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ON THE FUNDAMENTAL THEOREMS OF (α, β)-PYTHAGOREAN FUZZY IDEALS OF RINGS

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Abstract. An (α, β)-Pythagorean fuzzy set is a modern approach to handling ambiguity. This article represents the perception of an (α, β)-Pythagorean fuzzy coset of any (α, β)-Pythagorean fuzzy ideal of rings. We demonstrate several characteristics of (α, β)-Pythagorean fuzzy cosets. Moreover, we explain the (α, β)-Pythagorean fuzzy quotient ring of (α, β)-Pythagorean fuzzy ideals of any ring. Furthermore, we present the isomorphism theorems of (α, β)-Pythagorean fuzzy ideals.

Keywords: (α, β)-Pythagorean fuzzy set, (α, β)-Pythagorean fuzzy coset, (α, β)-Pythagorean fuzzy ideal, (α, β)-Pythagorean fuzzy quotient ring.

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1. Introduction

Uncertainty is an unavoidable aspect of our lives. Modern mathematical frameworks often grapple with imprecision and uncertainty. Making a straightforward decision is not always possible. When it comes to dealing with inaccuracy in decision-making circumstances, we have a huge difficulty. Zadeh [23] proposed the idea of fuzzy sets for handling vagueness. Fuzzy sets deal with such problems by assigning a membership to these objects. Rosenfeld [20] was the pioneer of the conception of fuzzy subgroups and ideals.

In 1982, various features of fuzzy ideals were explored by Liu [14, 15]. Ren [21] studied quotient fuzzy rings and fuzzy ideals. Dixit et al. [9] looked at many characteristics of fuzzy rings. Near-ring fuzzy ideals were introduced by Kim [13]. In 2012, Ceven [8] created N-ideals of rings. Only membership values are not always sufficient for making decisions. To put it another way, there may be hesitancy or uncertainty concerning the degree of membership of objects. In 1986, by associating non-membership degrees, Atanassov [2] constructed intuitionistic fuzzy sets. A fuzzy ring with intuitionistic properties was proposed by Hur et al. [11]. Intuitionistic fuzzy subrings and intuitionistic fuzzy ideals were

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given additional attention by Banerjee and Basnet [3]. In 2017, Hayat et al. [12] introduced bipolar anti-fuzzy h-ideals in hemi-rings. In 2020, Mehmood et al. [16, 17] proposed the structure of M-hazy rings and studied their fuzzy convexities. Gulzar et al. [1, 10] developed fuzzy subrings in Q-complex environment in 2021. In 2013, Yager [22] defined the terminology of the Pythagorean fuzzy set. Pythagorean fuzzy set is a powerful technique for accurately and precisely modelling ambiguity and uncertainty rather than intuitionistic fuzzy sets. Numerous results relevant to Pythagorean fuzzy sets were presented by [19].

In 2021, Ghorai and Bhunia [4] studied (α,β)-Pythagorean fuzzy Sets, where $\alpha,\beta\in[0,1]$ with $\alpha^2+\beta^2\in[0,1]$. (α,β)-Pythagorean fuzzy sets are more precise than intuitionistic fuzzy sets and Pythagorean fuzzy sets. The significance of (α,β)-Pythagorean fuzzy subgroups was discussed by them. Bhunia et al. [5] proved Lagrange's theorem in the (α,β)-Pythagorean fuzzy subgroups in 2021. The concept of ideals is very significant within the framework of ring theory. Isomorphism preserves ring structures. Isomorphic rings have identical ring-theoretic features. Isomorphism theorems play a vital role in the classical ring theory. In 2022, Bhunia et al. [6] introduced the notion of (α,β)-Pythagorean fuzzy subrings and (α,β)- Pythagorean fuzzy ideals of rings. They have described the ring homomorphisms and direct product of ideals of rings in (α,β)-Pythagorean fuzzy sets. Although quotient rings and fundamental theorems of ring isomorphisms are important, there is a lack of attention on these topics in the (α,β)-Pythagorean fuzzy environment in the existing literature. As a result, a natural question arises: what are the (α,β)-Pythagorean fuzzy versions of all the fundamental theorems of ring isomorphisms? These research gaps paved the way for further studies on (α,β)-Pythagorean fuzzy ideals of rings. The manuscript's objectives are as follows:

- First to introduce the perception of an (α,β)-Pythagorean fuzzy coset of any (α,β)-Pythagorean fuzzy ideal.
- Secondly to develop the concept of (α,β)-Pythagorean fuzzy quotient ring of (α,β)-Pythagorean fuzzy ideals.
- Finally describe the isomorphism theorems of (α,β)-Pythagorean fuzzy ideals.

This paper's outline is as follows: Section 2 discusses a few important concepts and definitions. Section 3 introduces the significance of (α,β)-Pythagorean fuzzy coset of an (α,β)-Pythagorean fuzzy ideal. Section 3 also deals with (α,β)-Pythagorean fuzzy quotient ring of (α,β)-Pythagorean fuzzy ideal. Section 4 describes first, second and third isomorphism theorem of (α,β)-Pythagorean fuzzy ideals. We present a conclusion in Section 5.

2. Preliminaries

Several significant terminologies and concepts are introduced in this section.

Definition 2.1. [2] *An intuitionistic fuzzy set of a crisp set $\mathfrak{J}$ has the structure* $\mathfrak{F}=\{(\wp,\mu(\wp),\nu(\wp)|\wp\in\mathfrak{J}\}$, *where* $\mu(\wp),\nu(\wp)\in[0,1]$ *with* $0\le\mu(\wp)+\nu(\wp)\le 1$.

Definition 2.2. [22] *A Pythagorean fuzzy set of a crisp set $\mathfrak{J}$ has the structure* $\psi=\{(\wp,\mu(\wp),\nu(\wp)|\wp\in\mathfrak{J}\}$, *where* $0\le\mu^2(\wp)+\nu^2(\wp)\le 1$.

Definition 2.3. [4] *An (α,β)-Pythagorean fuzzy set of a crisp set $\mathfrak{J}$ is* $\psi^*=\{(\wp,\mu^\alpha(\wp),\nu^\beta(\wp)|\wp\in\mathfrak{J}\}$ *with* $0\le(\mu^\alpha(\wp))^2+\left(\nu^\beta(\wp)\right)^2\le 1$.
Here, $\mu^\alpha(\wp)=\mu(\wp)\wedge\alpha$, $\nu^\beta(\wp)=\nu(\wp)\vee\beta$ *and* $0\le\alpha^2+\beta^2\le 1$.

Proposition 2.1. [4] *Let ψ_1^*, ψ_2^* be two (α,β)-Pythagorean fuzzy sets in $\mathfrak{J}$. Then*

(1) $\psi_1^*\cup\psi_2^*=\left\{\left(\wp,\mu_1^\alpha(\wp)\vee\mu_2^\alpha(\wp),\nu_1^\beta(\wp)\wedge\nu_2^\beta(\wp)\right)|\wp\in\mathfrak{J}\right\}$

(2) $\psi_1^* \cap \psi_2^* = \left\{ \left(\wp, \mu_1^\alpha(\wp) \wedge \mu_2^\alpha(\wp), \nu_1^\beta(\wp) \vee \nu_2^\beta(\wp) \right) | \wp \in \mathfrak{J} \right\}$
(3) $\psi_1^* \subseteq \psi_2^*$ *if* $\mu_1^\alpha(\wp) \leq \mu_2^\alpha(\wp)$ *and* $\nu_1^\beta(\wp) \geq \nu_2^\beta(\wp)$, $\forall \wp \in \mathfrak{J}$
(4) $\psi_1^* = \psi_2^*$ *if* $\mu_1^\alpha(\wp) = \mu_2^\alpha(\wp)$ *and* $\nu_1^\beta(\wp) = \nu_2^\beta(\wp)$, $\forall \wp \in \mathfrak{J}$.

Definition 2.4. [4] *Assume ψ^* forms an (α, β)-Pythagorean fuzzy set on $\mathfrak{J}$. Then for $\eta, \zeta \in [0,1]$, $\psi^*_{(\eta,\zeta)} = \{\wp \in \mathfrak{J} | \eta \leq \mu^\alpha(\wp)$ and $\nu^\beta(\wp) \leq \zeta\}$ is referred as a (α, β)-Pythagorean fuzzy level subset of ψ^*.*

Definition 2.5. [18] *A commutative ring $(\mathfrak{J}, +, \cdot)$, is a set $\mathfrak{J}$ together with two binary operation $+$ and $\cdot$ defined on $\mathfrak{J}$ such that the following axioms are satisfied:*
(1) $(\mathfrak{J}, +)$ *is an abelian group*
(2) $\wp_1 \cdot (\wp_2 \cdot \wp_3) = (\wp_1 \cdot \wp_2) \cdot \wp_3$ *for all* $\wp_1, \wp_2, \wp_3 \in \mathfrak{J}$
(3) $\wp_1 \cdot (\wp_2 + \wp_3) = (\wp_1 \cdot \wp_2) + (\wp_1 \cdot \wp_3)$ *and* $(\wp_1 + \wp_2) \cdot \wp_3 = (\wp_1 \cdot \wp_3) + (\wp_2 \cdot \wp_3)$ *hold for all* $\wp_1, \wp_2, \wp_3 \in \mathfrak{J}$
(4) $(\wp_1 \cdot \wp_2) = (\wp_2 \cdot \wp_1)$ *for all* $\wp_1, \wp_2 \in \mathfrak{J}$.

A commutative ring together with the multiplicative identity element is called a commutative ring with unity (CRU).

Definition 2.6. [6] *Let $(\mathfrak{J}, +, \cdot)$ be a ring and ψ^* be an (α, β)-Pythagorean fuzzy set on $\mathfrak{J}$. Now, ψ^* is an (α, β)-Pythagorean fuzzy ideal of $\mathfrak{J}$ if*
(1) $\mu^\alpha(\wp_1 - \wp_2) \geq \mu^\alpha(\wp_1) \wedge \mu^\alpha(\wp_2)$ *and* $\nu^\beta(\wp_1 - \wp_2) \leq \nu^\beta(\wp_1) \vee \nu^\beta(\wp_2)$, $\forall \wp_1, \wp_2 \in \mathfrak{J}$
(2) $\mu^\alpha(\wp_1 \cdot \wp_2) \geq \mu^\alpha(\wp_1) \vee \mu^\alpha(\wp_2)$ *and* $\nu^\beta(\wp_1 \cdot \wp_2) \leq \nu^\beta(\wp_1) \wedge \nu^\beta(\wp_2)$, $\forall \wp_1, \wp_2 \in \mathfrak{J}$.

Theorem 2.1. [6] *If a ring $(\mathfrak{J}, +, \cdot)$ has an (α, β)-Pythagorean fuzzy ideal ψ^*, $\mathfrak{P} = \{\wp \in \mathfrak{J} | \psi^*(\wp) = \psi^*(0)\}$ is an ideal of the ring $(\mathfrak{J}, +, \cdot)$.*

3. (α, β)-Pythagorean fuzzy cosets and (α, β)-Pythagorean fuzzy quotient ring

This section enlightens the concepts of (α, β)-Pythagorean fuzzy cosets as well as the (α, β)-Pythagorean fuzzy quotient rings of an (α, β)-Pythagorean fuzzy ideal.

Firstly, we define (α, β)-Pythagorean fuzzy cosets of an (α, β)-Pythagorean fuzzy ideal.

Definition 3.1. *Assume ψ^* is an (α, β)-Pythagorean fuzzy ideal of $(\mathfrak{J}, +, \cdot)$. Then for $\wp_1 \in \mathfrak{J}$, (α, β)-Pythagorean fuzzy coset of the (α, β)-Pythagorean fuzzy ideal ψ^* is represented by $\wp_1 + \psi^*$ and is characterised with $(\wp_1 + \psi^*) = (\wp_1 + \mu^\alpha, \wp_1 + \nu^\beta)$ where, $(\wp_1 + \mu^\alpha)(\wp_2) = \mu^\alpha(\wp_1 - \wp_2)$ and $(\wp_1 + \nu^\beta)(\wp_2) = \nu^\beta(\wp_1 - \wp_2)$ for all $\wp_1, \wp_2 \in \mathfrak{J}$.*

Example 3.1. *Consider the ring $(\mathbb{Z}_3, +_3, \cdot_3)$.*

Also, consider an (α, β)-Pythagorean fuzzy ideal $\psi^* = (\mu^\alpha, \nu^\beta)$ of $(\mathbb{Z}_3, +_3, \cdot_3)$ by

$$\mu^\alpha(z) = \begin{cases} 0.8, & \text{when } z = \bar{0} \\ 0.7, & \text{elsewhere} \end{cases}$$

and

$$\nu^\beta(z) = \begin{cases} 0.4, & \text{when } z = \bar{0} \\ 0.6, & \text{elsewhere.} \end{cases}$$

Take an element $z = \bar{2}$ of $\mathbb{Z}_3$.

Then the (α, β)-Pythagorean fuzzy coset $(\bar{2} + \psi^*)$ of the (α, β)-Pythagorean fuzzy ideal ψ^* is given by

$(\bar{2} + \psi^*) = (\bar{2} + \mu^\alpha, \bar{2} + \nu^\beta)$
$= \{(z, \bar{2} + \mu^\alpha, \bar{2} + \nu^\beta) | z \in \mathbb{Z}_3\}$
$= \{(\bar{0}, (\bar{2} + \mu^\alpha)(\bar{0}), (\bar{2} + \nu^\beta)(\bar{0})), (\bar{1}, (\bar{2} + \mu^\alpha)(\bar{1}), (\bar{2} + \nu^\beta)(\bar{1})), (\bar{2}, (\bar{2} + \mu^\alpha)(\bar{2}), (\bar{2} + \nu^\beta)(\bar{2}))\}$

$= \{(\bar{0}, \mu^{\alpha}(\bar{2}), \nu^{\beta}(\bar{2})), (\bar{1}, \mu^{\alpha}(\bar{1}), \nu^{\beta}(\bar{1})), (\bar{2}, \mu^{\alpha}(\bar{0}), \nu^{\beta}(\bar{0}))\}$
$= \{(\bar{0}, 0.7, 0.6), (\bar{1}, 0.7, 0.6), (\bar{2}, 0.8, 0.4)\}$.

Now, we will describe some important properties of (α,β)-Pythagorean fuzzy cosets of an (α,β)-Pythagorean fuzzy ideal.

Proposition 3.1. *Suppose ψ^* is an (α,β)-Pythagorean fuzzy ideal of $(\mathfrak{J},+,\cdot)$. Then $0+\psi^* = \psi^*$.*

Proof. According to the meaning of (α,β)-Pythagorean fuzzy coset, $(0+\psi^*) = (0+\mu^{\alpha}, 0+\nu^{\beta})$.
Now, $(0+\mu^{\alpha})(\wp) = \mu^{\alpha}(0-\wp) = \mu^{\alpha}(-\wp) = \mu^{\alpha}(\wp)$.
Again, $(0+\nu^{\beta})(\wp) = \nu^{\beta}(0-\wp) = \nu^{\beta}(-\wp) = \nu^{\beta}(\wp)$.
Therefore $(0+\mu^{\alpha}, 0+\nu^{\beta}) = (\mu^{\alpha}, \nu^{\beta})$.
Hence $0+\psi^* = \psi^*$. □

Theorem 3.1. *Suppose ψ^* is an (α,β)-Pythagorean fuzzy ideal of $(\mathfrak{J},+,\cdot)$. Then $\wp+\psi^* = \psi^*$ iff $\psi^*(\wp) = \psi^*(0)$, where $\wp \in \mathfrak{J}$.*

Proof. We assume that $\wp+\psi^* = \psi^*$ for $\wp \in \mathfrak{J}$.
Then $(\wp+\psi^*)(m) = \psi^*(m)$ for an arbitrary $m \in \mathfrak{J}$.
So, $(\wp+\mu^{\alpha})(m) = \mu^{\alpha}(m)$ and $(\wp+\nu^{\beta})(m) = \nu^{\beta}(m)$.
Therefore $\mu^{\alpha}(\wp-m) = \mu^{\alpha}(m)$ and $\nu^{\beta}(\wp-m) = \nu^{\beta}(m)$.
We put $m=0$ in the above relation.
Then we get $\mu^{\alpha}(\wp) = \mu^{\alpha}(0)$ and $\nu^{\beta}(\wp) = \nu^{\beta}(0)$.
Hence $\psi^*(\wp) = \psi^*(0)$.
Conversely, let $\psi^*(\wp) = \psi^*(0)$. Then $\mu^{\alpha}(\wp) = \mu^{\alpha}(0)$ and $\nu^{\beta}(\wp) = \nu^{\beta}(0)$.
For an arbitrary $t \in \mathfrak{J}$, $(\wp+\mu^{\alpha})(t) = \mu^{\alpha}(\wp-t) \geq \mu^{\alpha}(\wp) \wedge \mu^{\alpha}(t) = \mu^{\alpha}(0) \wedge \mu^{\alpha}(t) = \mu^{\alpha}(t)$ and $(\wp+\nu^{\beta})(t) = \nu^{\beta}(\wp-t) \leq \nu^{\beta}(\wp) \vee \nu^{\beta}(t) = \nu^{\beta}(0) \vee \nu^{\beta}(t) = \nu^{\beta}(t)$.
Also, $\mu^{\alpha}(t) = \mu^{\alpha}(t-\wp+\wp) \geq \mu^{\alpha}(t-\wp) \wedge \mu^{\alpha}(\wp) = \mu^{\alpha}(t-\wp) = (\wp+\mu^{\alpha})(t)$.
Similarly, $\nu^{\beta}(t) = \nu^{\beta}(t-\wp+\wp) \leq \nu^{\beta}(t-\wp) \vee \nu^{\beta}(\wp) = \nu^{\beta}(t-\wp) = (\wp+\nu^{\beta})(t)$.
Therefore $(\wp+\mu^{\alpha})(t) = \mu^{\alpha}(t)$ and $(\wp+\nu^{\beta})(t) = \nu^{\beta}(t)$ for $t \in \mathfrak{J}$. Hence $\wp+\psi^* = \psi^*$. □

Proposition 3.2. *Assume ψ^* is an (α,β)-Pythagorean fuzzy ideal of $(\mathfrak{J},+,\cdot)$. If $\psi^*(\wp_1-\wp_2) = \psi^*(0)$, then $\psi^*(\wp_1) = \psi^*(\wp_2)$ for all $\wp_1, \wp_2 \in \mathfrak{J}$.*

Proof. Here, ψ^* is an (α,β)-Pythagorean fuzzy ideal of $(\mathfrak{J},+,\cdot)$.
For any $\wp_1, \wp_2 \in \mathfrak{J}$, we have

$$\begin{aligned} \mu^{\alpha}(\wp_2) &= \mu^{\alpha}(\wp_2-\wp_1+\wp_1) \\ &\geq \mu^{\alpha}(\wp_2-\wp_1) \wedge \mu^{\alpha}(\wp_1) \\ &= \mu^{\alpha}(0) \wedge \mu^{\alpha}(\wp_1) \\ &= \mu^{\alpha}(\wp_1). \end{aligned}$$

By interchanging $\wp_1$ with $\wp_2$ in the above relation, we have $\mu^{\alpha}(\wp_1) \geq \mu^{\alpha}(\wp_2)$.
Therefore $\mu^{\alpha}(\wp_1) = \mu^{\alpha}(\wp_2)$.
Again,

$$\begin{aligned} \nu^{\beta}(\wp_2) &= \nu^{\beta}(\wp_2-\wp_1+\wp_1) \\ &\leq \nu^{\beta}(\wp_2-\wp_1) \vee \nu^{\beta}(\wp_1) \\ &= \nu^{\beta}(0) \vee \nu^{\beta}(\wp_1) \\ &= \nu^{\beta}(\wp_1). \end{aligned}$$

Similarly, $\nu^{\beta}(\wp_1) \leq \nu^{\beta}(\wp_2)$. Therefore $\nu^{\beta}(\wp_1) = \nu^{\beta}(\wp_2)$.

Hence $\psi^*(\wp_1) = \psi^*(\wp_2)$ for all $\wp_1, \wp_2 \in \mathfrak{J}$. □

In the next theorem, we will prove that the set of all (α,β)-Pythagorean fuzzy cosets of an (α,β)-Pythagorean fuzzy ideal forms a ring under certain binary operations.

Theorem 3.2. *Suppose ψ^* is an (α,β)-Pythagorean fuzzy ideal of $(\mathfrak{J},+,\cdot)$ and $\frac{\wp}{\psi^*}$ is the set of all (α,β)-Pythagorean fuzzy cosets of ψ^*. Then $\frac{\mathfrak{J}}{\psi^*}$ is a ring under the operations $(\wp_1+\psi^*)+(\wp_2+\psi^*) = (\wp_1+\wp_2+\psi^*)$ and $(\wp_1+\psi^*)\cdot(\wp_2+\psi^*) = (\wp_1\cdot\wp_2+\psi^*)$ for all $\wp_1, \wp_2 \in \mathfrak{J}$.*

Proof. Suppose $\wp_1+\psi^* = \wp_3+\psi^*$ and $\wp_2+\psi^* = \wp_4+\psi^*$ for $\wp_1,\wp_2,\wp_3,\wp_4 \in \mathfrak{J}$.
Then $\psi^*(\wp_1-\wp_3) = \psi^*(0)$ and $\psi^*(\wp_2-\wp_4) = \psi^*(0)$.
Now for $\wp \in \mathfrak{J}$,

$$\begin{aligned}
((\wp_1+\nu^\beta)+(\wp_2+\nu^\beta))(\wp) &= ((\wp_1+\wp_2)+\nu^\beta)(\wp)\\
&= \nu^\beta(\wp-\wp_1-\wp_2)\\
&= \nu^\beta(\wp-\wp_1-\wp_2+\wp_3-\wp_3+\wp_4-\wp_4)\\
&= \nu^\beta((\wp-\wp_3-\wp_4)-(\wp_1-\wp_3)-(\wp_2-\wp_4))\\
&\leq \nu^\beta((\wp-\wp_3-\wp_4)\vee\nu^\beta(\wp_1-\wp_3)\vee\nu^\beta(\wp_2-\wp_4)\\
&= \nu^\beta((\wp-\wp_3-\wp_4)\vee\nu^\beta(0)\vee\nu^\beta(0)\\
&= \nu^\beta((\wp-\wp_3-\wp_4)\\
&= ((\wp_3+\wp_4)+\nu^\beta)(\wp)\\
&= ((\wp_3+\nu^\beta)+(\wp_4+\nu^\beta))(\wp).
\end{aligned}$$

Similarly, $((\wp_3+\nu^\beta)+(\wp_4+\nu^\beta))(\wp) \leq ((\wp_1+\nu^\beta)+(\wp_2+\nu^\beta))(\wp)$.
Therefore $(\wp_1+\nu^\beta)+(\wp_2+\nu^\beta) = (\wp_3+\nu^\beta)+(\wp_4+\nu^\beta)$.
In the same manner, we can prove that $(\wp_1+\mu^\alpha)+(\wp_2+\mu^\alpha) = (\wp_3+\mu^\alpha)+(\wp_4+\mu^\alpha)$.
Again,

$$\begin{aligned}
((\wp_1+\nu^\beta)\cdot(\wp_2+\nu^\beta))(\wp) &= (\wp_1\cdot\wp_2+\nu^\beta)(\wp)\\
&= \nu^\beta(\wp-\wp_1\cdot\wp_2)\\
&= \nu^\beta(\wp-\wp_1\cdot\wp_2+\wp_3\cdot\wp_4-\wp_3\cdot\wp_4)\\
&= \nu^\beta((\wp-\wp_3\cdot\wp_4)-(\wp_1\cdot\wp_2-\wp_3\cdot\wp_4))\\
&\leq \nu^\beta(\wp-\wp_3\cdot\wp_4)\vee\nu^\beta(\wp_1\cdot\wp_2-\wp_3\cdot\wp_4).
\end{aligned}$$

Now,

$$\begin{aligned}
\nu^\beta(\wp_1\cdot\wp_2-\wp_3\cdot\wp_4) &= \nu^\beta(\wp_1\cdot\wp_2-\wp_2\cdot\wp_3+\wp_2\cdot\wp_3-\wp_3\cdot\wp_4)\\
&= \nu^\beta(\wp_2\cdot(\wp_1-\wp_3)+\wp_3\cdot(\wp_2-\wp_4))\\
&\leq \nu^\beta(\wp_2\cdot(\wp_1-\wp_3))\vee\nu^\beta(\wp_3(\wp_2-\wp_4))\\
&\leq (\nu^\beta(\wp_2)\wedge\nu^\beta(\wp_1-\wp_2))\vee(\nu^\beta(\wp_3)\wedge\nu^\beta(\wp_2-\wp_4))\\
&= (\nu^\beta(\wp_2)\wedge\nu^\beta(0))\vee(\nu^\beta(\wp_3)\wedge\nu^\beta(0))\\
&= \nu^\beta(0).
\end{aligned}$$

So, $\nu^\beta(\wp_1\cdot\wp_2-\wp_3\cdot\wp_4) = \nu^\beta(0)$.
Therefore $((\wp_1+\nu^\beta)\cdot(\wp_2+\nu^\beta))(\wp) \leq \nu^\beta(\wp-\wp_3\cdot\wp_4)\vee\nu^\beta(0) = \nu^\beta(\wp-\wp_3\cdot\wp_4) = (\wp_3\cdot\wp_4+\nu^\beta)(\wp) = ((\wp_3+\nu^\beta)\cdot(\wp_4+\nu^\beta))(\wp)$.

Similarly, $((\wp_3+\nu^\beta)\cdot(\wp_4+\nu^\beta))(\wp) \le ((\wp_1+\nu^\beta)\cdot(\wp_2+\nu^\beta))(\wp)$.
Therefore $(\wp_1+\nu^\beta)\cdot(\wp_2+\nu^\beta) = (\wp_3+\nu^\beta)\cdot(\wp_4+\nu^\beta)$.
In the same manner, $(\wp_1+\mu^\alpha)\cdot(\wp_2+\mu^\alpha) = (\wp_3+\mu^\alpha)\cdot(\wp_4+\mu^\alpha)$.
Thus the above operations are well defined.

Now, $((\wp_1+\psi^*)+(\wp_2+\psi^*))+(\wp_3+\psi^*) = (\wp_1+\wp_2+\psi^*)+(\wp_3+\psi^*) = \wp_1+\wp_2+\wp_3+\psi^* = (\wp_1+\psi^*)+(\wp_2+\wp_3+\psi^*) = (\wp_1+\psi^*)+((\wp_2+\psi^*)+(\wp_3+\psi^*))$.

Similarly, $((\wp_1+\psi^*)\cdot(\wp_2+\psi^*))\cdot(\wp_3+\psi^*) = (\wp_1+\psi^*)\cdot((\wp_2+\psi^*)\cdot(\wp_3+\psi^*))$.

Therefore operations $+$ and $\cdot$ defined on $\dfrac{\mathfrak{J}}{\psi^*}$ are associative.

Here, $0+\psi^*$ is the additive inverse of $\dfrac{\mathfrak{J}}{\psi^*}$ as $(\wp+\psi^*)+(0+\psi^*) = (\wp+\psi^*) = (0+\psi^*)+(\wp+\psi^*)$.

For an arbitrary element $\wp+\psi^*$ of $\dfrac{\mathfrak{J}}{\psi^*}$, we have $(-\wp)+\psi^*$ such that $(\wp+\psi^*)+(-\wp+\psi^*) = (0+\psi^*) = (-\wp+\psi^*)+(\wp+\psi^*)$. Therefore $(-\wp)+\psi^*$ is the inverse of $\wp+\psi^*$ in $\dfrac{\mathfrak{J}}{\psi^*}$.

Now, $(\wp_1+\psi^*)+(\wp_2+\psi^*) = (\wp_1+\wp_2)+\psi^* = (\wp_2+\wp_1)+\psi^* = (\wp_2+\psi^*)+(\wp_1+\psi^*)$. Therefore $+$ is commutative on $\dfrac{\mathfrak{J}}{\psi^*}$.

Also, $((\wp_1+\psi^*)+(\wp_2+\psi^*))\cdot(\wp_3+\psi^*) = ((\wp_1+\wp_2)+\psi^*)\cdot(\wp_3+\psi^*) = (\wp_1+\wp_2)\cdot\wp_3+\psi^* = \wp_1\cdot\wp_3+\wp_2\cdot\wp_3+\psi^* = (\wp_1\cdot\wp_3+\psi^*)+(\wp_2\cdot\wp_3+\psi^*) = (\wp_1+\psi^*)\cdot(\wp_3+\psi^*)+(\wp_2+\psi^*)\cdot(\wp_3+\psi^*)$ and $(\wp_1+\psi^*)\cdot((\wp_2+\psi^*)+(\wp_3+\psi^*)) = (\wp_1+\psi^*)\cdot(\wp_2+\wp_3+\psi^*) = \wp_1\cdot(\wp_2+\wp_3)+\psi^* = \wp_1\cdot\wp_2+\wp_1\cdot\wp_3+\psi^* = (\wp_1\cdot\wp_2+\psi^*)+(\wp_1\cdot\wp_3+\psi^*) = (\wp_1+\psi^*)\cdot(\wp_2+\psi^*)+(\wp_1+\psi^*)\cdot(\wp_3+\psi^*)$. Thus distributive law holds on $\dfrac{\mathfrak{J}}{\psi^*}$.

Hence $\dfrac{\mathfrak{J}}{\psi^*}$ forms a ring under the operations $+$ and $\cdot$ defined on it. □

Definition 3.2. *The ring* $(\dfrac{\mathfrak{J}}{\psi^*},+,\cdot)$ *in the Theorem 3.2, is called the* (α,β)*-Pythagorean fuzzy quotient ring of* ψ^*.

Example 3.2. *Take the* (α,β)*-Pythagorean fuzzy ideal* ψ^* *of the ring* $(\mathbb{Z}_3,+_3,\cdot_3)$ *in Example 3.1.*

Then $\dfrac{\mathfrak{J}}{\psi^*} = \{\bar{0}+\psi^*, \bar{1}+\psi^*, \bar{2}+\psi^*\}$. *Now,*

$(\bar{0}+\psi^*) = (\bar{0}+\mu^\alpha, \bar{0}+\nu^\beta)$
$= \{(z, \bar{0}+\mu^\alpha, \bar{0}+\nu^\beta) | z \in \mathbb{Z}_3\}$
$= \{(\bar{0}, (\bar{0}+\mu^\alpha)(\bar{0}), (\bar{0}+\nu^\beta)(\bar{0})), (\bar{1}, (\bar{0}+\mu^\alpha)(\bar{1}), (\bar{0}+\nu^\beta)(\bar{1})), (\bar{2}, (\bar{0}+\mu^\alpha)(\bar{2}), (\bar{0}+\nu^\beta)(\bar{2}))\}$
$= \{(\bar{0}, \mu^\alpha(\bar{0}), \nu^\beta(\bar{0})), (\bar{1}, \mu^\alpha(\bar{1}), \nu^\beta(\bar{1})), (\bar{2}, \mu^\alpha(\bar{2}), \nu^\beta(\bar{2}))\}$
$= \{(\bar{0}, 0.8, 0.4), (\bar{1}, 0.7, 0.6), (\bar{2}, 0.7, 0.6)\}$,
$(\bar{1}+\psi^*) = (\bar{1}+\mu^\alpha, \bar{1}+\nu^\beta)$
$= \{(z, \bar{1}+\mu^\alpha, \bar{1}+\nu^\beta) | z \in \mathbb{Z}_3\}$
$= \{(\bar{0}, (\bar{1}+\mu^\alpha)(\bar{0}), (\bar{1}+\nu^\beta)(\bar{0})), (\bar{1}, (\bar{1}+\mu^\alpha)(\bar{1}), (\bar{1}+\nu^\beta)(\bar{1})), (\bar{2}, (\bar{1}+\mu^\alpha)(\bar{2}), (\bar{1}+\nu^\beta)(\bar{2}))\}$
$= \{(\bar{0}, \mu^\alpha(\bar{1}), \nu^\beta(\bar{1})), (\bar{1}, \mu^\alpha(\bar{0}), \nu^\beta(\bar{0})), (\bar{2}, \mu^\alpha(\bar{1}), \nu^\beta(\bar{1}))\}$
$= \{(\bar{0}, 0.7, 0.6), (\bar{1}, 0.8, 0.4), (\bar{2}, 0.7, 0.6)\}$,
and $(\bar{2}+\psi^*) = \{(\bar{0}, 0.7, 0.6), (\bar{1}, 0.7, 0.6), (\bar{2}, 0.8, 0.4)\}$.

Clearly, the operation $+$ *and* $\cdot$ *defined on* $\dfrac{\mathfrak{J}}{\psi^*}$ *are associative.*

Here, $\bar{0}+\psi^*$ *is the zero element and* $\bar{1}+\psi^*$ *is the identity of* $\dfrac{\mathfrak{J}}{\psi^*}$.

Now, $(\bar{1}+\psi^*)+(\bar{2}+\psi^*)=(\bar{2}+\psi^*)+(\bar{1}+\psi^*)=(\bar{0}+\psi^*)$*. So,* $(\bar{1}+\psi^*)$ *and* $(\bar{2}+\psi^*)$ *are inverse of each other. It also shows that* $+$ *is commutative.*

Also, $(\bar{1}+\psi^*)\cdot(\bar{2}+\psi^*)=(\bar{2}+\psi^*)\cdot(\bar{1}+\psi^*)=(\bar{2}+\psi^*)$*. So,* $\cdot$ *is commutative.*

Here, $((\bar{0}+\psi^*)+(\bar{1}+\psi^*))\cdot(\bar{2}+\psi^*)=((\bar{0}+\bar{1})+\psi^*)\cdot(\bar{2}+\psi^*)=(\bar{0}+\bar{1})\cdot\bar{2}+\psi^*=\bar{0}\cdot\bar{2}+\bar{1}\cdot\bar{2}+\psi^*=(\bar{0}\cdot\bar{2}+\psi^*)+(\bar{1}\cdot\bar{2}+\psi^*)=(\bar{0}+\psi^*)\cdot(\bar{2}+\psi^*)+(\bar{1}+\psi^*)\cdot(\bar{2}+\psi^*)$ *and* $(\bar{0}+\psi^*)\cdot((\bar{1}+\psi^*)+(\bar{2}+\psi^*))=(\bar{0}+\psi^*)\cdot(\bar{1}+\bar{2}+\psi^*)=\bar{0}\cdot(\bar{1}+\bar{2})+\psi^*=\bar{0}\cdot\bar{1}+\bar{0}\cdot\bar{2}+\psi^*=(\bar{0}\cdot\bar{1}+\psi^*)+(\bar{0}\cdot\bar{2}+\psi^*)=(\bar{0}+\psi^*)\cdot(\bar{1}+\psi^*)+(\bar{0}+\psi^*)\cdot(\bar{2}+\psi^*)$*. Thus distributive law holds on* $\frac{\mathfrak{J}}{\psi^*}$.

Hence $\left(\frac{\mathfrak{J}}{\psi^*},+,\cdot\right)$ *is a unity-based commutative ring.*

Remark 3.1. *If the ring* $(\mathfrak{J},+,\cdot)$ *as in the Theorem 3.2, is a commutative ring with unity (CRU) then* $\frac{\mathfrak{J}}{\psi^*}$ *forms a CRU.*

Now we'll see under what conditions the (α,β)-Pythagorean fuzzy ideal becomes constant.

Proposition 3.3. *Suppose* ψ^* *is an* (α,β)*-Pythagorean fuzzy ideal of* $(\mathfrak{J},+,\cdot)$ *then* ψ^* *is constant on every coset of* $\mathfrak{D}=\{\wp\in\mathfrak{J}|\mu^\alpha(\wp)=\mu^\alpha(0),\nu^\beta(\wp)=\nu^\beta(0)\}$ *in* $\mathfrak{J}$.

Proof. Here, $\mathfrak{J}$ has an ideal $\mathfrak{D}$.

For any $\wp\in\mathfrak{J}$, $\wp+\mathfrak{D}$ is a coset of $\mathfrak{D}$.

Suppose, $x\in\wp+D$ then $x=\wp+d$ where $d\in\mathfrak{D}$.

So, $\mu^\alpha(x-\wp)=\mu^\alpha(0)$ and $\nu^\beta(x-\wp)=\nu^\beta(0)$.

This represents that $\mu^\alpha(x)=\mu^\alpha(\wp)$ and $\nu^\beta(x)=\nu^\beta(\wp)$.

Thus for all $x\in\wp+D$, $\psi^*(x)=\psi^*(\wp)$.

Hence ψ^* is constant on every coset of $\mathfrak{D}$. □

Proposition 3.4. *Assume* ψ^* *is an* (α,β)*-Pythagorean fuzzy ideal of* $(\mathfrak{J},+,\cdot)$*. Then* $(\wp+\psi^*)_{(\eta,\zeta)}=\wp+\psi^*_{(\eta,\zeta)}$ *for* $\wp\in\mathfrak{J}$ *where,* $\eta\le\mu^\alpha(0)$ *and* $\zeta\ge\nu^\beta(0)$.

Proof. Here, $\psi^*_{(\eta,\zeta)}=\{\wp\in\mathfrak{J}|\mu^\alpha(\wp)\ge\eta,\nu^\beta(\wp)\le\zeta\}$.

So, $(\wp+\psi^*)_{(\eta,\zeta)}=(\wp+\mu^\alpha,\wp+\nu^\beta)_{(\eta,\zeta)}$.

Suppose, $\wp_1\in(w+\psi^*)_{(\eta,\zeta)}$.

Therefore $(\wp+\mu^\alpha)(\wp_1)\ge\eta$ and $(\wp+\nu^\beta)(\wp_1)\le\zeta$.

This implies that $\mu^\alpha(\wp_1-\wp)\ge\eta$ and $\nu^\beta(\wp_1-\wp)\le\zeta$.

Thus $\wp_1-\wp\in\psi^*_{(\eta,\zeta)}$. Hence $\wp_1\in\wp+\psi^*_{(\eta,\zeta)}$.

Again, let $\wp_1\in\wp+\psi^*_{(\eta,\zeta)}$.

Then $\wp_1-\wp\in\psi^*_{(\eta,\zeta)}$.

So, $\mu^\alpha(\wp_1-\wp)\ge\eta$ and $\nu^\beta(\wp_1-\wp)\le\zeta$.

Therefore $\wp_1\in(\wp+\psi^*)_{(\eta,\zeta)}$.

Hence $(\wp+\psi^*)_{(\eta,\zeta)}=\wp+\psi^*_{(\eta,\zeta)}$, for $\wp\in\mathfrak{J}$. □

In the following proposition, we will demonstrate a crucial property of the restricted (α,β)-Pythagorean fuzzy ideals. This property will be used to prove the second isomorphism theorem of (α,β)-Pythagorean fuzzy Ideals.

Proposition 3.5. *Suppose* ψ^* *is an* (α,β)*-Pythagorean fuzzy ideal of* $(\mathfrak{J},+,\cdot)$ *and* $\mathfrak{J}$ *has an ideal* $\mathfrak{Z}$*. Then* $\psi^*|_{\mathfrak{Z}}$ *is an* (α,β)*-Pythagorean fuzzy ideal of* $\mathfrak{Z}$ *and* $\frac{\mathfrak{J}}{\psi^*}$ *has an ideal* $\frac{\mathfrak{Z}}{\psi^*}$.

Proof. Clearly, $\psi^*|_{\mathfrak{Z}}$ is an (α, β)-Pythagorean fuzzy ideal of $\mathfrak{Z}$.

Since, $\mathfrak{I}$ has an ideal $\mathfrak{Z}$ then $\mathfrak{Z} \subseteq \mathfrak{I}$, $\mathfrak{z}_1 + \mathfrak{z}_2 \in \mathfrak{Z}$ and $\wp.\mathfrak{z}_1 \in \mathfrak{Z}$ for $\wp \in \mathfrak{I}, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{Z}$.

Here, $\dfrac{\mathfrak{Z}}{\psi^*} = \{(\mathfrak{z} + \psi^*) | \mathfrak{z} \in \mathfrak{Z}\}$ and $\dfrac{\mathfrak{I}}{\psi^*} = \{(\wp + \psi^*) | \wp \in \mathfrak{I}\}$.

Clearly, $\dfrac{\mathfrak{Z}}{\psi^*} \subseteq \dfrac{\mathfrak{I}}{\psi^*}$.

Let $\mathfrak{z}_1 + \psi^*, \mathfrak{z}_2 + \psi^* \in \dfrac{\mathfrak{Z}}{\psi^*}$.

Then $(\mathfrak{z}_1 + \psi^*) + (\mathfrak{z}_2 + \psi^*) = (\mathfrak{z}_1 + \mathfrak{z}_2 + \psi^*) \in \dfrac{\mathfrak{Z}}{\psi^*}$.

Also, $(\wp + \psi^*) \cdot (\mathfrak{z}_1 + \psi^*) = (\wp \cdot \mathfrak{z}_1 + \psi^*) \in \dfrac{\mathfrak{Z}}{\psi^*}$.

Hence $\dfrac{\mathfrak{Z}}{\psi^*}$ is an ideal of $\dfrac{\mathfrak{I}}{\psi^*}$. □

4. Isomorphism Theorems of (α, β)-Pythagorean fuzzy ideals

Isomorphism is of great importance in every branch of abstract algebra. Ring isomorphism is a bijective homomorphism between two rings. The algebraic structures of two isomorphic rings are identical. Isomorphism theorems are one of the most significant theorems in ring theory. These theorems demonstrate the connection between ring homomorphism and quotient rings.

In this section, we will represent the (α, β)-Pythagorean fuzzy versions of these theorems.

Theorem 4.1. (First isomorphism theorem of (α, β)-Pythagorean fuzzy ideals) *Let $(\mathfrak{I}_1, +, \cdot)$ and $(\mathfrak{I}_2, +, \cdot)$ be two rings and ψ^* be an (α, β)-Pythagorean fuzzy ideal of the ring $(\mathfrak{I}_2, +, \cdot)$. If $\Omega : \mathfrak{I}_1 \to \mathfrak{I}_2$ is an onto homomorphism of rings then $\dfrac{\mathfrak{I}_1}{\Omega^{-1}(\psi^*)} \cong \dfrac{\mathfrak{I}_2}{\psi^*}$.*

Proof. We have $\Omega^{-1}(\psi^*)(\wp) = (\Omega^{-1}(\mu^\alpha)(\wp), \Omega^{-1}(\nu^\beta)(\wp))$ for all $\wp \in \mathfrak{I}_1$, where $\Omega^{-1}(\mu^\alpha)(\wp) = \mu^\alpha(\Omega(\wp))$ and $\Omega^{-1}(\nu^\beta)(\wp) = \nu^\beta(\Omega(\wp))$.

Now, we construct a map $\lambda : \dfrac{\mathfrak{I}_1}{\Omega^{-1}(\psi^*)} \to \dfrac{\mathfrak{I}_2}{\psi^*}$ by

$$\lambda(\wp + \Omega^{-1}(\psi^*)) = \Omega(\wp) + \psi^*$$

for $\wp \in \mathfrak{I}_1$.

That is $\lambda(\wp + \Omega^{-1}(\mu^\alpha)) = \Omega(\wp) + \mu^\alpha$ and $\lambda(\wp + \Omega^{-1}(\nu^\beta)) = \Omega(\wp) + \nu^\beta$ for $\wp \in \mathfrak{I}_1$.

At first, we have to show that λ is a well defined map.

Assume $\wp_1 + \Omega^{-1}(\mu^\alpha) = \wp_2 + \Omega^{-1}(\mu^\alpha)$ for $\wp_1, \wp_2 \in \mathfrak{I}_1$.

So, $(\wp_1 - \wp_2) + \Omega^{-1}(\mu^\alpha) = \Omega^{-1}(\mu^\alpha)$.

This implies that $\Omega^{-1}(\mu^\alpha)(\wp_1 - \wp_2) = \Omega^{-1}(\mu^\alpha)(0)$.

Therefore $\mu^\alpha(\Omega(\wp_1 - \wp_2)) = \mu^\alpha(\Omega(0))$.

So, $\mu^\alpha(\Omega(\wp_1) - \Omega(\wp_2)) = \mu^\alpha(0')$, where $0'$ is the additive identity of $\mathfrak{I}_2$.

Therefore by Theorem 3.1, $(\Omega(\wp_1) - \Omega(\wp_2)) + \mu^\alpha = \mu^\alpha$.

Thus $\Omega(\wp_1) + \mu^\alpha = \Omega(\wp_2) + \mu^\alpha$.

Similarly, we can produce that for $\wp_1 + \Omega^{-1}(\nu^\beta) = \wp_2 + \Omega^{-1}(\nu^\beta)$, $\Omega(\wp_1) + \nu^\beta = \Omega(\wp_2) + \nu^\beta$.

Hence λ is well-defined.

Now, we have to show that λ is a ring-homomorphism. So, for $\wp_1, \wp_2 \in \mathfrak{J}_1$,

$$\begin{aligned}
\lambda((\wp_1 + \Omega^{-1}(\psi^*)) + (\wp_2 + \Omega^{-1}(\psi^*))) &= \lambda((\wp_1 + \wp_2) + f^{-1}(\psi^*)) \\
&= \Omega(\wp_1 + \wp_2) + \psi^* \\
&= \Omega(\wp_1) + \Omega(\wp_2) + \psi^* \\
&= (\Omega(\wp_1) + \psi^*) + (\Omega(\wp_2) + \psi^*) \\
&= \lambda(\wp_1 + \Omega^{-1}(\psi^*)) + \lambda(\wp_2 + \Omega^{-1}(\psi^*))
\end{aligned}$$

and

$$\begin{aligned}
\lambda((\wp_1 + \Omega^{-1}(\psi^*)) \cdot (\wp_2 + \Omega^{-1}(\psi^*))) &= \lambda((\wp_1 \cdot \wp_2) + \Omega^{-1}(\psi^*)) \\
&= \Omega(\wp_1 \cdot \wp_2) + \psi^* \\
&= (\Omega(\wp_1) \cdot \Omega(\wp_2)) + \psi^* \\
&= (\Omega(\wp_1) + \psi^*) \cdot (\Omega(\wp_2) + \psi^*) \\
&= \lambda(\wp_1 + \Omega^{-1}(\psi^*)) \cdot \lambda(\wp_2 + \Omega^{-1}(\psi^*)).
\end{aligned}$$

Suppose $\wp_2 + \psi^*$ be an element of $\frac{\mathfrak{J}_2}{\psi^*}$. Since $\Omega : \mathfrak{J}_1 \to \mathfrak{J}_2$ is an onto homomorphism, then there exists a $\wp_1 \in \mathfrak{J}_1$ such that $\Omega(\wp_1) = \wp_2$.

Therefore $\lambda(\wp_1 + \Omega^{-1}(\psi^*)) = \Omega(\wp_1) + \psi^* = \wp_2 + \psi^*$.

Thus λ is an onto homomorphism.

Let $\Omega(\wp_1) + \psi^* = \Omega(\wp_2) + \psi^*$ for $\wp_1, \wp_2 \in \mathfrak{J}_1$.

so, $(\Omega(\wp_1) - \Omega(\wp_2)) + \psi^* = \psi^*$.

Then by Theorem 3.1, $\psi^*(\Omega(\wp_1) - \Omega(\wp_2)) = \psi^*(0')$.

Therefore $\Omega^{-1}(\psi^*)(\wp_1 - \wp_2) = \Omega^{-1}(\psi^*)(0)$.

Again by Theorem 3.1, $(\wp_1 - \wp_2) + \Omega^{-1}(\psi^*) = \Omega^{-1}(\psi^*)$.

Thus $\wp_1 + \Omega^{-1}(\psi^*) = \wp_2 + \Omega^{-1}(\psi^*)$.

Therefore λ is a one-one homomorphism.

Hence $\frac{\mathfrak{J}_1}{\Omega^{-1}(\psi^*)} \cong \frac{\mathfrak{J}_2}{\psi^*}$. □

Theorem 4.2. *Suppose ψ^* is an (α, β)-Pythagorean fuzzy ideal of $(\mathfrak{J}, +, \cdot)$. Then $\frac{\mathfrak{J}}{\mathfrak{P}} \cong \frac{\mathfrak{J}}{\psi^*}$, where $\mathfrak{P} = \{\wp \in \mathfrak{J} | \psi^*(\wp) = \psi^*(0)\}$.*

Proof. we construct a map $\lambda : \mathfrak{J} \to \frac{\mathfrak{J}}{\psi^*}$ by $\lambda(\wp) = \wp + \psi^*$ for $\wp \in \mathfrak{J}$.

Suppose $\wp_1, \wp_2 \in \mathfrak{J}$.

Then $\lambda(\wp_1 + \wp_2) = \wp_1 + \wp_2 + \psi^* = (\wp_1 + \psi^*) + (\wp_2 + \psi^*) = \lambda(\wp_1) + \lambda(\wp_2)$.

Also, $\lambda(\wp_1 \cdot \wp_2) = \wp_1 \cdot \wp_2 + \psi^* = (\wp_1 + \psi^*) \cdot (\wp_2 + \psi^*) = \lambda(\wp_1) \cdot \lambda(\wp_2)$.

Thus λ is a ring-homomorphism.

Now, $\lambda(0) = \psi^*$ and for $\wp + \psi^* \in \frac{\mathfrak{J}}{\psi^*}$, $\exists \wp \in \mathfrak{J}$ such that $\lambda(\wp) = \wp + \psi^*$.

Hence λ is an isomorphism. Here,

$$\begin{aligned}
Ker(\lambda) &= \{\wp \in \mathfrak{J} | \lambda(\wp) = 0 + \psi^*\} \\
&= \{\wp \in \mathfrak{J} | \wp + \psi^* = \psi^*\} \\
&= \{\wp \in \mathfrak{J} | \psi^*(\wp) = \psi^*(0)\} \\
&= \mathfrak{P}.
\end{aligned}$$

Therefore by the First Isomorphism theorem of ring, $\dfrac{\mathfrak{J}}{ker(\lambda)} \cong Im(\lambda)$. Hence $\dfrac{\mathfrak{J}}{\mathfrak{P}} \cong \dfrac{\mathfrak{J}}{\psi^*}$.
□

This theorem established a relation between a quotient ring and an (α,β)-Pythagorean fuzzy quotient ring of a ring.

Theorem 4.3. (Second isomorphism theorem of (α,β)-Pythagorean fuzzy ideals) *Suppose $\psi_1^* = (\mu_1^\alpha, \nu_1^\beta)$ and $\psi_2^* = (\mu_2^\alpha, \nu_2^\beta)$ are (α,β)-Pythagorean fuzzy ideals of $(\mathfrak{J},+,\cdot)$ with $\psi_1^*(0) = \psi_2^*(0)$. Then $\dfrac{(P_1+P_2)}{\psi_2^*} \cong \dfrac{P_1}{(\psi_1^*\cap\psi_2^*)}$ where, $P_1 = \{\wp \in \mathfrak{J} | \mu_1^\alpha(\wp) = \mu_1^\alpha(0), \nu_1^\beta(\wp) = \nu_1^\beta(0)\}$ and $P_2 = \{\wp \in \mathfrak{J} | \mu_2^\alpha(\wp) = \mu_2^\alpha(0), \nu_2^\beta(\wp) = \nu_2^\beta(0)\}$.*

Proof. Since ψ_1^* and ψ_2^* are (α,β)-Pythagorean fuzzy ideals of $(\mathfrak{J},+,\cdot)$, $(\psi_1^*\cap\psi_2^*)$ is a (α,β)-Pythagorean fuzzy ideal of $(\mathfrak{J},+,\cdot)$.

Then by Proposition 3.5, $(\psi_1^*\cap\psi_2^*)$ is also a (α,β)-Pythagorean fuzzy ideal of P_1.

Again, P_1 and P_2 are ideals of $(\mathfrak{J},+,\cdot)$ then (P_1+P_2) is an ideal of $(\mathfrak{J},+,\cdot)$.

Then by Proposition 3.5, ψ_2^* is a (α,β)-Pythagorean fuzzy ideal of (P_1+P_2).

Suppose $x \in (P_1+P_2)$. Then $x = \mathfrak{a}+\mathfrak{b}$ where, $\mathfrak{a} \in P_1$ and $\mathfrak{b} \in P_2$.

Now, we construct a map $f : \dfrac{(P_1+P_2)}{\psi_2^*} \to \dfrac{P_1}{(\psi_1^*\cap\psi_2^*)}$ by

$$f(x+\psi_2^*) = \mathfrak{a} + (\psi_1^*\cap\psi_2^*).$$

Suppose $x_1 = \mathfrak{a}_1+\mathfrak{b}_1$, $x_2 = \mathfrak{a}_2+\mathfrak{b}_2$ are elements of (P_1+P_2) where, $\mathfrak{a}_1,\mathfrak{a}_2 \in P_1$ and $\mathfrak{b}_1,\mathfrak{b}_2 \in P_2$.

First of all, we will show f is well-defined.

Let $x_1+\psi_2^* = x_2+\psi_2^*$. So, $(x_1-x_2)+\psi_2^* = \psi_2^*$.

Therefore $\psi_2^*(x_1-x_2) = \psi_2^*(0)$.

So, $\psi_2^*((\mathfrak{a}_1-\mathfrak{a}_2)-(\mathfrak{b}_2-\mathfrak{b}_1)) = \psi_2^*(0)$.

This shows that $\psi_2^*(\mathfrak{a}_1-\mathfrak{a}_2) = \psi_2^*(\mathfrak{b}_2-\mathfrak{b}_1) = \psi_2^*(0)$.

Again,

$$\begin{aligned}
(\psi_1^*\cap\psi_2^*)(\mathfrak{a}_1-\mathfrak{a}_2) &= (\mu_1^\alpha\wedge\mu_2^\alpha, \nu_1^\beta\vee\nu_2^\beta)(\mathfrak{a}_1-\mathfrak{a}_2)\\
&= (\mu_1^\alpha(\mathfrak{a}_1-\mathfrak{a}_2)\wedge\mu_2^\alpha(\mathfrak{a}_1-\mathfrak{a}_2), \nu_1^\beta(\mathfrak{a}_1-\mathfrak{a}_2)\vee\nu_2^\beta(\mathfrak{a}_1-\mathfrak{a}_2))\\
&= (\mu_1^\alpha(0)\wedge\mu_2^\alpha(0), \nu_1^\beta(0)\vee\nu_2^\beta(0))\\
&= (\psi_1^*\cap\psi_2^*)(0).
\end{aligned}$$

Therefore $\wp_1+(\psi_1^*\cap\psi_2^*) = \wp_2+(\psi_1^*\cap\psi_2^*)$. Thus f is well-defined.

Now, we will present that f is a ring-homomorphism. So,

$$\begin{aligned}
f((x_1+\psi_2^*)+(x_2+\psi_2^*)) &= f(x_1+x_2+\psi_2^*)\\
&= (\mathfrak{a}_1+\mathfrak{a}_2)+(\psi_1^*\cap\psi_2^*)\\
&= (\mathfrak{a}_1+(\psi_1^*\cap\psi_2^*))+(\mathfrak{a}_2+(\psi_1^*\cap\psi_2^*))\\
&= f(x_1+\psi_2^*)+f(x_2+\psi_2^*).
\end{aligned}$$

Again,

$$\begin{aligned}
f((x_1+\psi_2^*)\cdot(x_2+\psi_2^*)) &= f(x_1\cdot x_2+\psi_2^*)\\
&= \mathfrak{a}_1\cdot\mathfrak{a}_2+(\psi_1^*\cap\psi_2^*)\\
&= (\mathfrak{a}_1+(\psi_1^*\cap\psi_2^*))\cdot(\mathfrak{a}_2+(\psi_1^*\cap\psi_2^*))\\
&= f(x_1+\psi_2^*)\cdot f(x_2+\psi_2^*).
\end{aligned}$$

For every $\mathfrak{a}+(\psi_1^*\cap\psi_2^*) \in \dfrac{P_1}{(\psi_1^*\cap\psi_2^*)}$, consider an element $\mathfrak{b} \in P_2$ then we have $x = \mathfrak{a}+\mathfrak{b} \in (P_1+P_2)$. Therefore $f(x+\psi_2^*) = \mathfrak{a}+(\psi_1^*\cap\psi_2^*)$. Thus f is an onto ring-homomorphism.

To show f is a one-to-one ring-homomorphism, we assume $\mathfrak{a}_1+(\psi_1^*\cap\psi_2^*) = \mathfrak{a}_2+(\psi_1^*\cap\psi_2^*)$.

So, $(\psi_1^*\cap\psi_2^*)(\mathfrak{a}_1-\mathfrak{a}_2) = (\psi_1^*\cap\psi_2^*)(0)$.

Therefore $(\mu_1^\alpha\cap\mu_2^\alpha)(\mathfrak{a}_1-\mathfrak{a}_2) = (\mu_1^\alpha\cap\mu_2^\alpha)(0)$ and $(\nu_1^\beta\cap\nu_2^\beta)(\mathfrak{a}_1-\mathfrak{a}_2) = (\nu_1^\beta\cap\nu_2^\beta)(0)$.

This implies that $\mu_1^\alpha(\mathfrak{a}_1-\mathfrak{a}_2)\wedge\mu_2^\alpha(\mathfrak{a}_1-\mathfrak{a}_2) = \mu_1^\alpha(0)\wedge\mu_2^\alpha(0)$ and $\nu_1^\beta(\mathfrak{a}_1-\mathfrak{a}_2)\vee\nu_2^\beta(\mathfrak{a}_1-\mathfrak{a}_2) = \nu_1^\beta(0)\vee\nu_2^\beta(0)$.

Thus $\mu_2^\alpha(\mathfrak{a}_1-\mathfrak{a}_2) = \mu_2^\alpha(0)$ and $\nu_2^\beta(\mathfrak{a}_1-\mathfrak{a}_2) = \nu_2^\beta(0)$. Now,

$$\begin{aligned}
\mu_2^\alpha(x_1-x_2) &= \mu_2^\alpha((\mathfrak{a}_1+\mathfrak{b}_1)-(\mathfrak{a}_2+\mathfrak{b}_2)) \\
&= \mu_2^\alpha((\mathfrak{a}_1-\mathfrak{a}_2)-(\mathfrak{b}_2-\mathfrak{b}_1)) \\
&\geq \mu_2^\alpha(\mathfrak{a}_1-\mathfrak{a}_2)\wedge\mu_2^\alpha(\mathfrak{b}_2-\mathfrak{b}_1) \\
&= \mu_2^\alpha(0)\wedge\mu_2^\alpha(0) \\
&= \mu_2^\alpha(0).
\end{aligned}$$

Therefore $\mu_2^\alpha(x_1-x_2) = \mu_2^\alpha(0)$. Similarly, $\nu_2^\beta(x_1-x_2) = \nu_2^\beta(0)$.

Thus $\psi_2^*(x_1-x_2) = \psi_2^*(0)$ which implies $x_1+\psi_2^* = x_2+\psi_2^*$.

Hence $\dfrac{(P_1+P_2)}{\psi_2^*} \cong \dfrac{P_1}{(\psi_1^*\cap\psi_2^*)}$. □

Theorem 4.4. (Third isomorphism theorem of (α,β)-Pythagorean fuzzy ideals) *Suppose ψ_1^*, ψ_2^* are (α,β)-Pythagorean fuzzy ideals of $(\mathfrak{J},+,\cdot)$ with $\psi_2^*\subseteq\psi_1^*$ and $\psi_1^*(0) = \psi_2^*(0)$. Then $\dfrac{(\frac{\mathfrak{J}}{\psi_2^*})}{(\frac{P_1}{\psi_2^*})} \cong \dfrac{\mathfrak{J}}{\psi_1^*}$ where, $P_1 = \{\wp\in\mathfrak{J}|\psi_1^*(\wp) = \psi_1^*(0)\}$.*

Proof. Since $\psi_2^*\subseteq\psi_1^*$ and $\psi_1^*(0) = \psi_2^*(0)$.

Then $\mu_1^\alpha(\wp)\geq\mu_2^\alpha(\wp)$ and $\nu_1^\beta(\wp)\leq\nu_2^\beta(\wp)$ for all $\wp\in\mathfrak{J}$. Also, $\mu_1^\alpha(0) = \mu_2^\alpha(0)$ and $\nu_1^\beta(0) = \nu_2^\beta(0)$.

We construct a map $\lambda : \frac{\mathfrak{J}}{\psi_2^*}\to\frac{\mathfrak{J}}{\psi_1^*}$ by $\lambda(\wp+\psi_2^*) = (\wp+\psi_1^*)$ for $\wp\in\mathfrak{J}$.

Therefore $\lambda(\wp+\mu_2^\alpha) = (\wp+\mu_1^\alpha)$ and $\lambda(\wp+\nu_2^\beta) = (\wp+\nu_1^\beta)$ for $\wp\in\mathfrak{J}$.

Suppose $\wp_1,\wp_2\in\mathfrak{J}$ and $\wp_1+\mu_2^\alpha = \wp_2+\mu_2^\alpha$.

Then $\mu_2^\alpha(\wp_1-\wp_2) = \mu_2^\alpha(0)$.

So, $\mu_1^\alpha(\wp_1-\wp_2)\geq\mu_2^\alpha(\wp_1-\wp_2) = \mu_2^\alpha(0) = \mu_1^\alpha(0)$.

Therefore $\mu_1^\alpha(\wp_1-\wp_2) = \mu_1^\alpha(0)$.

This implies that $\wp_1+\mu_1^\alpha = \wp_2+\mu_1^\alpha$.

Similarly, $\wp_1+\nu_2^\beta = \wp_2+\nu_2^\beta$ implies $\wp_1+\nu_1^\beta = \wp_2+\nu_1^\beta$.

Therefore λ is well-defined.

Now, $\lambda((\wp_1+\psi_2^*)+(\wp_2+\psi_2^*)) = \lambda(\wp_1+\wp_2+\psi_2^*) = \wp_1+\wp_2+\psi_1^* = (\wp_1+\psi_1^*)+(\wp_2+\psi_1^*) = \lambda(\wp_1+\psi_2^*)+\lambda(\wp_2+\psi_2^*)$ and $\lambda((\wp_1+\psi_2^*)\cdot(\wp_2+\psi_2^*)) = \lambda(\wp_1\cdot\wp_2+\psi_2^*) = \wp_1\cdot\wp_2+\psi_1^* = (\wp_1+\psi_1^*)\cdot(\wp_2+\psi_1^*) = \lambda(\wp_1+\psi_2^*)\cdot\lambda(\wp_2+\psi_2^*)$ for all $\wp_1,\wp_2\in\mathfrak{J}$.

This proves that λ represents a ring-homomorphism.

Clearly, λ is an isomorphism.

So,

$$\begin{aligned} Ker(\lambda) &= \{(\wp+\psi_2^*)\in\frac{\mathfrak{J}}{\psi_2^*}|\lambda(\wp+\psi_2^*)=0+\psi_1^*\} \\ &= \{(\wp+\psi_2^*)\in\frac{\mathfrak{J}}{\psi_2^*}|\wp+\psi_1^*=\psi_1^*\} \\ &= \{(\wp+\psi_2^*)\in\frac{\mathfrak{J}}{\psi_2^*}|\psi_1^*(\wp)=\psi_1^*(0)\} \\ &= \{(\wp+\psi_2^*)\in\frac{\mathfrak{J}}{\psi_2^*}|\wp\in P_1\} \\ &= \frac{P_1}{\psi_2^*}. \end{aligned}$$

Therefore by the First Isomorphism theorem of rings, $\dfrac{(\frac{\mathfrak{J}}{\psi_2^*})}{(\frac{P_1}{\psi_2^*})}\cong\dfrac{\mathfrak{J}}{\psi_1^*}$. □

5. Conclusion

Bhunia et al. [6] developed (α,β)-Pythagorean fuzzy ideals of rings. They have discussed various aspects of it. Factor rings and isomorphism play an important role in classical ring theory. However, these topics receive no attention in (α,β)-Pythagorean fuzzy ideals. As a result, the goal of this paper is to fill these research gaps. Here, we have introduced the perception of (α,β)-Pythagorean fuzzy cosets of (α,β)-Pythagorean fuzzy ideals. The characteristics of (α,β)-Pythagorean fuzzy cosets have been briefly discussed. We have shown that an (α,β)-Pythagorean fuzzy ideal is constant under certain condition. Further, we have defined the (α,β)-Pythagorean fuzzy quotient ring of an (α,β)-Pythagorean fuzzy ideal. We have seen that the restricted (α,β)-Pythagorean fuzzy ideal is also an (α,β)-Pythagorean fuzzy ideal of any ring. Finally, we have proved the fundamental theorems of ring isomorphism in (α,β)-Pythagorean fuzzy ideals.

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Supriya Bhunia for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.15, N.7.

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