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SIGNED SUM CORDIAL LABELING OF GRAPHS

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Abstract. The notion of signed product cordial labeling was introduced in 2011 and further studied by several researchers. Inspired by this notion, we define a new concept namely signed sum cordial labeling as follows: A vertex labeling of a graph G, $f : V(G) \rightarrow \{-1,+1\}$ with induced edge labeling $f^* : E(G) \rightarrow \{-2,0,+2\}$ defined by $f^*(uv) = f(u) + f(v)$ is signed sum cordial labeling if $|v_f(-1) - v_f(+1)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i,j \in \{-2,0,+2\}$, where $v_f(-1)$ is the number of vertices labeled with -1, $v_f(+1)$ is the number of vertices labeled with +1, $e_{f^*}(-2)$ is the number of edges labeled with -2, $e_{f^*}(0)$ is the number of edges labeled with 0 and $e_{f^*}(+2)$ is the number of edges labeled with +2. A graph G is signed sum cordial if it admits signed sum cordial labeling. In this paper, we investigate the signed sum cordial behaviour of some standard graphs.

Keywords: cordial labeling, signed cordial labeling, signed product cordial labeling, signed sum cordial labeling.

AMS Subject Classification: 05C78.

1. Introduction

All graphs $G = (V(G), E(G))$ considered here are simple, finite, connected and undirected. Rosa [12] first proposed the idea of graph labeling in 1967 and it is one of the major fields of graph theory. Since then, many graph labeling methods have been introduced and studied by various researchers. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Gallian [5] regularly updates the development in the field of graph labeling in his survey popularly known as a dynamic survey on graph labeling. Cordial labeling, one of the popular labelings was introduced by Cahit [2] and this concept has been extensively studied by several researchers, see [5].

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Harary [7] introduced the concept of signed cordinality. Devaraj et al. [3] took the initiative to prove that the complete graph, Peterson graph, flower graph, Jahangir graph and book graph are signed cordial graphs. Jayapal Baskar Babujee et al. [8] extended the concept of signed cordial labeling and introduced the concept of signed product cordial labeling as follows: A vertex labeling of a graph G, $f : V(G) \to \{-1, +1\}$ with induced edge labeling $f^* : E(G) \to \{-1, +1\}$ defined by $f^*(uv) = f(u)f(v)$ is called a signed product cordial labeling if $|v_f(-1) - v_f(+1)| \leq 1$ and $|e_{f^*}(-1) - e_{f^*}(+1)| \leq 1$, where $v_f(-1)$ is the number of vertices labeled with -1, $v_f(+1)$ is the number of vertices labeled with +1, $e_{f^*}(-1)$ is the number of edges labeled with -1 and $e_{f^*}(+1)$ is the number of edges labeled with +1. A graph G is signed product cordial if it admits signed product cordial labeling. Since then new results on signed product cordial labeling have been published by many researchers. For further information, we suggest the reader to refer [1], [4], [9], [10], [11], [13], [14] and [15].

Motivated by the concept of signed product cordial labeling [8], we introduce a new variation of cordial labeling namely signed sum cordial labeling as follows: A vertex labeling of a graph G, $f : V(G) \to \{-1, +1\}$ with induced edge labeling $f^* : E(G) \to \{-2, 0, +2\}$ defined by $f^*(uv) = f(u) + f(v)$ is signed sum cordial labeling if $|v_f(-1) - v_f(+1)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j \in \{-2, 0, +2\}$, where $v_f(-1)$ is the number of vertices labeled with -1, $v_f(+1)$ is the number of vertices labeled with +1, $e_{f^*}(-2)$ is the number of edges labeled with -2, $e_{f^*}(0)$ is the number of edges labeled with 0 and $e_{f^*}(+2)$ is the number of edges labeled with +2. A graph G is signed sum cordial if it admits signed sum cordial labeling. Signed sum cordial graphs have variety of applications in missile guidance code, study of X-ray crystallography, design good radar type codes, communication network, convolution codes with optimal autocorrelation properties, to determine ideal circuit layouts, and more. These applications demonstrate the potential of signed sum cordial labeling in solving real-world problems and optimizing complex systems.

We follow the basic notations and terminology of graph theory as in [6] and also we use the following graph structures to prove our main results.

The pan is the graph obtained by joining a cycle C_n to a pendant vertex. The sunlet $C_n \odot K_1$ is the graph obtained by attaching n-pendant edges to the cycle C_n. The triangular snake T_n is obtained from a path $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new vertex u_i, for $1 \leq i \leq n-1$. The bistar $B_{n,n}$ is the graph obtained by attaching the apex vertices of two copies of $K_{1,n}$ by an edge. The comb $P_n \odot K_1$ is the graph obtained from a path by attaching a pendant egde to each vertex of the path. In this paper, we prove that the graphs such as path, cycle, pan, triangular snake, sunlet, bistar and comb are signed sum cordial graphs.

2. Main Results

Theorem 2.1. *The path P_n is a signed sum cordial graph for all $n \geq 1$.*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ be the vertex set and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$ be the edge set of P_n.

Define vertex labeling $f : V(P_n) \to \{-1, +1\}$ as

$$f(v_i) = \begin{cases} +1 & ; i \equiv 1, 2, 3 \pmod 6 \\ -1 & ; i \equiv 0, 4, 5 \pmod 6 \end{cases} ; 1 \leq i \leq n-2,$$

$$f(v_{n-1}) = \begin{cases} +1 & ; n \equiv 2, 3 \pmod 6 \\ -1 & ; \text{otherwise}, \end{cases}$$

$$f(v_n) = \begin{cases} +1 & ; n \equiv 1 \pmod 6 \\ -1 & ; \text{otherwise}. \end{cases}$$

The induced edge labeling $f^*: E(P_n) \to \{-2, 0, +2\}$ is given by

$$f^*(v_iv_{i+1}) = \begin{cases} +2 & ; i \equiv 1,2 \pmod 6 \\ 0 & ; i \equiv 0,3 \pmod 6 \\ -2 & ; i \equiv 4,5 \pmod 6 \end{cases} ; 1 \leq i \leq n-3,$$

$$f^*(v_{n-2}v_{n-1}) = \begin{cases} -2 & ; n \equiv 0,1 \pmod 6 \\ 0 & ; n \equiv 2,4,5 \pmod 6 \\ +2 & ; n \equiv 3 \pmod 6, \end{cases}$$

$$f^*(v_{n-1}v_n) = \begin{cases} 0 & ; n \equiv 1,2,3 \pmod 6 \\ -2 & ; \text{otherwise}. \end{cases}$$

We observe that,
$v_f(-1) = v_f(+1) + 1 = \lfloor \frac{n}{2} \rfloor$; n is odd,
$v_f(-1) = v_f(+1) = \frac{n}{2}$; n is even,
$e_{f^*}(-2) + 1 = e_{f^*}(0) = e_{f^*}(+2) + 1 = \frac{n}{3} - 1$; $n \equiv 0 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = \lfloor \frac{n}{3} \rfloor$; $n \equiv 1,4 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) = \lfloor \frac{n}{3} \rfloor$; $n \equiv 2 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) + 1 = \frac{n}{3} - 1$; $n \equiv 3 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) + 1 = \lfloor \frac{n}{3} \rfloor$; $n \equiv 5 \pmod 6$.
Thus, we have $|v_f(+1) - v_f(-1)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$; $i, j \in \{-2, 0, +2\}$. Hence, the path P_n admits signed sum cordial labeling. □

Example 2.1. *A signed sum cordial labeling of P_7 is shown in Figure 1.*

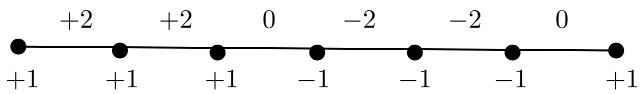

Figure 1: Signed sum cordial labeling of P_7

Theorem 2.2. *The cycle C_n is a signed sum cordial graph if and only if $n \not\equiv 3 \pmod 6$ for all $n \geq 3$.*

Proof. Let $V(C_n) = \{v_i : 1 \leq i \leq n\}$ be the vertex set and $E(C_n) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\}$ be the edge set of C_n.
For $n \not\equiv 3 \pmod 6$
Define vertex labeling $f: V(C_n) \to \{-1, +1\}$ as

$$f(v_i) = \begin{cases} +1 & ; i = 1,2 \\ -1 & ; i = 3,4 \end{cases} ; n = 4,$$

$$f(v_i) = \begin{cases} +1 & ; i \equiv 1,2,3 \pmod 6 \\ -1 & ; i \equiv 0,4,5 \pmod 6 \end{cases} ; n \equiv 0,1,5 \pmod 6,$$

$$f(v_i) = \begin{cases} +1 & ; 1 \leq i \leq 4 \\ -1 & ; 5 \leq i \leq 8 \end{cases} ; n \equiv 2,4 \pmod 6,$$

$$f(v_i) = \begin{cases} +1 & ; i \equiv 3,4,5 \pmod 6 \\ -1 & ; i \equiv 0,1,2 \pmod 6 \end{cases} ; 9 \leq i \leq n-1, n \equiv 2,4 \pmod 6,$$

$f(v_n) = -1$; $n \equiv 2,4 \pmod 6$.
The induced edge labeling $f^*: E(C_n) \to \{-2, 0, +2\}$ is given by

$$f^*(v_iv_{i+1}) = \begin{cases} +2 & ; i \equiv 1,2 \pmod 6 \\ 0 & ; i \equiv 0,3 \pmod 6 \\ -2 & ; i \equiv 4,5 \pmod 6 \end{cases} ; 1 \leq i \leq n-2, n \equiv 0,1,5 \pmod 6,$$

$$f^*(v_iv_{i+1}) = \begin{cases} +2 & ; 1 \leq i \leq 3 \\ 0 & ; i = 4,8 \\ -2 & ; 5 \leq i \leq 7 \end{cases} ; n \equiv 2,4 \pmod 6,$$

$$f^*(v_iv_{i+1}) = \begin{cases} +2 & ; i \equiv 3,4 \pmod 6 \\ 0 & ; i \equiv 2,5 \pmod 6 \\ -2 & ; i \equiv 0,1 \pmod 6 \end{cases} ; 9 \le i \le n-2, n \equiv 2,4 \pmod 6,$$

$$f^*(v_{n-1}v_n) = \begin{cases} 0 & ; n \equiv 1,4 \pmod 6 \\ -2 & ; \text{otherwise,} \end{cases}$$

$$f^*(v_nv_1) = \begin{cases} +2 & ; n \equiv 1 \pmod 6 \\ 0 & ; \text{otherwise.} \end{cases}$$

We observe that,
$v_f(-1) = v_f(+1) + 1 = \lfloor \frac{n}{2} \rfloor$; n is odd,
$v_f(-1) = v_f(+1) = \frac{n}{2}$; n is even,
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = \frac{n}{3}$; $n \equiv 0 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) + 1 = \lfloor \frac{n}{3} \rfloor$; $n \equiv 1 \pmod 6$,
$e_{f^*}(-2) + 1 = e_{f^*}(0) = e_{f^*}(+2) + 1 = \lfloor \frac{n}{3} \rfloor$; $n \equiv 2 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) = \lfloor \frac{n}{3} \rfloor$; $n \equiv 4 \pmod 6$,
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) + 1 = \lfloor \frac{n}{3} \rfloor$; $n \equiv 5 \pmod 6$.
Thus, we have $|v_f(+1) - v_f(-1)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1$; $i, j \in \{-2, 0, +2\}$. Hence, the cycle C_n admits signed sum cordial labeling if $n \not\equiv 3 \pmod 6$.
For $n \equiv 3 \pmod 6$
In order to satisfy the vertex condition for a signed sum cordial graph, we assign label +1 to either at least $\lfloor \frac{n}{2} \rfloor$ or at most $\lceil \frac{n}{2} \rceil$ vertices out of n vertices. Suppose, we assign label +1 to at least $\lfloor \frac{n}{2} \rfloor$ vertices, which results in $\frac{n}{3}$ - 1 edges with label -2, $\frac{n}{3}$ + 1 edges with label 0 and $\frac{n}{3}$ edges with label +2 out of n edges. Thus, we get $|e_{f^*}(0) - e_{f^*}(-2)| = 2 > 1$. Otherwise, we assign label +1 to at most $\lceil \frac{n}{2} \rceil$ vertices, which results in $\frac{n}{3}$ edges with label -2, $\frac{n}{3}$ + 1 edges with label 0 and $\frac{n}{3}$ - 1 edges with label +2 out of n edges. Thus, we get $|e_{f^*}(0) - e_{f^*}(+2)| > 1$. Therefore, in both cases the edge condition for a signed sum cordial graph is not satisfied. Hence, the cycle C_n does not admit signed sum cordial labeling if $n \equiv 3 \pmod 6$. □

Example 2.2. *A signed sum cordial labeling of C_8 is shown in Figure 2.*

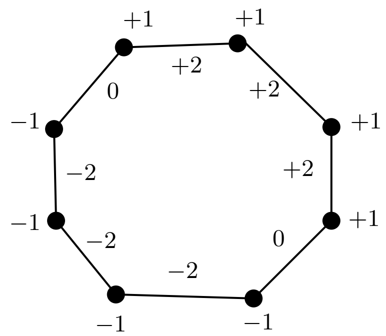

Figure 2: Signed sum cordial labeling of C_8

Theorem 2.3. *The n-pan graph is a signed sum cordial graph for all $n \ge 3$.*

Proof. Let G be a n-pan graph. Let $V(G) = \{v_i : 1 \le i \le n\} \cup \{v\}$ be the vertex set and $E(G) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_nv_1, v_nv\}$ be the edge set of G.
Define vertex labeling $f : V(G) \to \{-1, +1\}$ as

$$f(v_i) = \begin{cases} +1 & ; i \equiv 1,2,3 \pmod 6 \\ -1 & ; i \equiv 0,4,5 \pmod 6 \end{cases} ; 1 \le i \le n-2,$$

$$f(v_{n-1}) = \begin{cases} +1 & ; n \equiv 3,4 \pmod 6 \\ -1 & ; \text{otherwise,} \end{cases}$$

$$f(v_n) = \begin{cases} +1 & ; n \equiv 1,2 \pmod 6 \\ -1 & ; \text{otherwise,} \end{cases}$$

$f(v_{n+1}) = -1.$

The induced edge labeling $f^* : E(G) \to \{-2, 0, +2\}$ is given by

$$f^*(v_i v_{i+1}) = \begin{cases} +2 & ; i \equiv 1,2 \pmod 6 \\ 0 & ; i \equiv 0,3 \pmod 6 \\ -2 & ; i \equiv 4,5 \pmod 6 \end{cases} ; 1 \le i \le n-3,$$

$$f^*(v_{n-2}v_{n-1}) = \begin{cases} -2 & ; n \equiv 0,1,2 \pmod 6 \\ 0 & ; n \equiv 5 \pmod 6 \\ +2 & ; n \equiv 3,4 \pmod 6), \end{cases}$$

$$f^*(v_{n-1}v_n) = \begin{cases} -2 & ; n \equiv 0,5 \pmod 6 \\ 0 & ; \text{otherwise,} \end{cases}$$

$$f^*(v_n v_1) = \begin{cases} +2 & ; n \equiv 1,2 \pmod 6 \\ 0 & ; \text{otherwise,} \end{cases}$$

$$f^*(v_n v) = \begin{cases} 0 & ; n \equiv 1,2 \pmod 6 \\ -2 & ; \text{otherwise.} \end{cases}$$

We observe that,
$v_f(-1) = v_f(+1) + 1 = \frac{n}{2}; n \equiv 0 \pmod 4),$
$v_f(-1) = v_f(+1) = \lceil \frac{n}{2} \rceil; n \equiv 1,3 \pmod 4),$
$v_f(-1) + 1 = v_f(+1) = \frac{n}{2}; n \equiv 2 \pmod 4),$
$e_{f^*}(-2) + 1 = e_{f^*}(0) = e_{f^*}(+2) = \frac{n}{3}; n \equiv 0 \pmod 6),$
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) + 1 = \lfloor \frac{n}{3} \rfloor; n \equiv 1,4 \pmod 6),$
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = \lceil \frac{n}{3} \rceil; n \equiv 2,5 \pmod 6),$
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) = \frac{n}{3}; n \equiv 3 \pmod 6).$
Thus, we have $|v_f(+1) - v_f(-1)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1$; $i, j \in \{-2, 0, +2\}$.
Hence, the n-pan graph admits signed sum cordial labeling. □

Example 2.3. *A signed sum cordial labeling of 8-pan graph is shown in Figure 3.*

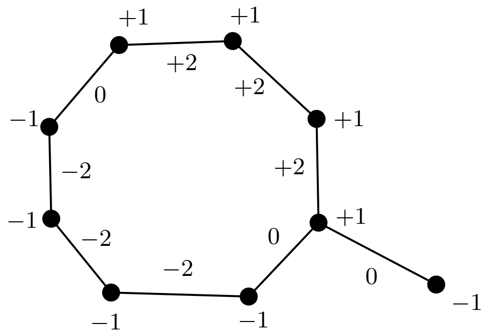

Figure 3: Signed sum cordial labeling of 8-pan graph

Theorem 2.4. *A triangular snake graph T_n is a signed sum cordial graph if and only if n is odd and $n > 3$.*

Proof. Let $V(T_n) = \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n-1\}$ be the vertex set and $E(T_n) = \{v_i v_{i+1}, v_i u_i, u_i v_{i+1} : 1 \le i \le n-1\}$ be the edge set of T_n.
Let n be an odd integer, $n > 3$.
Define vertex labeling $f : V(T_n) \to \{-1, +1\}$ as

$$f(v_i) = f(u_i) = \begin{cases} +1 & ; i \equiv 1,2 \pmod 4 \\ -1 & ; i \equiv 0,3 \pmod 4 \end{cases} ; 1 \le i \le n-4,$$

$f(u_{n-3}) = +1,$
$f(v_{n-1}) = f(u_{n-1}) = -1,$
$$f(v_{n-2}) = f(u_{n-2}) = \begin{cases} -1 & ; n \equiv 1 \pmod 4 \\ +1 & ; n \equiv 3 \pmod 4), \end{cases}$$
$$f(v_{n-3}) = f(v_n) = \begin{cases} +1 & ; n \equiv 1 \pmod 4 \\ -1 & ; n \equiv 3 \pmod 4). \end{cases}$$
The induced edge labeling $f^* : E(T_n) \to \{-2, 0, +2\}$ is given by
$$f^*(v_iv_{i+1}) = f^*(u_iv_{i+1}) = \begin{cases} +2 & ; i \equiv 1 \pmod 4 \\ 0 & ; i \equiv 0,2 \pmod 4 \\ -2 & ; i \equiv 3 \pmod 4 \end{cases} ; 1 \le i \le n-4,$$
$$f^*(v_{n-2}v_{n-1}) = f^*(u_{n-2}v_{n-1}) = \begin{cases} -2 & ; n \equiv 1 \pmod 4 \\ 0 & ; n \equiv 3 \pmod 4), \end{cases}$$
$$f^*(v_{n-1}v_n) = f^*(u_{n-1}v_n) = \begin{cases} 0 & ; n \equiv 1 \pmod 4 \\ -2 & ; n \equiv 3 \pmod 4), \end{cases}$$
$f^*(v_{n-3}v_{n-2}) = 0,$
$$f^*(u_{n-3}v_{n-2}) = \begin{cases} 0 & ; n \equiv 1 \pmod 4 \\ +2 & ; n \equiv 3 \pmod 4), \end{cases}$$
$$f^*(v_iu_i) = \begin{cases} +2 & ; i \equiv 1,2 \pmod 4 \\ -2 & ; i \equiv 0,3 \pmod 4) \end{cases} ; 1 \le i \le n-4,$$
$$f^*(v_{n-3}u_{n-3}) = \begin{cases} +2 & ; n \equiv 1 \pmod 4 \\ 0 & ; n \equiv 3 \pmod 4), \end{cases}$$
$$f^*(v_{n-2}u_{n-2}) = \begin{cases} -2 & ; n \equiv 1 \pmod 4 \\ +2 & ; n \equiv 3 \pmod 4), \end{cases}$$
$f^*(v_{n-1}u_{n-1}) = -2.$
We observe that,
$v_f(-1) = v_f(+1) + 1 = n-1,$
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = n-1$, if $n \equiv 1,3 \pmod 4$.
Thus, we have $|v_f(+1) - v_f(-1)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1$ for $i, j \in \{-2, 0, +2\}$. Hence, a triangular snake graph T_n admits signed sum cordial labeling if n is odd.
For n=3
In order to satisfy both vertex and edge conditions for a signed sum cordial graph, we assign +1 to either at least 2 vertices or at most 3 vertices out of 5 vertices, which results in every 2 edges are labeled with -2, 0, +2 out of 6 edges. If the vertices of any one of the triangles is labeled with either +1 or -1, which results in 3 edges with labels +2 or -2. If the vertices of both the triangles are labeled with +1 and -1, which results in 4 edges with label 0. Therefore, in either case, the edge condition for signed sum cordial graph is not satisfied. Hence, a triangular snake graph T_n does not admit signed sum cordial labeling if $n = 3$.
For n is even
In order to satisfy both vertex and edge conditions for a signed sum cordial graph, we assign +1 to either at least n-1 vertices or at most n vertices out of 2n-1 vertices, which results in each n-1 edges, i.e., odd number of edges is labeled with -2, 0, +2 out of 3(n-1) edges. If we assign labels +1 and -1 to the vertices of any one of the triangles, which results in 2 edges with label 0 in the same triangle. Thus, we get even number of edges labeled with 0 in T_n. Therefore, the edge condition for signed sum cordial graph is not satisfied. Hence, a triangular snake graph T_n does not admit signed sum cordial labeling if n is even.

□

Example 2.4. *A signed sum cordial labeling of T_7 is shown in Figure 5.*

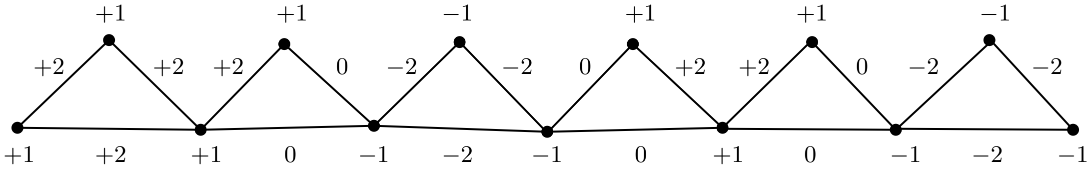

Figure 4: Signed sum cordial labeling of T_7

Theorem 2.5. *The sunlet graph $C_n \odot K_1$ is a signed sum cordial graph if and only if $n \not\equiv 3 \pmod 6$ for all $n \geq 3$.*

Proof. Let $V(C_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and $E(C_n \odot K_1) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{u_iv_i : 1 \leq i \leq n\}$ be the edge set of $C_n \odot K_1$.
For $n \not\equiv 3 \pmod 6$
Define vertex labeling $f : V(C_n \odot K_1) \to \{-1, +1\}$ as

$$f(u_i) = \begin{cases} +1 & ; i \equiv 1,2 \pmod 4 \\ -1 & ; i \equiv 0,3 \pmod 4 \end{cases} \quad \begin{array}{l} ; 1 \leq i \leq 2\lceil \frac{n}{3} \rceil,\ n \equiv 0,4,5 \pmod 6; \\ \ \ 1 \leq i \leq 2\lfloor \frac{n}{3} \rfloor,\ n \equiv 1,2 \pmod 6, \end{array}$$

$$f(u_i) = \begin{cases} +1 & ; i \text{ is odd} \\ -1 & ; i \text{ is even} \end{cases} \quad \begin{array}{l} ; 2\lceil \frac{n}{3} \rceil + 1 \leq i \leq n,\ n \equiv 0,4,5 \pmod 6; \\ \ \ 2\lceil \frac{n}{3} \rceil + 1 \leq i \leq n,\ n \equiv 1,2 \pmod 6, \end{array}$$

$f(v_i) = \ f(u_i) \ ; 1 \leq i \leq n-1,$
$f(v_n) = \ -1.$
The induced edge labeling $f^* : E(C_n \odot K_1) \to \{-2, 0, +2\}$ is given by

$$f^*(u_iu_{i+1}) = \begin{cases} +2 & ; i \equiv 1 \pmod 4 \\ 0 & ; i \equiv 0,2 \pmod 4 \\ -2 & ; i \equiv 3 \pmod 4 \end{cases} \quad \begin{array}{l} ; 1 \leq i \leq 2\lceil \frac{n}{3} \rceil,\ n \equiv 0,4,5 \pmod 6; \\ \ \ 1 \leq i \leq 2\lfloor \frac{n}{3} \rfloor,\ n \equiv 1,2 \pmod 6, \end{array}$$

$$f^*(u_iu_{i+1}) = \quad 0 \quad \begin{array}{l} ; 2\lceil \frac{n}{3} \rceil + 1 \leq i \leq n-1,\ n \equiv 1,2 \pmod 6; \\ \ \ 2\lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n-1,\ n \equiv 0,4,5 \pmod 6, \end{array}$$

$$f^*(u_nu_1) = \begin{cases} 0 & ; n \text{ is even} \\ +2 & ; n \text{ is odd}, \end{cases}$$

$$f^*(u_iv_i) = \begin{cases} +2 & ; i \equiv 1,2 \pmod 4 \\ -2 & ; i \equiv 0,3 \pmod 4 \end{cases} \quad \begin{array}{l} ; 1 \leq i \leq 2\lceil \frac{n}{3} \rceil,\ n \equiv 0,4,5 \pmod 6; \\ \ \ 1 \leq i \leq 2\lfloor \frac{n}{3} \rfloor,\ n \equiv 1,2 \pmod 6, \end{array}$$

$$f^*(u_iv_i) = \begin{cases} +2 & ; i \text{ is odd} \\ -2 & ; i \text{ is even} \end{cases} \quad \begin{array}{l} ; 2\lceil \frac{n}{3} \rceil + 1 \leq i \leq n-1,\ n \equiv 0,4,5 \pmod 6; \\ \ \ 2\lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n-1,\ n \equiv 1,2 \pmod 6, \end{array}$$

$$f^*(u_nv_n) = \begin{cases} -2 & ; n \text{ is even} \\ 0 & ; n \text{ is odd}. \end{cases}$$

We observe that,
$v_f(-1) = v_f(+1) = n,$
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = \frac{2n}{3}; n \equiv 0 \pmod 6),$
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) + 1 = 2\lfloor \frac{n}{3} \rfloor; n \equiv 1 \pmod 6),$
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) = 2\lfloor \frac{n}{3} \rfloor + 1; n \equiv 2 \pmod 6),$
$e_{f^*}(-2) + 1 = e_{f^*}(0) = e_{f^*}(+2) + 1 = 2\lfloor \frac{n}{3} \rfloor; n \equiv 4 \pmod 6),$
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) + 1 = 2\lfloor \frac{n}{3} \rfloor + 1; n \equiv 5 \pmod 6).$

Thus, we have $|v_f(+1) - v_f(-1)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j \in \{-2, 0, +2\}$. Hence, the sunlet graph $C_n \odot K_1$ admits signed sum cordial labeling if $n \not\equiv 3 \pmod 6$.
For $n \equiv 3 \pmod 6$
In order to satisfy the vertex condition for a signed sum cordial graph, we assign label +1 to exactly n vertices out of 2n vertices, which results in $\frac{2n}{3}$ - 1 edges with label -2, $\frac{2n}{3} + 1$ edges with label 0 and $\frac{2n}{3}$ edges with label +2 out of 2n edges. Thus, we get $|e_{f^*}(-2) - e_{f^*}(0)| = 2 > 1$. Therefore, the edge condition for a signed sum cordial graph is not satisfied. Hence, the sunlet graph $C_n \odot K_1$ does not admit signed sum cordial labeling if $n \equiv 3 \pmod 6$.

□

Example 2.5. *A signed sum cordial labeling of $C_8 \odot K_1$ is shown in Figure 4.*

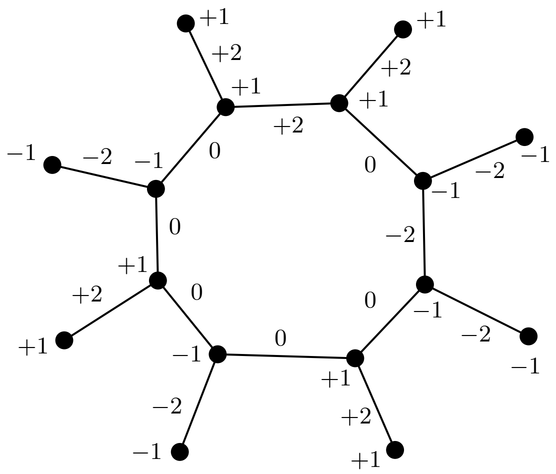

Figure 5: Signed sum cordial labeling of $C_8 \odot K_1$

Theorem 2.6. *The bistar graph $B_{n,n}$ is a signed sum cordial graph for all $n \geq 1$.*

Proof. Let $V(B_{n,n}) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{v, u\}$ be the vertex set and $E(B_{n,n}) = \{vv_i, uu_i : 1 \leq i \leq n\} \cup \{vu\}$ be the edge set of $B_{n,n}$.
Define vertex labeling $f : V(B_{n,n}) \to \{-1, +1\}$ as

$$f(v_i) = \begin{cases} +1 & ; i \equiv 1, 2 \pmod 3 \\ -1 & ; i \equiv 0 \pmod 3), \end{cases}$$

$$f(v) = +1,$$
$$f(u) = -1,$$

$$f(u_i) = \begin{cases} -1 & ; i \equiv 1, 2 \pmod 3 \\ +1 & ; i \equiv 0 \pmod 3). \end{cases}$$

The induced edge labeling $f^* : E(B_{n,n}) \to \{-2, 0, +2\}$ is given by

$$f^*(vv_i) = \begin{cases} +2 & ; i \equiv 1, 2 \pmod 3 \\ 0 & ; i \equiv 0 \pmod 3), \end{cases}$$

$$f^*(vu) = 0,$$

$$f^*(uu_i) = \begin{cases} -2 & ; i \equiv 1, 2 \pmod 3 \\ 0 & ; i \equiv 0 \pmod 3). \end{cases}$$

We observe that,
$v_f(-1) = v_f(+1) = n + 1$,
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = \frac{2n+1}{3}$, if $n \equiv 1 \pmod 3$,

$e_{f^*}(-2)+1 = e_{f^*}(0) = e_{f^*}(+2)+1 = \frac{2n-1}{3}$, if $n \equiv 2 \pmod 3$,
$e_{f^*}(-2) = e_{f^*}(0)+1 = e_{f^*}(+2) = \frac{2n}{3}$, if $n \equiv 0 \pmod 3$.
Thus, we have $|v_f(+1) - v_f(-1)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$; $i, j \in \{-2, 0, +2\}$.
Hence, the bistar graph $B_{n,n}$ admits signed sum cordial labeling. □

Example 2.6. *A signed sum cordial labeling of $B_{5,5}$ is shown in Figure 6.*

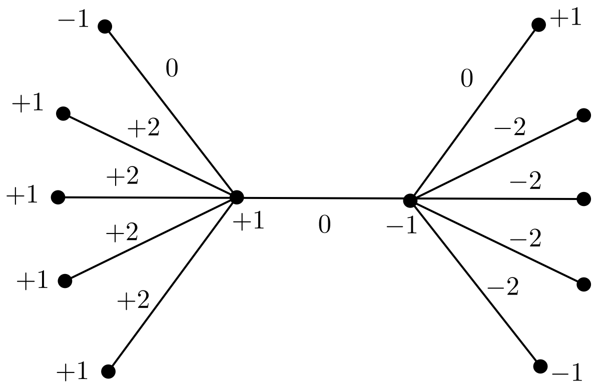

Figure 6: Signed sum cordial labeling of $B_{5,5}$

Theorem 2.7. *The comb graph $P_n \odot K_1$ is signed sum cordial graph for all $n \geq 1$.*

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and $E(P_n \odot K_1) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i u_i : 1 \leq i \leq n\}$ be the edge set of $P_n \odot K_1$.
Define vertex labeling $f : V(P_n \odot K_1) \to \{-1, +1\}$ as

$$f(u_i) = \begin{cases} +1 & ; 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ -1 & ; \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq 2\lfloor \frac{n}{3} \rfloor \end{cases} ; n \equiv 0, 1 \pmod 3,$$

$$f(u_i) = \begin{cases} +1 & ; 1 \leq i \leq \lceil \frac{n}{3} \rceil \\ -1 & ; \lceil \frac{n}{3} \rceil + 1 \leq i \leq 2\lceil \frac{n}{3} \rceil \end{cases} ; n \equiv 2 \pmod 3,$$

$$f(u_i) = \begin{cases} +1 & ; i \text{ is odd} \\ -1 & ; i \text{ is even} \end{cases} ; \frac{2n}{3} + 1 \leq i \leq n, \ n \equiv 0 \pmod 3,$$

$$f(u_i) = \begin{cases} -1 & ; i \text{ is odd} \\ +1 & ; i \text{ is even} \end{cases} ; 2\lfloor \frac{n}{3} \rfloor + 1 \leq i \leq n, \ n \equiv 1 \pmod 3; \ 2\lceil \frac{n}{3} \rceil + 1 \leq i \leq n, \ n \equiv 2 \pmod 3,$$

$$f(v_i) = \begin{cases} +1 & ; 1 \leq i \leq \lceil \frac{n}{3} \rceil \\ -1 & ; \lceil \frac{n}{3} \rceil + 1 \leq i \leq 2\lceil \frac{n}{3} \rceil, \end{cases}$$

$$f(v_i) = \begin{cases} -1 & ; i \text{ is odd} \\ +1 & ; i \text{ is even} \end{cases} ; \frac{2n}{3} + 1 \leq i \leq n, \ n \equiv 0 \pmod 3,$$

$$f(v_i) = \begin{cases} +1 & ; i \text{ is odd} \\ -1 & ; i \text{ is even} \end{cases} ; 2\lceil \frac{n}{3} \rceil + 1 \leq i \leq n, \ n \equiv 1, 2 \pmod 3.$$

The induced edge labeling $f^* : E(P_n \odot K_1) \to \{-2, 0, +2\}$ is given by

$$f^*(v_i v_{i+1}) = \begin{cases} 0 & ; i = n-2 \\ -2 & ; i = n-1 \end{cases} ; n = 3, 4,$$

$$f^*(v_1 v_2) = +2 \quad ; n = 4,$$

$$f^*(v_i v_{i+1}) = \begin{cases} +2 & ; 1 \leq i \leq \frac{n}{3} - 1 \\ -2 & ; \frac{n}{3} \leq i \leq \frac{2n}{3} \\ 0 & ; \frac{2n}{3} + 1 \leq i \leq n-1 \end{cases} ; n \equiv 0 \pmod 3,$$

$$f^*(v_i v_{i+1}) = \begin{cases} +2 & ; 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ 0 & ; i = \lfloor \frac{n}{3} \rfloor + 1, \ 2\lfloor \frac{n}{3} \rfloor + 2 \leq i \leq n-1 \\ -2 & ; \lfloor \frac{n}{3} \rfloor + 2 \leq i \leq 2\lfloor \frac{n}{3} \rfloor + 1 \end{cases} ; n \equiv 1, 2 \pmod 3,$$

$$f^*(v_iu_i) = \begin{cases} +2 & ; 1 \le i \le \lfloor \frac{n}{3} \rfloor \\ 0 & ; i = \lfloor \frac{n}{3} \rfloor + 1,\ 2\lfloor \frac{n}{3} \rfloor + 2 \le i \le n \\ -2 & ; \lfloor \frac{n}{3} \rfloor + 2 \le i \le 2\lfloor \frac{n}{3} \rfloor + 1 \end{cases} ; n \equiv 1 \text{ (mod 3)},$$

$$f^*(v_iu_i) = \begin{cases} +2 & ; 1 \le i \le \lceil \frac{n}{3} \rceil \\ -2 & ; \lceil \frac{n}{3} \rceil + 1 \le i \le 2\lceil \frac{n}{3} \rceil \\ 0 & ; 2\lceil \frac{n}{3} \rceil + 1 \le i \le n \end{cases} ; n \equiv 0, 2 \text{ (mod 3)}.$$

We observe that,
$v_f(-1) = v_f(+1) = n$,
$e_{f^*}(-2) + 1 = e_{f^*}(0) + 1 = e_{f^*}(+2) = \frac{2n}{3} - 1;\ n \equiv 0\ (mod\ 3)$,
$e_{f^*}(-2) = e_{f^*}(0) + 1 = e_{f^*}(+2) = \frac{2(n-1)}{3};\ n \equiv 1\ (mod\ 3)$,
$e_{f^*}(-2) = e_{f^*}(0) = e_{f^*}(+2) = \frac{2n-1}{3};\ n \equiv 2\ (mod\ 3)$.
Thus, in each case, we have $|v_f(+1) - v_f(-1)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1;\ i, j \in \{-2, 0, +2\}$. Hence, the comb graph $P_n \odot K_1$ admits signed sum cordial labeling. □

Example 2.7. *A signed sum cordial labeling of $P_6 \odot K_1$ is shown in Figure 7.*

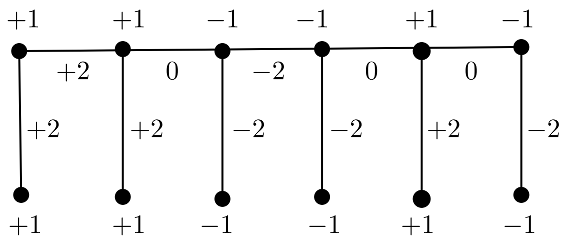

Figure 7: Signed sum cordial labeling of $P_6 \odot K_1$

3. Conclusion

The study on the labeling of different graph structures is one of the potential areas of research. In this paper we introduce a new concept namely 'signed sum cordial labeling' and prove that the graphs such as path, cycle, pan, triangular snake, sunlet, bistar and comb admit signed sum cordial labeling. Examples are given at the end of each theorem for better understanding of the labeling pattern defined in each theorem. To find new results on signed sum cordial labeling for other graph families is an open area of research.

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