\quad\cosh\left(Q\right)=\frac{e^{Q}+e^{-Q}}{2}$$

for $Q\in\mathbb{C}^{n\times n}$. Using the Binet formulas in equation (3) and (4), Bahşi and Solak [4] defined the symmetrical hyperbolic Fibonacci sine and cosine matrix functions

$$sFs\left(Q\right)=\frac{\psi^{Q}-\psi^{-Q}}{\sqrt{5}},\qquad cFs\left(Q\right)=\frac{\psi^{Q}+\psi^{-Q}}{\sqrt{5}},$$

and the symmetrical hyperbolic Lucas sine and cosine matrix function

$$sLs\left(Q\right)=\psi^{Q}-\psi^{-Q},\qquad cLs\left(Q\right)=\psi^{Q}+\psi^{-Q},$$

where $\psi=\dfrac{1+\sqrt{5}}{2}$ and $Q\in\mathbb{C}^{n\times n}$. Bahşi and Mersin [2] introduced hyperbolic Horadam sine and cosine matrix functions

$$sW\left(Q\right)=\frac{A\psi^{2Q}-Bv^{2Q}\psi^{-2Q}}{\sqrt{u^{2}+4v}},\quad cW\left(Q\right)=\frac{A\psi^{2Q+I}+Bv^{2Q+I}\psi^{-2Q-I}}{\sqrt{u^{2}+4v}},$$

and their symmetrical forms

$$sWs\left(Q\right)=\frac{A\psi^{Q}-Bv^{Q}\psi^{-Q}}{\sqrt{u^{2}+4v}},\quad cWs\left(Q\right)=\frac{A\psi^{Q}+Bv^{Q}\psi^{-Q}}{\sqrt{u^{2}+4v}},\tag{5}$$

where ψ is as in equation (2), u and v are non-zero real numbers such that $u^2+4v>0$, $Q\in\mathbb{C}^{n\times n}$ and I is the identity matrix of order n.

There are some identities for $u,v\in\mathbb{R}^{+}$, $Q\in\mathbb{C}^{n\times n}$ and the identity matrix I of order n [4]:

(1) $v^{0}=I$, for $0\in\mathbb{C}^{n\times n}$,
(2) $v^{I}=vI$, $v^{mI}=\left(v^{m}\right)I$ and $v^{mI}v^{nI}=v^{(m+n)I}$, for $m,n\in\mathbb{Z}$,
(3) $\left(v^{Q}\right)^{m}=v^{mQ}$, for $m\in\mathbb{Z}$,
(4) $v^{\left(Q^{T}\right)}=\left(v^{Q}\right)^{T}$, where Q^{T} is the transpose matrix of Q,
(5) $\left(v^{Q}\right)^{-1}=v^{-Q}$, where $\left(v^{Q}\right)^{-1}$ is the inverse of the matrix v^{Q},
(6) $\dfrac{dv^{Qt}}{dt}=\ln\left(v\right)Qv^{Qt}$, where $\dfrac{dv^{Qt}}{dt}$ is the derivative of the matrix v^{Qt},
(7) $v^{Q_1+Q_2}=v^{Q_1}v^{Q_2}$, where Q_1 and Q_2 are commutable matrices of order n,
(8) $u^{Q_1}v^{Q_2}=v^{Q_2}u^{Q_1}$, where Q_1 and Q_2 are commutable matrices of order n.

Furthermore, Bahşi and Mersin [2] obtained the equalities

(1) $\psi^{2I}=u\psi^{I}+vI$,

(2) $v^{I}\psi^{-2I} = I - u\psi^{-I}$,
(3) $\psi^{I} - v\psi^{-I} = uI$,
(4) $2v^{I} + \psi^{2I} + v^{2I}\psi^{-2I} = \left(u^2 + 4v\right) I$,

where ψ is as in equation (2), u and v are non-zero real numbers such that $u^2 + 4v > 0$ and I is the identity matrix of order n. The series expansions

$$e^{Q} = I + Q + \frac{Q^2}{2!} + \frac{Q^3}{3!} + \dots$$

and

$$\psi^{Q} = I + \ln(\psi) Q + \frac{(\ln(\psi))^2}{2!} Q^2 + \frac{(\ln(\psi))^3}{3!} Q^3 + \dots$$

are valid for $Q \in \mathbb{C}^{n\times n}$ and the identity matrix I of order n [4].

Recently, many researchers have studied a new type of number, hybrid numbers [17, 19, 33, 34, 35, 37, 39]. Özdemir [25] introduced the hybrid number system, which extends the concepts of complex, hyperbolic, and dual numbers. It sets by

$$\mathbb{K} = \{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = hi = \epsilon + i\}.$$

Szynal-Liana [33] defined the n-th Horadam hybrid number H_n, by the formula

$$H_n = W_n + iW_{n+1} + \epsilon W_{n+2} + hW_{n+3}$$

and obtained the Binet formula for H_n as

$$H_n = \frac{A\underline{\psi}\psi^n - B\underline{\zeta}\zeta^n}{\psi - \zeta},$$

where $\underline{\psi} = 1 + i\psi + \epsilon\psi^2 + h\psi^3$ and $\underline{\zeta} = 1 + i\zeta + \epsilon\zeta^2 + h\zeta^3$. Şentürk et al. [38] studied on Horadam hybrid numbers and obtained the exponential generating function, Poisson generating function, generating matrix, Vajda's, Catalan's, Cassini's, and d'Ocagne's identities for these numbers. Szynal-Liana and Wloch [36] introduced Fibonacci hybrid numbers and derived some of their properties using classical Fibonacci identities. In [20], the Fibonacci divisor hybrid numbers are defined with the help of the Fibonacci divisor numbers, generalizing the Fibonacci hybrid numbers introduced in [36]. Mersin [23] defined a class of hybrid numbers characterized by components derived from hyperbolic Horadam functions and explored several properties. The author introduced hybrid sine and cosine functions based on hyperbolic Horadam numbers and their symmetrical forms and investigated their recurrence relations, derivatives, the Pythagorean theorem, and identities such as the Cassini and De Moivre.

Inspired by the previous papers, we present the hybrid versions of hyperbolic Horadam sine and cosine matrix functions, together with their symmetrical counterparts, and classical analogues. Furthermore, we examine the various properties associated with these functions.

2. Main Results

Definition 2.1. *The hybrid forms of the hyperbolic Horadam sine and cosine matrix functions are defined as*

$$sHW(Q) = \frac{A\underline{\psi}\psi^{2Q} - B\underline{\zeta}v^{2Q}\psi^{-2Q}}{\sqrt{u^2 + 4v}}$$

and

$$cHW(Q) = \frac{A\underline{\psi}\psi^{2Q+I} + B\underline{\zeta}v^{2Q+I}\psi^{-2Q-I}}{\sqrt{u^2 + 4v}},$$

respectively, where $Q \in \mathbb{C}^{n\times n}$, I *is the identity matrix of order* n, $\underline{\psi} = I + i\psi^{I} + \epsilon\psi^{2I} + h\psi^{3I}$, $\underline{\zeta} = I + i\zeta^{I} + \epsilon\zeta^{2I} + h\zeta^{3I}$, ψ *and* ζ *are as in equation* (2).

Note that we use ψ, ζ, $\underline{\psi}$ and $\underline{\zeta}$ as in Definition 2.1 throughout the paper. Also, throughout the paper, we occasionally resort to the abbreviations RHS and LHS, representing the right-hand side and left-hand side of the corresponding equation, respectively.

Now, we define the symmetrical versions of the hybrid forms of the hyperbolic Horadam sine and cosine matrix functions.

Definition 2.2. *The hybrid forms of the symmetrical hyperbolic Horadam sine and cosine matrix functions are defined by*

$$sHWs\,(Q) = \frac{A\underline{\psi}\psi^{Q} - B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}}$$

and

$$cHWs\,(Q) = \frac{A\underline{\psi}\psi^{Q} + B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}},$$

respectively, where $Q \in \mathbb{C}^{n\times n}$ *and* I *is the identity matrix of order* n.

For $0 \in \mathbb{C}^{n\times n}$ and the identity matrix I of order n, we have

$$sHWs\,(0) = \frac{A\underline{\psi} - B\underline{\zeta}}{\sqrt{u^2+4v}}I, \quad cHWs\,(0) = \frac{A\underline{\psi} + B\underline{\zeta}}{\sqrt{u^2+4v}}I,$$

$$sHWs\,(I) = \frac{A\underline{\psi}\psi - B\underline{\zeta}v\psi^{-1}}{\sqrt{u^2+4v}}I \quad \text{and} \quad cHWs\,(I) = \frac{A\underline{\psi}\psi + B\underline{\zeta}v\psi^{-1}}{\sqrt{u^2+4v}}I.$$

The series expansions for $sHWs\,(Q)$ and $cHWs\,(Q)$ are

$$\begin{aligned} sHWs(Q) \quad &= (\sqrt{u^2+4v})^{-1}\Bigg\{A(I + \ln(\psi)Q + \frac{(\ln(\psi))^2Q^2}{2!} + \frac{(\ln(\psi))^3Q^3}{3!} + \ldots)\underline{\psi} \\ &\qquad - B(I + \ln(\frac{v}{\psi})Q + \frac{(\ln(\frac{v}{\psi}))^2Q^2}{2!} + \frac{(\ln(\frac{q}{\psi}))^3Q^3}{3!} + \ldots)\underline{\zeta}\Bigg\} \end{aligned}$$

and

$$\begin{aligned} cHWs(Q) \quad &= (\sqrt{u^2+4v})^{-1}\Bigg\{A(I + \ln(\psi)Q + \frac{(\ln(\psi))^2Q^2}{2!} + \frac{(\ln(\psi))^3Q^3}{3!} + \ldots)\underline{\psi} \\ &\qquad + B(I + \ln(\frac{v}{\psi})Q + \frac{(\ln(\frac{v}{\psi}))^2Q^2}{2!} + \frac{(\ln(\frac{q}{\psi}))^3Q^3}{3!} + \ldots)\underline{\zeta}\Bigg\}, \end{aligned}$$

respectively, where $Q \in \mathbb{C}^{n\times n}$ and I is the identity matrix of order n.

Theorem 2.1. *The relations between the symmetrical hyperbolic Horadam sine and cosine matrix functions and their hybrid forms are as follows*

(i) $sHWs(Q) = sWs(Q) + isWs(Q+I) + \epsilon sWs(Q+2I) + hsWs(Q+3I)$,
(ii) $cHWs(Q) = cWs(Q) + icWs(Q+I) + \epsilon cWs(Q+2I) + hcWs(Q+3I)$.

Proof. Considering the equalities in (5), we have

(i)

$$\begin{aligned} RHS &= \frac{A\psi^{Q} - Bv^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} + i\frac{A\psi^{Q+I} - Bv^{Q+I}\psi^{-Q-I}}{\sqrt{u^2+4v}} \\ &\quad + \epsilon\frac{A\psi^{Q+2I} - Bv^{Q+2I}\psi^{-Q-2I}}{\sqrt{u^2+4v}} + h\frac{A\psi^{Q+3I} - Bv^{Q+3I}\psi^{-Q-3I}}{\sqrt{u^2+4v}} \\ &= \frac{A\underline{\psi}\psi^{Q} - B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} \\ &= sHWs\left(Q\right). \end{aligned}$$

(ii) The proof is similar to that in (i).

□

Theorem 2.2. *The hybrid forms of the symmetrical hyperbolic Horadam sine and cosine matrix functions satisfy the following recursive relations*

(i) $sHWs\left(Q+2I\right) = ucHWs\left(Q+I\right) + vsHWs\left(Q\right)$,

(ii) $cHWs\left(Q+2I\right) = usHWs\left(Q+I\right) + vcHWs\left(Q\right)$.

Proof. (i)

$$\begin{aligned} RHS &= u\left(\frac{A\underline{\psi}\psi^{Q+I} + B\underline{\zeta}v^{Q+I}\psi^{-Q-I}}{\sqrt{u^2+4v}}\right) + v\left(\frac{A\underline{\psi}\psi^{Q} - B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}}\right) \\ &= \frac{A\underline{\psi}\psi^{Q}\left(u\psi^{I} + vI\right) + B\underline{\zeta}v^{Q+I}\psi^{-Q}\left(u\psi^{-I} - I\right)}{\sqrt{u^2+4v}} \\ &= \frac{A\underline{\psi}\psi^{Q+2I} - B\underline{\zeta}v^{Q+2I}\psi^{-Q-2I}}{\sqrt{u^2+4v}} \\ &= sHWs\left(Q+2I\right). \end{aligned}$$

(ii) The proof is similar to that in (i).

□

Theorem 2.3. *The Cassini identities for the hybrid forms of the symmetrical hyperbolic Horadam sine and cosine matrix functions are*

(i) $\left(sHWs\left(Q\right)\right)^2 - cHWs\left(Q+I\right)cHWs\left(Q-I\right) = -AB\underline{\psi\zeta}v^{Q-I}$,

(ii) $\left(cHWs\left(Q\right)\right)^2 - sHWs\left(Q+I\right)sHWs\left(Q-I\right) = AB\underline{\psi\zeta}v^{Q-I}$.

Proof. (i)

$$\begin{aligned} LHS &= \frac{\left(A\underline{\psi}\psi^{Q} - B\underline{\zeta}v^{Q}\psi^{-Q}\right)^2}{u^2+4v} \\ &\quad - \frac{\left(A\underline{\psi}\psi^{Q+I} + B\underline{\zeta}v^{Q+I}\psi^{-Q-I}\right)\left(A\underline{\psi}\psi^{Q-I} + B\underline{\zeta}v^{Q-I}\psi^{-Q+I}\right)}{u^2+4v} \\ &= \frac{-AB\underline{\psi\zeta}v^{Q-I}\left(2v^{I} + \psi^{2I} + v^{2I}\psi^{-2I}\right)}{u^2+4v} \\ &= -AB\underline{\psi\zeta}v^{Q-I}. \end{aligned}$$

(ii) The proof is similar to that in (i).

□

Theorem 2.4. *The Pythagorean theorem holds true for the hybrid forms of the symmetrical hyperbolic Horadam sine and cosine matrix functions, as*

$$(cHWs\,(Q))^2 - (sHWs\,(Q))^2 = \frac{4AB\underline{\psi}\underline{\zeta}v^Q}{u^2+4v}.$$

Proof. The proof is clear from Definition 2.2. □

Theorem 2.5. *The De Moivre identities for the hybrid forms of the symmetrical hyperbolic Horadam sine and cosine matrix functions are as follows*

(i) $(cHWs\,(Q) + sHWs\,(Q))^n = \left(\dfrac{2A\underline{\psi}}{\sqrt{u^2+4v}}\right)^{n-1} (cHWs\,(nQ) + sHWs\,(nQ)),$

(ii) $(cHWs\,(Q) - sHWs\,(Q))^n = \left(\dfrac{2B\underline{\zeta}}{\sqrt{u^2+4v}}\right)^{n-1} (cHWs\,(nQ) - sHWs\,(nQ)).$

Proof. (i)

$$\begin{aligned}
LHS &= \left(\frac{A\underline{\psi}\psi^Q + B\underline{\zeta}v^Q\psi^{-Q}}{\sqrt{u^2+4v}} + \frac{A\underline{\psi}\psi^Q - B\underline{\zeta}v^Q\psi^{-Q}}{\sqrt{u^2+4v}}\right)^n \\
&= \left(\frac{2A\underline{\psi}\psi^Q}{\sqrt{u^2+4v}}\right)^n \\
&= \left(\frac{2A\underline{\psi}}{\sqrt{u^2+4v}}\right)^{n-1} \left(\frac{2A\underline{\psi}\psi^{nQ}}{\sqrt{u^2+4v}}\right) \\
&= \left(\frac{2A\underline{\psi}}{\sqrt{u^2+4v}}\right)^{n-1} \left(\frac{A\underline{\psi}\psi^{nQ} + B\underline{\zeta}v^{nQ}\psi^{-nQ}}{\sqrt{u^2+4v}} + \frac{A\underline{\psi}\psi^{nQ} - B\underline{\zeta}v^{nQ}\psi^{-nQ}}{\sqrt{u^2+4v}}\right) \\
&= \left(\frac{2A\underline{\psi}}{\sqrt{u^2+4v}}\right)^{n-1} (cHWs\,(nQ) + sHWs\,(nQ))\,.
\end{aligned}$$

(ii) The proof is similar to that in (i).

□

Theorem 2.6. *The nth derivatives of the hybrid forms of the symmetrical hyperbolic Horadam sine and cosine matrix functions have the following expressions*

$$\text{(i)}\ \ (sHWs\,(Q))^{(n)} = \begin{cases} (\ln(\psi))^n\, cHWs\,(Q) - \dfrac{\left(\ln\left(\dfrac{v}{\psi}\right)\right)^n + (\ln(\psi))^n}{\sqrt{u^2+4v}} B\underline{\zeta}v^Q\psi^{-Q}, & n \text{ is odd} \\[2ex] (\ln(\psi))^n\, sHWs\,(Q) - \dfrac{\left(\ln\left(\dfrac{v}{\psi}\right)\right)^n - (\ln(\psi))^n}{\sqrt{u^2+4v}} B\underline{\zeta}v^Q\psi^{-Q}, & n \text{ is even,} \end{cases}$$

$$\text{(ii)}\ \ (cHWs(Q))^{(n)} = \begin{cases} (\ln(\psi))^n sHWs(Q) + \dfrac{(\ln(\dfrac{v}{\psi}))^n + (\ln(\psi))^n}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q}, & n \textit{ is odd} \\ (\ln(\psi))^n cHWs(Q) + \dfrac{(\ln(\dfrac{v}{\psi}))^n - (\ln(\psi))^n}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q}, & n \textit{ is even.} \end{cases}$$

Proof. (i) We apply the induction method to n. Since

$$\begin{aligned} (sHWs(Q))' &= \left(\frac{A\underline{\psi}\psi^Q - B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}} \right)' \\ &= \frac{A\underline{\psi}\psi^Q \ln(\psi) - B\underline{\zeta} v^Q \psi^{-Q} \left(\ln(v) - \ln(\psi)\right)}{\sqrt{u^2+4v}} \\ &= \ln(\psi) \left(\frac{A\underline{\psi}\psi^Q + B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}} \right) - \frac{B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}} \ln(v) \\ &= \ln(\psi)\, cHWs(Q) - \frac{\ln\left(\dfrac{v}{\psi}\right) + \ln(\psi)}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} \end{aligned}$$

and

$$\begin{aligned} (sHWs(x))'' &= \left(\ln(\psi) \frac{A\underline{\psi}\psi^Q + B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}} - \frac{\ln\left(\dfrac{v}{\psi}\right) + \ln(\psi)}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} \right)' \\ &= \ln(\psi) \frac{A\underline{\psi}\psi^Q \ln(\psi) + B\underline{\zeta} v^Q \psi^{-Q} \left(\ln(v) - \ln(\psi)\right)}{\sqrt{u^2+4v}} \\ &\quad - \frac{\ln\left(\dfrac{v}{\psi}\right) + \ln(\psi)}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} \left(\ln(v) - \ln(\psi)\right) \\ &= (\ln(\psi))^2 \left(\frac{A\underline{\psi}\psi^Q - B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}} \right) + \frac{B\underline{\zeta} v^Q \psi^{-Q} \left(\ln(v)\ln(\psi)\right)}{\sqrt{u^2+4v}} \\ &\quad - \frac{B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}} \left(\ln\left(\frac{v}{\psi}\right)^2 + \ln(v)\ln(\psi) - (\ln(\psi))^2 \right) \\ &= (\ln(\psi))^2 sHWs(Q) - \frac{\left(\ln\left(\dfrac{v}{\psi}\right)\right)^2 - (\ln(\psi))^2}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q}, \end{aligned}$$

the result is true for $n = 1$ and $n = 2$. Let k is an odd number and the result is true for $n = k$. For the even number $n = k + 1$:

$$\left((sHWs(Q))^{(k)}\right)' = \left((\ln(\psi))^k cHWs(Q) - \frac{\left(\ln\left(\frac{v}{\psi}\right)\right)^k + (\ln(\psi))^k}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} \right)'$$

$$= (\ln(\psi))^k \frac{A\underline{\psi}\psi^Q \ln(\psi) + B\underline{\zeta} v^Q \psi^{-Q} (\ln(v) - \ln(\psi))}{\sqrt{u^2+4v}}$$

$$- \frac{\left(\ln\left(\frac{v}{\psi}\right)\right)^k + (\ln(\psi))^k}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} (\ln(v) - \ln(\psi))$$

$$= (\ln(\psi))^{k+1} \frac{A\underline{\psi}\psi^Q - B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}}$$

$$- \frac{\left(\ln\left(\frac{v}{\psi}\right)\right)^{k+1} - (\ln(\psi))^{k+1}}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q}.$$

Assume that k is an even number and the result holds true for $n = k$. Finally, we show that the result also holds true for the odd number $n = k + 1$:

$$\left((sHWs(Q))^{(k)}\right)' = \left((\ln(\psi))^k sHWs(Q) - \frac{\left(\ln\left(\frac{v}{\psi}\right)\right)^k - (\ln(\psi))^k}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} \right)'$$

$$= (\ln(\psi))^k \frac{A\underline{\psi}\psi^Q \ln(\psi) - B\underline{\zeta} v^Q \psi^{-Q} (\ln(v) - \ln(\psi))}{\sqrt{u^2+4v}}$$

$$- \frac{\left(\ln\left(\frac{v}{\psi}\right)\right)^k - (\ln(\psi))^k}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q} (\ln(v) - \ln(\psi))$$

$$= (\ln(\psi))^{k+1} \frac{A\underline{\psi}\psi^Q + B\underline{\zeta} v^Q \psi^{-Q}}{\sqrt{u^2+4v}}$$

$$- \frac{\left(\ln\left(\frac{v}{\psi}\right)\right)^{k+1} + (\ln(\psi))^{k+1}}{\sqrt{u^2+4v}} B\underline{\zeta} v^Q \psi^{-Q}.$$

(ii) The proof is similar to that in (i).

□

3. The Hybrid Form of the Quasi-Sine Horadam Matrix Functions

Bahşi and Mersin [2] have defined quasi-sine Horadam matrix function as follows

$$WW(Q) = \frac{A\psi^Q - \cos(\pi Q) B v^Q \psi^{-Q}}{\sqrt{u^2+4v}},$$

where $Q \in \mathbb{C}^{n\times n}$.

Next, we provide the definition of the hybrid form of the quasi-sine Horadam matrix function.

Definition 3.1. *The function*

$$HWW\left(Q\right) = \frac{A\underline{\psi}\psi^{Q} - \cos\left(\pi Q\right) B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}}$$

is called the hybrid form of the quasi-sine Horadam matrix function, where $Q \in \mathbb{C}^{n\times n}$.

The hybrid form of the quasi-sine hyperbolic Horadam matrix function satisfies the following relations.

Theorem 3.1.

$$HWW\left(Q+2I\right) = uHWW\left(Q+I\right) + vHWW\left(Q\right).$$

Proof. Considering the equalities

$$\cos(\pi Q + 2\pi I) = \cos(\pi Q) = -\ \cos(\pi Q + \pi I),$$

we have

$$\begin{aligned}
RHS &= u\left(\frac{A\underline{\psi}\psi^{Q+I} - \cos(\pi(Q+I))B\underline{\zeta}v^{Q+I}\psi^{-Q-I}}{\sqrt{u^2+4v}}\right) \\
&\quad + v\left(\frac{A\underline{\psi}\psi^{Q} - \cos(\pi Q)B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}}\right) \\
&= \frac{A\underline{\psi}\psi^{Q}(u\psi^{I} + vI) - \cos(\pi Q + 2\pi I)B\underline{\zeta}v^{Q+I}\psi^{-Q-I}(vIv^{-I}\psi^{I} - uI)}{\sqrt{u^2+4v}} \\
&= \frac{A\underline{\psi}\psi^{Q+2I} - \cos(\pi(Q+2I))B\underline{\zeta}v^{Q+2I}\psi^{-Q-2I}}{\sqrt{u^2+4v}} \\
&= HWW(Q+2I).
\end{aligned}$$

□

Theorem 3.2.

$$\left(HWW\left(Q\right)\right)^2 - HWW\left(Q+I\right)HWW\left(Q-I\right) = -AB\underline{\psi\zeta}v^{Q-I}\cos\left(\pi Q\right).$$

Proof. By using

$$\cos(\pi(Q+I)) = \cos(\pi(Q-I)) = -\cos(\pi Q),$$

we have

$$\begin{aligned}
LHS &= \left(\frac{A\underline{\psi}\psi^{Q} - \cos(\pi Q)B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}}\right)^2 \\
&\quad - \left[\left(\frac{A\underline{\psi}\psi^{Q+I} - \cos(\pi(Q+I))B\underline{\zeta}v^{Q+I}\psi^{-Q-I}}{\sqrt{u^2+4v}}\right)\right. \\
&\qquad \left.\times\left(\frac{A\underline{\psi}\psi^{Q-I} - \cos(\pi(Q-I))B\underline{\zeta}v^{Q-I}\psi^{-Q+I}}{\sqrt{u^2+4v}}\right)\right].
\end{aligned}$$

Then,

$$
\begin{aligned}
LHS &= \frac{A^2\underline{\psi}^2\psi^{2Q} - 2A\underline{\psi}\psi^{Q}B\underline{\zeta}v^{Q}\psi^{-Q}\cos(\pi Q) + \cos^2(\pi Q)B^2\underline{\zeta}^2v^{2Q}\psi^{-2Q}}{u^2+4v} \\
&\quad - \frac{A^2\underline{\psi}^2\psi^{2Q} - A\underline{\psi}\psi^{Q+I}B\underline{\zeta}v^{Q-I}\psi^{-Q+I}\cos(\pi(Q-I)}{u^2+4v} \\
&\quad + \frac{A\underline{\psi}\psi^{Q-I}B\underline{\zeta}v^{Q+I}\psi^{-Q-I}\cos(\pi(Q+I))}{u^2+4v} \\
&\quad - \frac{\cos(\pi(Q+I))\cos(\pi(Q-I))B^2\underline{\zeta}^2v^{Q+I}\psi^{-Q-I}v^{Q-I}\psi^{-Q+I}}{u^2+4v} \\
&= \frac{-2AB\underline{\psi\zeta}v^{Q}\cos(\pi Q) + AB\underline{\psi\zeta}\psi^{2I}v^{Q-I}\cos(\pi(Q-I))}{u^2+4v} \\
&\quad + \frac{AB\underline{\psi\zeta}\psi^{-2I}v^{Q+I}\cos(\pi(Q+I))}{u^2+4v} \\
&= -AB\underline{\psi\zeta}v^{Q-I}\cos(\pi Q)\frac{(2v^{I}+\psi^{2I}+v^{2I}\psi^{-2I})}{u^2+4v} \\
&= -AB\underline{\psi\zeta}v^{Q-I}\cos(\pi Q).
\end{aligned}
$$

□

4. The Hybrid Form of the Three-Dimensional Horadam Matrix Spiral

Bahşi and Mersin [2] defined the matrix form of the three-dimensional Horadam spiral by the rules

$$
\begin{aligned}
CWW(Q) &= \frac{A\psi^{Q} - \cos(\pi Q)Bv^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} + i\frac{\sin(\pi Q)Bv^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} \\
&= \frac{A\psi^{Q} + ie^{i\pi\left(\frac{1}{2}I-Q\right)}Bv^{Q}\psi^{-Q}}{\sqrt{u^2+4v}},
\end{aligned}
$$

where $Q \in \mathbb{C}^{n\times n}$ and $i^2=-1$.

Definition 4.1. *The hybrid form of the three-dimensional Horadam matrix spiral is defined by*

$$
\begin{aligned}
CHWW(Q) &= \frac{A\underline{\psi}\psi^{Q} - \cos(\pi Q)B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} + i\frac{\sin(\pi Q)B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} \\
&= \frac{A\underline{\psi}\psi^{Q} + ie^{i\pi\left(\frac{1}{2}I-Q\right)}B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}},
\end{aligned}
$$

where $Q \in \mathbb{C}^{n\times n}$ *and* $i^2=-1$.

Theorem 4.1. *The hybrid form of the three-dimensional Horadam matrix spiral satisfies the following recursive relation*

$$CHWW\left(Q+2I\right) = uCHWW\left(Q+I\right) + vCHWW\left(Q\right).$$

Proof.

$$\begin{aligned}
& uCHWW\left(Q+I\right) + vCHWW\left(Q\right) \\
= \quad & u\frac{A\underline{\psi}\psi^{Q+I} + ie^{i\pi\left(\frac{1}{2}I-Q-I\right)}B\underline{\zeta}v^{Q+I}\psi^{-Q-I}}{\sqrt{u^2+4v}} + v\frac{A\underline{\psi}\psi^{Q} + ie^{i\pi\left(\frac{1}{2}I-Q\right)}B\underline{\zeta}v^{Q}\psi^{-Q}}{\sqrt{u^2+4v}} \\
& = \frac{A\underline{\psi}\psi^{Q}\left(u\psi^{I}+vI\right) + ie^{-i\pi Q}B\underline{\zeta}v^{Q+I}\psi^{-Q-I}\left(ue^{-i\pi\frac{1}{2}I} + e^{i\pi\frac{1}{2}I}\psi\right)}{\sqrt{u^2+4v}} \\
& = \frac{A\underline{\psi}\psi^{Q}\left(u\psi^{I}+vI\right) + ie^{-i\pi Q}B\underline{\zeta}v^{Q+I}\psi^{-Q-I}\left(\psi^{I}-uI\right)i}{\sqrt{u^2+4v}} \\
& = \frac{A\underline{\psi}\psi^{Q+2I} + ie^{-i\pi Q - i\frac{3\pi}{2}I}B\underline{\zeta}v^{Q+2I}\psi^{-Q-2I}}{\sqrt{u^2+4v}} \\
& = \frac{A\underline{\psi}\psi^{Q+2I} + ie^{i\pi\left(\frac{1}{2}I-Q-2I\right)}B\underline{\zeta}v^{Q+2I}\psi^{-Q-2I}}{\sqrt{u^2+4v}} \\
& = CHWW\left(Q+2I\right).
\end{aligned}$$

□

5. Conclusion

This paper presented the hybrid forms of hyperbolic Horadam matrix functions and their corresponding symmetrical structures. Various properties, such as recurrence relations, derivatives, and identities like those of Cassini and De Moivre, are examined in detail. Additionally, the hybrid forms of quasi-sine Horadam matrix functions and the three-dimensional Horadam matrix spiral are introduced. Given the growing interest among researchers in the hybrid number system, adapting this system to hyperbolic Horadam matrix functions is expected to provide alternative approaches to studying their properties and applications, while also contributing to the analytical and numerical solutions of differential equations. This adaptation may offer more flexible and efficient solution methods, particularly for complex and nonlinear equations, enhancing accuracy and accelerating the solution process.

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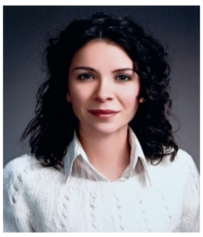

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