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NEW APPROACH TO EVALUATE THE FAILURE PROBABILITY OF A *K-OUT-OF-N* SYSTEM UNDER HARMFUL SHOCKS

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ABSTRACT. This paper introduces a novel approach to calculating the failure probability of k-out-of-n systems under harmful shocks, which occur randomly and affect at least one component. Traditional methods for this calculation face challenges such as high computational costs and complexity, especially for large systems. The new approach simplifies the computation by developing an easily executable algorithm, saving time and resources. The proposed formula has been shown to be equivalent to existing ones. Additionally, it has been demonstrated that the failure probability under certain conditions follows a binomial distribution, with system design influencing the failure probability for each shock.

Keywords: Failure probability, Binomial distribution, Shocks, $k-out-of-n$ system.

AMS Subject Classification: 62E99, 62E15, 62P30, 90B25

1. Introduction

The $k-out-of-n : G$ systems ($k \leq n$) under consideration consist of n identical components (each having the same probability of failure) that are independent (the failure of one component does not affect the others) and fail if at least $(n-k+1)$ components fail. Such systems are very important in fail-safe designs. Given their significance, many researchers have studied their reliability [1–7] and optimal design [8–12] under various conditions.

These types of redundant systems are often used in industrial, mechanical, and electronic fields, as well as in mechanisms subject to demanding conditions, such as those caused by voltage drops, pressure, high temperatures, and various other factors.

These conditions can expose $k-out-of-n$ systems to shocks of varying degrees, increasing the risk of rapid breakdown, especially when considered shocks are harmful (i.e., each shock destroys at least one system component). Hence, it is of great interest to both

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manufacturers and companies to determine the failure probability of the entire system after each shock, particularly in sensitive industries such as the aircraft industry, to plan and execute maintenance schemes.

The concept of cumulative damage shock models has been discussed when shocks occur according to the Poisson process [13,14], the renewal process [15,16], and more generalized stochastic processes, such as the birth process, Lévy process, and general counting process [17,20]. Many other stochastic models describing material damage due to shocks were presented in details [21]. The concept of damage is not limited to the industrial domain; indeed, it has been adopted by several other disciplines, such as the biological sciences. For example, we cite the work of Dolan et al. [22], who present an integrated stochastic model of DNA damage repair and gamma irradiation-induced cellular senescence, explaining how DNA repair dynamics and p53/p21 signaling control cell fate after DNA damage.

The analysis of damage models for redundant systems, such as $k-out-of-n$ systems and, more generally, systems consisting of multiple components, is more complicated given their their special design and composition. In particular, for the $k-out-of-n$ systems, several authors have focused on the magnitudes of shocks and have considered the system to fail when its strength exceeds a critical level in the case of identical and nonidentical components [23]. Eryilmaz and Devrim [24] considered the critical level to be the number of components destroyed by shocks. The number of destroyed components was modeled as a random variable, with each shock assumed to be harmful. The probability of system failure at the time of each shock was determined by calculating the total number of cases that cause the system to fail upon the occurrence of each shock. This paper deals specifically with binary systems, where each component can only be in one of two states: functioning or failed. We reconsider the same assumptions considered by Eryilmaz and Devrim [24] to devise a novel method for deriving a straightforward formula to calculate the failure probability of a $k-out-of-n$ system exposed to harmful shocks. This is achieved through a detailed examination of results obtained from algorithmic execution. Employing this method promises cost and time savings by reducing reliance on costly and complex computations. Furthermore, we prove that, under these assumptions, the failure probability of a $k-out-of-n$ system follows a binomial distribution.

The remainder of the paper is organized as follows: In Section 2, after introducing the necessary notations and abbreviations, we describe the shock model for the $k-out-of-n$ system. Section 3 is dedicated to determining the analytical expressions for failure cases and failure probabilities of the $k-out-of-n$ system under harmful shocks, with a $3-out-of-5$ system considered as a case study. In Section 4, we propose an algorithm to compute the probability of system failure at the time of each shock. Section 5 derives the formula for calculating the number of cases that cause system failure upon the occurrence of each shock and the corresponding probability. In Section 6, we discuss the importance of system design in avoiding early failures. The paper concludes with some remarks in Section 7.

2. Model description

The following notations will be used throughout this paper:

- Z_i : The random number of components destroyed by the i^{th} shock ($Z_0 = 0$ it implies that no component is considered faulty if there is no shock).
- Z : The total number of components destroyed at the time of the system failure, where $Z = \sum_{i=0}^{m} Z_i$ and m is the m^{th} shock.
- n_i : The number of cases where the system fails at the time of the i^{th} shock.

- p_i : The probability that system fails at the time of the i^{th} shock.

Consider a global system subjected to shocks, which sustains damage as a result of these shocks.
In reliability and system analysis, a shock is an external event that can impact a system's components, either instantaneously (e.g., power surges) or cumulatively (e.g., wear and tear). Shocks can cause damage, degrading the performance or reliability of the system. As an illustrative example, consider a telecommunications network where a lightning strike (shock) damages several routers (damage), potentially disrupting data flow.
As described in [14], let the random variables D_j $(j = 1, 2, \dots)$ denote the damage caused by the j^{th} shock, where $D_0 = 0$. It is assumed that the sequence D_j is non-negative, independent, and identically distributed.
Let $N(t)$ denote the random variable representing the total number of shocks up to time t $(t \geq 0)$. Then, define the random variable:

$$Z(t) = \sum_{j=0}^{N(t)} D_j \ ,$$

where $Z(t)$ represents the total damage at time t.
It is assumed that the system fails when the total damage exceeds a prespecified level K $(0 < K < \infty)$ for the first time.
The system under consideration is a k-out-of-n system with n identical components and is considered functional if, and only if, at least k components are operational, where $1 \leq k \leq n$.
Shocks occur at random times and are assumed to be harmful, implying that each shock destroys at least one component of the system. The total amount of damage suffered by the system at time t is given by the total number of failed components destroyed up to time t:

$$Z(t) = \sum_{j=0}^{N(t)} Z_j$$

The $k - out - of - n$ systems $(k \leq n)$ fails if at least $(n - k + 1)$ components fail which is equivalent to:

$$Z \equiv Z(t) \geq n - k + 1$$

Furthermore, since each shock affects at least one component, we can conclude that the maximum number of shocks that the system can support is $(n - k + 1)$.
To compute the probability that the system fails at the time of the i^{th} shock, where $i \in \{1, 2, \dots, n - k + 1\}$, as in Eryilmaz and Devrim [24], the key problem is to determine the number of cases that cause failure of the system upon the occurrence of each shock. For this purpose, we first identify all possible failure cases for a $k - out - of - n$ system under the assumptions described above.

3. Number of failure cases n_i and failure probabilities p_i of $k - out - of - n$ systems

In order to obtain the probability of the system failure at the time of each shock, we first need to determine the number of cases that causes failure of the system upon the occurrence of each shock. Let $\{Z_1, Z_2, \dots, Z_i\}$ represent the random number of components destroyed by the $\{1^{\text{st}}, 2^{\text{nd}}, \cdots, i^{\text{th}} shock\}$, and let $\{z_1, z_2, \dots, z_i\}$ be a realization of $\{Z_1, Z_2, \dots, Z_i\}$. For example, if Z_1 is the random variable modeling the number of components destroyed by the first shock, then z_1 refers to a specific observed value taken by Z_1. For instance, in

a system consisting of three components, z_1 can take the values 1, 2, or 3.
Let $\{n_1, n_2, \ldots, n_i\}$ denote the number of cases where the system fails at the time of the $\{1^{\text{st}}, 2^{\text{nd}}, \cdots, i^{th}\ shock\}$.
According to section 2, we can set:

$$\begin{cases} n_1 = \sum\limits_{cases} \big(z_1 \geq n-k+1\big) \\ \ \equiv Number\ of\ cases\ where\ z_1 \geq n-k+1 \\ n_2 = \sum\limits_{cases} \big(z_1 + z_2 \geq n-k+1; z_1 < n-k+1\big) \\ \ \equiv Number\ of\ cases\ where\ z_1 + z_2 \geq n-k+1\ and\ z_1 < n-k+1 \\ . \\ . \\ . \\ n_i = \sum\limits_{cases} \big(z_1 + \cdots + z_i \geq n-k+1; z_1 + \cdots + z_{i-1} < n-k+1\big) \\ \ \equiv Number\ of\ cases\ where\ z_1 + \cdots + z_i \geq n-k+1 \\ \ and\ z_1 + \cdots + z_{i-1} < n-k+1 \end{cases}$$

$$\begin{cases} p_1 = P\{Z(t) = Z_1 \geq n-k+1; z_1 > 0\} \\ = n_1 \Big/ \sum\limits_{i=1}^{n-k+1} n_i \\ p_2 = P\{Z(t) = Z_1 + Z_2 \geq n-k+1; z_1 < n-k+1\} \\ = n_2 \Big/ \sum\limits_{i=1}^{n-k+1} n_i \\ . \\ . \\ . \\ p_i = P\{Z(t) = \sum\limits_{j=0}^{i} Z_j = Z_1 + Z_2 + \cdots + Z_i \geq n-k+1; \sum\limits_{j=0}^{i-1} Z_j < n-k+1\} \\ = n_i \Big/ \sum\limits_{i=1}^{n-k+1} n_i \end{cases}$$

It is important to note that this formula for computing p_i is based on the assumption that the components are identical and fail independently; otherwise, it would not hold in its current form, and alternative approaches would be required.
For illustration, consider a 3-out-of-5 system subjected to shocks. Table 1 shows all possible failure cases, given that the maximum number of shocks for such a system is $n-k+1=3$. According to Table 1, we can notice that, for a $3-out-of-5$ system, there are three possibilities for failure after the accumulation of three shocks, six possibilities for failure after the accumulation of two shocks, and three possibilities for failure due to the first shock. Hence, we have $n_3 = 3$, $n_2 = 6$, and $n_1 = 3$.
The corresponding probabilities p_i are obtained as follows:
$p_3 = P\{Z = Z_1 + Z_2 + Z_3 \geq 3\} = n_3/(n_1+n_2+n_3) = 3/12 = 1/4$,
$p_2 = P\{Z = Z_1 + Z_2 \geq 3\} = n_2/(n_1+n_2+n_3) = 6/12 = 1/2$, and
$p_1 = P\{Z = Z_1 \geq 3\} = n_1/(n_1+n_2+n_3) = 3/12 = 1/4$

4. An algorithm for computing the number of failure cases and failure probabilities

Based on the results presented above, we describe, for fixed values of n and k, an algorithm that can be used to numerically compute the number of failure cases of a $k-$

TABLE 1. All possible failure cases for a $3-out-of-5$ system

Z_1	Z_2	Z_3	Z
1	1	1	$Z = Z_1 + Z_2 + Z_3$
1	1	2	$Z = Z_1 + Z_2 + Z_3$
1	1	3	$Z = Z_1 + Z_2 + Z_3$
1	2	-	$Z = Z_1 + Z_2$
1	3	-	$Z = Z_1 + Z_2$
1	4	-	$Z = Z_1 + Z_2$
2	1	-	$Z = Z_1 + Z_2$
2	2	-	$Z = Z_1 + Z_2$
2	3	-	$Z = Z_1 + Z_2$
3	-	-	$Z = Z_1$
4	-	-	$Z = Z_1$
5	-	-	$Z = Z_1$

$out-of-n$ system caused by the i^{th} shock and the corresponding failure probability p_i, where i belongs to $1, 2, \ldots, n-k+1$.

The algorithm was implemented on a PC with an Intel i7 processor and 16 GB of RAM. The code was written in MATLAB, and the computational time required for the algorithm for $n = 5$ and $k = 3$ ranged from a fraction of a second to a few seconds. However, for larger values (for example, $n = 10$ and $k = 6$), the time could increase to several seconds due to the exponential growth of the number of permutations.

Algorithm 1 An algorithm for computing the number of failure cases (n_i) due to the i^{th} shock and the corresponding failure probability (p_i)

Require: : $n \geq 1$; $0 < k \leq n$
Ensure: : Ncases: vector representing failure cases per shock.
Ensure: : P : vector representing the probability of the system failure per shock

$V \leftarrow \{1, 2, ..., n\}$;
$i \leftarrow n - k + 1$;
M $\leftarrow$ $permutations\_with\_replacement(V, i)$
while i ≥ 1 **do**
  while j $\leq RowCount(M)$ **do**
    if $Count(M[j, :]) < n - k + 1$ **then**
      $M \Leftarrow M.delete(j)$
    else$[Count(M[j, : -1]) > n]$
      $M \Leftarrow M.delete(j)$
    end if
    $j \leftarrow j + 1$
  end while
  $n_i \leftarrow RowCount(M)$;
  $Ncases \leftarrow concat(Ncases, n_i)$;
  $i \leftarrow i - 1$
end while
$P \leftarrow Ncases/Count(Ncases)$

Where:

- *permutations_with_replacement*(V, i): is a function that generates all possible tuples of length i where each element of the tuple can be any element from the vector V.
- *RowCount*(M): return the number of rows in the matrix M.
- *Count*$(M[j,:]$: count the sum of elements contained in the j^{th} row of the matrix M.
- *Count*$(M[j,:-1])$: count the sum of elements in the j^{th} row of the matrix M, excluding the last element of that row.
- *M.delete*(j) : remove the j^{th} row from the matrix M.
- *concat_vertically*$(Ncases, n_i)$: concatenates the new element n_i to the vector V.

In the following, we provide a concise description of the algorithm's steps, accompanied by an example to illustrate its application.

Detailed explanation of the algorithm steps

- **Line 1:** Initialize the vector $V = \{1, 2, \ldots, n\}$, representing the system components.
- **Line 2:** Set $i = n - k + 1$, which determines the initial tuple length for generating permutations with replacement.
- **Line 3:** Use the function `permutations_with_replacement(V, i)` to generate all possible tuples of length i from V.
- **Lines 4-16:** For each $i \geq 1$, refine the generated permutations M by filtering out invalid rows based on two criteria:
  - **Line 6:** Remove rows where the sum of elements is less than $n - k + 1$, ensuring that sufficient shocks occur.
  - **Line 8:** Remove rows where the sum of all elements except the last exceeds n, avoiding over-counting cases.
- **Line 13:** Count the remaining valid rows in M, which represent the number of failure cases (n_i) due to the i-th shock.
- **Lines 14-15:** Append n_i to the result vector `Ncases` and decrement i to process shorter tuple lengths.
- **Line 17:** Compute the probability vector P, where each element represents the failure probability due to the corresponding shock , which is equal to the number of failure cases (n_i) due to the i-th shock divided by the total number of cases.

An example to illustrate the algorithm Let $n = 4$, $k = 2$.

- **Step 1:** Initialize $V = \{1, 2, 3, 4\}$, $i = 4 - 2 + 1 = 3$.
- **Step 2:** Generate M as all permutations of length 3 with replacement from V:
$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ \vdots & & \\ 4 & 4 & 4 \end{bmatrix}.$$
  ($4^3 = 64$ rows initially).
- **Step 3:** Filter M using the conditions:
  - Remove rows where the sum of elements is less than $4 - 2 + 1 = 3$.
  - Remove rows where the sum of elements excluding the last exceeds 4.

  After filtering, let M' contain the valid rows.
- **Step 4:** Count the rows of M': $n_3 = \texttt{RowCount}(M')$. Append n_3 to `Ncases`.
- **Step 5:** Decrement i and repeat until $i = 1$.

Output At the end,

$$\texttt{Ncases} = [n_1, n_2, n_3, \dots] \quad \text{and} \quad P = \frac{\texttt{Ncases}}{\texttt{sum(Ncases)}}.$$

The following Table 2 gives an execution of the proposed algorithm for different values of n and k.

TABLE 2. Number of failure cases under shock for different values of k and n

k=1	n_1	n_2	n_3	n_4	n_5	n_6	n_7	k=2	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n=1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-
n=2	1	1	-	-	-	-	-	n=2	2	-	-	-	-	-	-
n=3	1	2	1	-	-	-	-	n=3	2	2	-	-	-	-	-
n=4	1	3	3	1	-	-	-	n=4	2	4	2	-	-	-	-
n=5	1	4	6	4	1	-	-	n=5	2	6	6	2	-	-	-
n=6	1	5	10	10	5	1	-	n=6	2	8	12	8	2	-	-
n=7	1	6	15	20	15	6	1	n=7	2	10	20	20	10	2	-
k=3	n_1	n_2	n_3	n_4	n_5	n_6	n_7	k=4	n_1	n_2	n_3	n_4	n_5	n_6	n_7
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
n=3	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
n=4	3	3	-	-	-	-	-	n=4	4	-	-	-	-	-	-
n=5	3	6	3	-	-	-	-	n=5	4	4	-	-	-	-	-
n=6	3	9	9	3	-	-	-	n=6	4	8	4	-	-	-	-
n=7	3	12	18	12	3	-	-	n=7	4	12	12	4	-	-	-

5. Formulas for computing the number of failure cases n_i and the corresponding failure probability p_i:

By analyzing the results presented in Table 2 given in Section 4, we observe a relationship between the number of failure cases caused by the i^{th} shock and the values of n, k and i (where i is the number of shocks). This relationship can be expressed as follows:

$$n_i = k\binom{n-k}{i-1}, \; for \; i \in \{1, 2, \cdots, n-k+1\}. \tag{1}$$

Since the maximum number of shocks the system can withstand is $n-k+1$, the total number of failure cases is:

$$\begin{aligned} \sum_{i=1}^{n-k+1} n_i &= k\Big[\binom{n-k}{0} + \binom{n-k}{1} + \cdots + \binom{n-k}{n-k}\Big] \\ &= k2^{n-k}. \end{aligned} \tag{2}$$

The number of failure cases of the $k-out-of-n$ system upon the occurrence of the i^{th} shock, as determined by Eryilmaz and Devrim [24], is given by:

$$\alpha(i) = \begin{cases} \sum\limits_{z_1=max(n-k+1,i)}^{n} \; \sum\limits_{z_2=i-1}^{min(n-k,z_1-1)} \binom{z_2-1}{i-2} & \text{for } i \geq 2, \\ k & \text{if } i = 1. \end{cases} \tag{3}$$

And the corresponding total number of failure cases is:

$$\sum_{i=1}^{n-k+1} \alpha(i) = k + \sum_{i=2}^{n-k+1} \sum_{z_1=max(n-k+1,i)}^{n} \sum_{z_2=i-1}^{min(n-k,z_1-1)} \binom{z_2-1}{i-2}. \tag{4}$$

In the following, we propose a simplified version of Eryilmaz and Devrim's [24] theorem, which facilitates more practical use in computing the number of failure cases for the $k-out-of-n$ system upon the occurrence of the i^{th} shock, as given by Eq. 1, and the total number of failure cases, as given by Eq. 2.

Proposition 5.1.

$$\sum_{z_1=max(n-k+1,i)}^{n} \sum_{z_2=i-1}^{min(n-k,z_1-1)} \binom{z_2-1}{i-2} = k\binom{n-k}{i-1}, \ for\ i \geq 2\ and\ n \geq k. \tag{5}$$

Proof. We have $1 \leq i \leq n-k+1$ then in Eq. 3 we can put:

$$z_1 = max(n-k+1, i) = n-k+1$$

And

$$min(n-k, z_1-1) = n-k$$

Then, $for\ i \geq 2$ the resulting $\alpha(i)$ becomes,

$$\begin{aligned}\alpha(i) &= \sum_{z_1=n-k+1}^{n} \sum_{z_2=i-1}^{n-k} \binom{z_2-1}{i-2} \\ &= k\sum_{z_2=i-1}^{n-k} \binom{z_2-1}{i-2}. \end{aligned} \tag{6}$$

Then we have to prove that:

$$k\sum_{z_2=i-1}^{n-k} \binom{z_2-1}{i-2} = k\binom{n-k}{i-1}. \tag{7}$$

For a fixed k, the proof of Eq. 7 is done by a recurrence on the number of components n .
It is clear that Eq. 7 is true for the first term $n = k$.
Let us assume that Eq. 7 is true for each $n > k$ and we have to prove that is true for $n+1$.
We have:

$$\begin{aligned}k\sum_{z_2=i-1}^{n-k+1} \binom{z_2-1}{i-2} &= k\Big[\sum_{z_2=i-1}^{n-k} \binom{z_2-1}{i-2} + \binom{n-k}{i-2}\Big] \\ &= k\Big[\binom{n-k}{i-1} + \binom{n-k}{i-2}\Big] \\ &= k\binom{n-k+1}{i-1}. \end{aligned} \tag{8}$$

Which completes the proof of Eq. 5. □

Proposition 5.2.

$$k + \sum_{i=2}^{n-k+1} \sum_{z_1=max(n-k+1,i)}^{n} \sum_{z_2=i-1}^{min(n-k,z_1-1)} \binom{z_2-1}{i-2} = k2^{n-k}, \ for\ n \geq k. \tag{9}$$

Proof. By the same way as described in the previous proof, Eq. 9 can be rewritten as follows,

$$k + k\sum_{i=2}^{n-k+1}\sum_{z_2=i-1}^{n-k}\binom{z_2-1}{i-2} = k2^{n-k}. \tag{10}$$

For a fixed k, the proof of Eq. 10 is also done by a recurrence on the number of components n.
It is clear that Eq. 10 is true for the first term $n = k$.
Let us assume that Eq. 10 is true for each $n > k$ and we have to prove that is true for $n + 1$.
We have:

$$\begin{aligned}
\sum_{i=1}^{n-k+2}\alpha(i) &= k + k\sum_{i=2}^{n-k+2}\sum_{z_2=i-1}^{n-k+1}\binom{z_2-1}{i-2} \\
&= k + k\sum_{i=2}^{n-k+1}\sum_{z_2=i-1}^{n-k+1}\binom{z_2-1}{i-2} + k\sum_{z_2=n-k+1}^{n-k+1}\binom{z_2-1}{n-k} \\
&= k + k\sum_{i=2}^{n-k+1}\sum_{z_2=i-1}^{n-k+1}\binom{z_2-1}{i-2} + k\binom{n-k}{n-k} \\
&= 2k + k\sum_{i=2}^{n-k+1}\sum_{z_2=i-1}^{n-k}\binom{z_2-1}{i-2} + k\sum_{i=2}^{n-k+1}\binom{n-k}{i-2} \\
&= k + k2^{n-k} + k\sum_{i=2}^{n-k+1}\binom{n-k}{i-2} \\
&= k2^{n-k} + k\Big[1 + \binom{n-k}{0} + \binom{n-k}{1} + \cdots + \binom{n-k}{n-k-1}\Big] \\
&= k2^{n-k} + k2^{n-k} \\
&= k2^{n-k+1}.
\end{aligned} \tag{11}$$

And then Eq. 9 is also proved. □

Additionally, for $i = 1$ we have:

$$n_1 = k\binom{n-k}{0} = k. \tag{12}$$

which is equal to the value of $\alpha(1) = k$ considered by Eryilmaz and Devrim [24].

Theorem 5.1. *Let S be a random variable representing the failure probability of a $k-out-of-n$ system under harmful shocks. Then, we have:*

$$S \sim Bin(n-k, \frac{1}{2}). \tag{13}$$

And the probability that the system fails at the time of the occurrence of the i^{th} shock $(i = 1, 2, \ldots, n-k+1)$ is:

$$P(S = i) = \binom{n-k}{i-1}\Big(\frac{1}{2}\Big)^{i-1}\Big(\frac{1}{2}\Big)^{n-k-i+1}. \tag{14}$$

Where, $Bin(n-k, \frac{1}{2})$ refers to the binomial distribution with parameters $n-k$ and $p = 1/2$.

Proof. The probability that the system fails at the time of the i^{th} shock can be calculated as follows:

$$\begin{aligned} p_i &= P(S=i) \\ &= \frac{\text{total number of failure cases upon the occurrence of the i-th shock}}{\text{total number of failure cases upon the occurrence of all shocks}} \\ &= \frac{n_i}{\sum\limits_{i=1}^{n-k+1} n_i}. \end{aligned}$$

Moreover, according to the results from Propositions 5.1 and 5.2, we have: $n_i = k\binom{n-k}{i-1}$ and $\sum\limits_{i=1}^{n-k+1} n_i = k2^{n-k}$ And then,

$$\begin{aligned} p_i &= \frac{k\binom{n-k}{i-1}}{k2^{n-k}} \\ &= \binom{n-k}{i-1}\frac{1}{2^{n-k}} \\ &= \binom{n-k}{i-1}\left(\frac{1}{2}\right)^{i-1}\left(\frac{1}{2}\right)^{n-k-i+1}. \end{aligned}$$

For every $i \in \{1, \cdots, n-k+1\}$.
And, it is clear that

$$\sum_{i=1}^{n-k+1} p_i = 1.$$

□

6. Impact of the $k - out - of - n$ system design on failure probabilities p_i:

$k - out - of - n$ systems are often used in highly demanding environments such as aeronautics, electronics, satellites, and, more generally, in industries exposed to severe shocks. In such cases, there is a high probability that the system may fail upon experiencing the first shocks. One solution to this type of problem involves employing a specific design for such systems to minimize the risk of failure from the initial shocks.
In this section, graphical results and interpretations are provided to illustrate the impact of $k - out - of - n$ system design on the failure probability, p_i, following the occurrence of the i^{th} shock.
Figure 1 illustrates the probability p_1 that a k-out-of-n system fails at the time of the first shock for different values of k and n. It is evident that, for a fixed k, the probability p_1 decreases as the total number of components n increases. A plausible interpretation of this decrease is that, when n is large enough, there is a higher likelihood that k components will survive the first shock.
In reality, in reliability theory, redundancy in $k - out - of - n$ systems enhances robustness by allowing the system to function as long as at least k components remain operational. The probability of the system failure during the first shock decreases as the number of components n increases (with k constant), as additional components provide greater resilience against initial failure.
Example: Consider a 3-out-of-5 system, where the system fails only if more than two components fail.

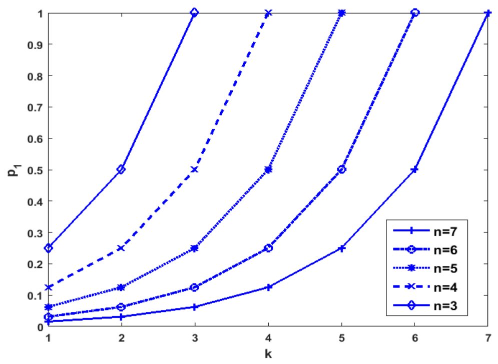

FIGURE 1. Failure probability p_1 for different values of k and n

- For a system with 5 components, the probability of failure during the first shock (i.e., more than two components failing) can be computed using the binomial distribution. The failure probability for 3 or more components failing would be:

$$P(\text{failure}) = P(3 \text{ fail}) + P(4 \text{ fail}) + P(5 \text{ fail}).$$

- For comparison, consider a $3 - out - of - 3$ system (no redundancy). The system will fail if any one of the three components fails, so the failure probability is simply the probability of at least one component failing, which is much higher than in the $3 - out - of - 5$ case.

According to this interpretation, to reduce the likelihood of failure resulting from the first shock, it is advisable to increase the number of components in a $k - out - of - n$ system, regardless of the value of k. This assumption is supported by Figure 2, which describes, for a fixed k, the evolution of the failure probability (p_1) as the number of system components increases.

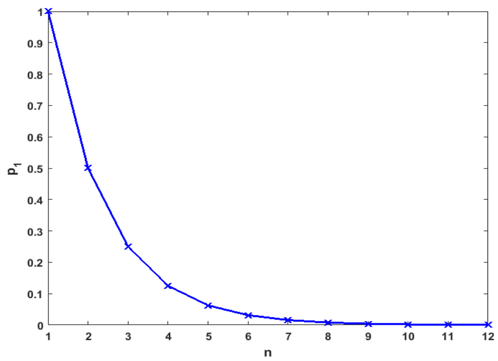

FIGURE 2. Variation of p_1 with increasing number of components

Figure 3 illustrates, for a fixed $k = 3$, the shapes of the failure probability curves (p_i) at the time of the i^{th} shock ($i = 2, 3, \cdots, 7$) for different values of n. It is clear that there exists a specific number of components for which the probability $(p_i)_{i \geq 2}$ is maximal. This result has an interesting interpretation: it is not always evident that adding components to the k-out-of-n system will decrease the probability $(p_i)_{i \geq 2}$ of the system failure.

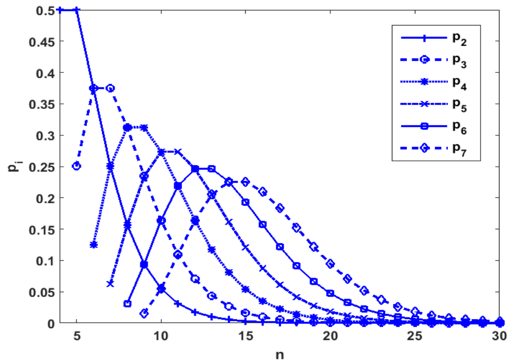

FIGURE 3. The variation of $(p_i)_{i\geq 2}$ with increasing number of components

7. Conclusions

This paper presents a new approach aimed at simplifying the computation of failure probability in $k-out-of-n$ systems subjected to harmful shocks. Through meticulous analysis of algorithmic results, we have proposed a simplification of the existing formulas to compute the failure probability, reducing the need for complex computations and saving both time and resources. Additionally, our findings suggest that, under certain assumptions, the failure probability of such systems follows a binomial distribution. Importantly, we highlight the influence of system design on the corresponding failure probability. Thus, the developed results can serve as a basis for selecting a suitable $k-out-of-n$ system that minimizes the risk of breakdowns during the initial shocks.

By providing an efficient and effective method for assessing failure probability, our approach enhances the safety and reliability of such systems while reducing costs and computational burdens. While this study focuses on binary systems, future research could explore the generalization of the proposed approach to multi-state systems, considering the additional complexity introduced by multiple component states.

On the other hand, future research could explore the case of non-identical components in k-out-of-n systems. Incorporating heterogeneous components, which contribute unequally to the system's reliability, presents additional challenges, as highlighted by previous studies such Shaked and Shanthikumar [25], Rezapour and Alamatsaz [26], Balakrishnan et al. [27] and Bhattacharyya et al. [28]. These challenges involve stochastic comparisons and the need for modified reliability models to account for varying component characteristics.

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