TWMS J. App. and Eng.Math.V.15, N.11, 2025, pp. 2638-2651

A LABELING APPROACH TO VERTEX N-MAGIC WEIGHTED TOTAL GRAPHS WITH VERTEX-ODD VALUES

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Abstract. The current text discusses the concept of odd vertex N-magic total graphs, a novel idea in the realm of vertex N-magic-type graphs. This idea is defined by N- magic with distinct degrees and identifies odd vertex labels relevant to this property. This manuscript offers a novel approach to labeling graphs, focusing on a specific type of graph. It outlines a method for creating labeling functions that meet specific requirements for vertex weights and magic constants.

This study explores the practical application of vertex N-magic labeling for modeling employee skills and extracurricular activity networks, where a graph-based approach is employed to examine the relationships and capabilities of employees within replicated corporate settings. Graph theory's versatility is demonstrated through its application in both theoretical research and real-world data analysis, highlighting the effectiveness of vertex N-magic total labeling. It concludes with an examination of a specific approach for utilizing the proposed labeling along with an unresolved issue.

Keywords: Vertex N-magic, (n,t)-kite graph, Weighted graph, Banana Tree.

AMS Subject Classification: 05C78, 05C90

1. Introduction

This research exclusively examines an undirected graph with unique vertex degrees. The degree of each vertex in a simple, finite, undirected graph G with p vertices and q edges is distinct, and the number of these distinct degrees is represented by N. The edge set and vertex set of a graph are denoted by $E(G)$ and $V(G)$ respectively. $N(v)$ is a set of adjacent vertices for a vertex v. Standard graph notations are employed in this study. Graph labeling involves labeling elements of a graph, like vertices and edges, assigned to a labeled sequence of numbers. A comprehensive overview of graph labeling is provided in [5], which covers various label types. The concept of magic labeling of graphs was invented by Sedlacek [14]. This was followed by the development of vertex-magic total labeling by MacDougall et al [7]. The study of magic graphs, as detailed in [8], provides a comprehensive overview. In [4, 15], tree graphs with vertex magic total labeling were discussed, delving into the intricacies of wheels in [13]. [1, 2, 3, 6, 10] and [16] provided

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§ Manuscript received: October 28, 2024; accepted: February 24, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.11;

explanations of vertex-magic labeling for regular and non-regular graphs, as well as a disjoint union of graphs, respectively. MacDougall, Miller, and Sugeng were responsible for the development of super-vertex-magic total labeling.

The pioneering work on vertex N-magic total labeling was done by Marimuthu and Kumar, as outlined in [9]. A vertex N-magic total labeling is a one-to-one map that takes the function from the set of vertices and edges of G, denoted by $V(G) \cup E(G)$, onto the consecutive integers $1, 2, ..., p+q$, with the property that the number of distinct degrees N is equal to the number of distinct magic constants k_i for $i \in \{1, 2, ..., N\}$, and the magic constants k_i must be in a strictly increasing order $k_1 < k_2 <, ..., < k_i$ for some $i \in \{1, 2, ..., N\}$. Here, N represents the number of distinct degrees of the vertices in a simple, finite, undirected graph G, whose vertices and edges are p and q, respectively. Moreover, it k_i represents the weighted sum of every vertex $u \in V(G)$ for some $i \in \{1, 2, ..., N\}$. Nishanthini and Jeyabalan's research on vertex N-magic total labeling graphs focused on establishing sharp bounds, specifically by examining the bijective function that maps vertices and edges to integers. In [11], Sharp bounds on Vertex N-magic total labeling graphs, I provide strict mathematical bounds for vertex N-magic total graphs and extend results to disjoint union graphs. Provides theoretical results on the upper and lower bounds of magic constants. Explores the labeling of various graph families, including trees and disjoint union graphs. Both refer to the importance of magic constants and their ordering in graph labeling. Specializes in theoretical bounds for various graph families. Emphasizes the mathematical strictness of proofs for establishing upper and lower limits of magic constants. Focused on the mathematical structure and application of Strong vertex N-magic total labeling in [12].

This manuscript presents a novel labeling technique called odd vertex N-magic total labeling for graphs. Total labeling with an odd vertex N-magic graph is referred to as odd vertex N -magic total labeling if all vertices of G are assigned only odd integer values. A graph that admits an odd vertex N-magic total labeling is called an odd vertex-magic total or odd VNMT graph. We have introduced a new labeling type in which vertex labels are odd and suggest a stepping process for building such graphs and highlighting constructive methods and applications in real life of odd vertex N-magic labeling. Specializes in a new graph labeling type known as odd vertex N-magic total labeling. Highlights labeling graphs with all vertex labels as odd integers. Introduces applications of personality development in employees in the form of extracurricular activity-related applications. Specializes in odd vertex N-magic total labeling, a special case in which only odd integer values are used for vertex labels, and introduces a variety of graph classes and shows how the odd vertex N-magic property can be established. These findings highlight the intricate relationships between vertex and edge structures and the corresponding labeling rules.

The following are the objectives of this article: (i) To assemble the graph labeling in various ways and offer a simple, effective stepping procedure for assessing Vertex N-magic total graphs with odd labels. (ii) To strengthen the weighted graphs using magic constants. (iii) To acquire magical constants by employing graph theoretical weighted graphs. The following result will be beneficial in proving some theorems.

Lemma 1.1. *The magic constant value for the vertex magic labeling $k > p + q$.*

Odd-Vertex N-magic Stepping Procedure

Consider a $G(p, q)$ graph with p vertices and q edges. The graph is divided into two sets denoted by vertices and edges. Initially, we determine the number of distinct degrees of a graph.

Step 1: Let v_i be the number of vertices in V. Label the vertices within $1, 2, 3, ..., (p+q)$.

Step 2: The remaining vertex labels were eliminated to compute edge labels. The number of edges in E is for each n.
Step 3: Decide the weighted sum of each vertex in $V(G)$ which provides a magic constant value k_i for all $i \in \{1, 2, ..., N\}$. Examine the magical constants for Vertex N-magic must be in a strict ascending order $k_1 < k_2 <, ..., k_i$ for all $i \in \{1, 2, ..., N\}$.
Step 4: The values at the vertex and edge labels are swapped whenever there is a discrepancy. Rearrange every vertex and edge on the graph and proceed to step 1.

This manuscript analyzes the connection between extracurricular participation and the development of specific skills in a learning context. Based on the developmental theory also argues that participation in extracurricular activities enhances overall cognitive development and academic success. The study focuses on a diverse group of employees, encompassing various backgrounds and skill sets, including personal interests.

Definition 1.1. *A magic labeling for a graph assigns integers to vertices and edges, with the property that the sum of the labels on an edge is equal to the sum of its endpoints' labels.*

Definition 1.2. *Vertex labeling assigns numerical identifiers to vertices and edges in a graph. The sum of the labels assigned to a vertex and the labels of its connected edges determines its magic number.*

Definition 1.3. *A path graph can be constructed with n vertices and $n-1$ edges, where each vertex and edge forms a line, for any integer n greater than or equal to 3.*

Definition 1.4. *A star graph $k_{1,n}$ can connect n vertices to a single isolated vertex with $n+1$ edges.*

Definition 1.5. *Generate a bi-star graph with $2n+2$ vertices and $2n+1$ edges by linking two star graphs of order n with a connection.*

Definition 1.6. *Each cycle of vertices is connected to one isolated vertex via edges, resulting in the sun graph C_n with $2n$ vertices and edges.*

Definition 1.7. *A graph that connects each n copy P_2 to the same root vertex is known as a $(n, 2)$-banana tree.*

Definition 1.8. *A (n, t) kite graph with $n+t$ vertices and edges is a cycle with a path connecting one of its vertices to the tail of the cycle.*

Definition 1.9. *The corona product of the graph P_n and the complete graph K_1, denoted by $P_n \odot K_1$, results in a graph where, for a path of length n, each vertex is connected to a pendant vertex with $2n$ vertices and $2n-1$ edges.*

2. Odd Vertex N-magic Total Labeling of Star, Sun, Banana Tree and (n, t)-Kite Graphs

In this part, I proved that VNMT labeling graphs with vertex odd values have specific properties and showed how to build such graphs using scheme constructs. In this section, I analyze the possibility of this labeling for a set of particular graphs, including the star graph, sun, banana tree, bi-star, and the (n, t)-kite graph.

Theorem 2.1. *Let G be any graph except for a path of even length $p \geq 6$. If the number of points exceeds the number of edges, then G admits an odd VNMT labeling.*

Proof. Assume it G is an odd VNMT graph that p is less than or equal to q. Then there exists an odd VNMT labeling f, with the vertex labels ranging from 1 to $2p-1$. Let $v_i \in V(G)$ be the vertex such that the labeling is $2p-1$. We demonstrate that the labels of all vertices are always less than the total number of graphs.

If not, then $f(v_i) \geq p+q > p+p = 2p$, contrary to the fact that the labels of vertex v_i are $2p-1$ and $p \leq q$. Thus, $f(v_i) < p+q$. Hence, G admits odd VNMT labeling if the number of points exceeds the number of edges. □

Corollary 2.1. *Every tree admits an odd vertex N-magic total except for the even length $n \geq 6$ of the path graph P_n.*

By the theorem above, it is clear that trees permit odd vertex N-magic. Consider the following path graph, which has the following vertices and edge sets:

$V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{v_i v_{i+1}, i = 1, 2, ..., n-1\}$.

As a result, our main goal is to affirm that vertices with the same degrees have the same magical constant values k_i for each $i \in \{1, 2, ..., N\}$.

All the vertices in the path graph have two distinct degrees. Define a function from $V(P_n) \cup E(P_n)$ onto $\{1, 2, ..., 2n-1\}$ as follows: For the odd length n of path graph,

Step 1: Assign the initial and end vertices to 1 and 3, respectively. The second vertex from left to right by $2n-1$ and v_{n-1} by 5.

Step 2: The first edge v_1v_2 by $n+1$ and edge v_{i+2} labeled by $2n-1-2i$, $i = 1, 2, ..., n-3, n > 3$. The edges are $v_{2i}v_{2i+1}$ labeled by $2i$ for $i = 1, 2, ..., \frac{n-3}{2}$ and $v_{2i+1}v_{2i+2}$ labeled $n+1+2i$, $i = 1, 2, ..., \frac{n-5}{2}$ respectively. Edge $v_{n-2}v_{n-1}$ by $2n-2$ for $n \geq 5$. The final edge $v_{n-1}v_n$ is $n-1$.

Step 3: The magic constants are $n+2$ and $3n+2$ respectively.

Note 1. *Path graph P_4 with the odd vertex N-magic contains magic constants* 7 *and* 13. *With $n \geq 6$, the labels swapped more frequently, but we could not set it up to accept the odd vertex N-magic labeling for the even length n. Two distinct magic constants, k_1 and k_2, are necessary. However, label rearrangement contradicts this condition. When rearranging labels, we obtain only odd vertex values with vertex N-magic.*

2.1. **Exploring Odd Vertex N-Magic Total Labeling in Specialized Graphs.**

Theorem 2.2. *Every $K_{1,n}$ with n greater than or equal to* 2 *is an odd vertex N-magic total graph.*

Proof. Let v be the star's central vertex, and v_i for each $i = 1, 2, .., n$ be the star graph's pendant vertices, considering $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ and and $E(K_{1,n}) = \{vv_n = e_i, i = 1, 2, ..., n\}$. Define the one-one and onto function as ψ to $\{1, 2, ..2n+1\}$ as follows:

Step 1: Initialize labels v_1 to 1. Assign $2i-1$ to all the vertices v_i for each $i = 2, 3, ..., n$ of $K_{1,n}$ and the remaining central vertex v, labelled $2n+1$.

Step 2: Next, we place the labels for edges vv_i by $2(n+1-i)$ for each $i = 1, 2, ..., n$.

Step 3: We discover that the weight persists $n(n+3)+1$ for the central vertex and $2n+1$ for n vertices. The magic constants are $2n+1$ and $n(n+3)+1$. □

Theorem 2.3. *For any n greater than or equal to* 3, *a Sun graph C_n^+ possesses an odd vertex N-magic labeling.*

Proof. Let the vertex set and edge set of sun graph C_n^+ be $\{u_1, u_2, ...u_n, v_1, v_2, ...v_n\}$ and $\{u_iv_i; 1 \leq i \leq n\} \cup \{v_iv_{i+1}, 1 \leq i \leq n-1\} \cup v_nv_1$.

For each $1 \leq i \leq n$, u_i is the vertices of degree one, and v_i is the vertices of cycle c_n. Define the bijective function ψ to $\{1, 2, ..., 4n\}$ as follows:

Step 1: Designate the labels $2i-1$ to the vertices v_i for each $1 \leq i \leq n$ and $4n-1-2i$ to the vertices u_i for each $1 \leq i \leq n-1$. Vertex u_n should be labeled as $4n-1$.
Step 2: The remaining labels of the Sun graph are as follows:

$$\begin{aligned} \psi(u_i u_{i+1}) &= 2i+2; \qquad 1 \leq i \leq n-1 \\ \psi(u_1 u_n) &= 2 \\ \psi(u_i v_i) &= 4n-2i+2; \qquad 1 \leq i \leq n \end{aligned}$$

Step 3: The magic constants are $4n+1$ and, $8n+3$ respectively. □

Theorem 2.4. *If a banana tree $Ban_{n,2}$ has at least two vertices, then it has an odd vertex N- magic total labeling.*

Proof. Let the vertices and the edges of the $(n, 2)$-banana tree be as follows:

$V(Ban_{n,2}) = \{v_1, v_2, ..v_n, w_1, w_2, ...w_n\}$ and $E(Ban_{n,2}) = \{v_i w_i; 1 \leq i \leq n\} \cup \{vv_i; 1 \leq j \leq n\}$ respectively. Consider v_i and w_j as the path's vertices for $1 \leq i \leq n$ and v as the root vertex. Define the bijective function $\psi : V(Ban_{n,2}) \cup E(Ban_{n,2}) \rightarrow \{1, 2, ..., 4n+1\}$ as follows:

Designate the labels $4n+3-2i$ to the vertices w_i and the edges $v_i w_i$ by $2i$ for each $1 \leq i \leq n$.
Case i: n is even and $n \geq 2$
Step 1: Label the next vertex v as $n+1$, while the depiction of the other vertices follows:

$$\psi(v_i) = \begin{cases} n+2i+1; & 1 \leq i \leq \frac{n}{2} \\ 2i-n-1; & \frac{n}{2}+1 \leq i \leq n \end{cases}$$

Step 2: The edge labeling of the graph is defined here.

$$\psi(vv_i) = \begin{cases} 4n-4i+2; & 1 \leq i \leq \frac{n}{2} \\ 6n-4i+4; & \frac{n}{2}+1 \leq i \leq n \end{cases}$$

Step 3: The magic constants $k_1 = 4n+3$ and $k_2 = 5n+3$ respectively. The weight calculation for the central vertex v is as follows:

$$\begin{aligned} k_3 &= n+1+\sum_{i=1}^{\frac{n}{2}}(4n-4i+2)+\sum_{i=\frac{n}{2}+1}^{n}(6n-4i+4) \\ &= n+1+\sum_{i=1}^{n}(6n+4)-\sum_{i=1}^{n}4i-\sum_{i=1}^{\frac{n}{2}}(2n+2) \\ &= n+1+n.(6n+4)-\frac{4n(n+1)}{2}-\frac{n.(2n+2)}{2} \\ &= 3n^2+2n+1 \end{aligned}$$

Case ii: n is odd and $n \geq 3$
Step 1: Designate the label $2n+1$ to the vertex v and the remaining vertices labelled in the following manner:

$$\psi(v_i) = \begin{cases} n+2i; & 1 \leq i \leq \frac{n-1}{2} \\ 2i-n; & \frac{n+1}{2} \leq i \leq n \end{cases}$$

Step 2: The remaining edge labeling is as follows:

$$\psi(vv_i) = \begin{cases} 4n-4i+2; & 1 \leq i \leq \frac{n-1}{2} \\ 6n-4i+2; & \frac{n+1}{2} \leq i \leq n \end{cases}$$

Step 3: The magic constants $k_1 = 4n+3$ and $k_2 = 5n+2$ respectively. The method for determining the weight of the central vertex v is as follows.

$$\begin{aligned}
k_3 &= (2n+1) + \sum_{i=1}^{\frac{n-1}{2}}(4n-4i+2) + \sum_{i=\frac{n+1}{2}}^{n}(6n-4i+4) \\
&= (2n+1) + \sum_{i=1}^{n}(6n-4i+2) - \sum_{i=1}^{\frac{n-1}{2}} 2n \\
&= n+1+n.(6n+2) - \frac{4n(n+1)}{2} - \frac{2n.(n-1)}{2} \\
&= 3n^2+3n+1
\end{aligned}$$

The magic constant for the central vertex v is $3n^2+3n+1$. □

Theorem 2.5. *An (n,t)- kite graph admits an odd vertex N-magic total labeling for $n \geq 3$ and $t \geq 2$.*

Proof. Let the vertices and the edges of $G = (n,t)$-kite graph be described as follows:

$V(G) = \{v_1, v_2, ..v_n, w_1, w_2, ...w_t\}$ and $E(G) = \{v_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{w_jw_{j+1}; 1 \leq j \leq t-1\} \cup \{v_1w_t\}$. Consider the cycle's vertices by v_i for $1 \leq i \leq n$. And w_j be the path's vertices for $1 \leq j \leq t$.

Define the bijective function $\psi : V(G) \cup E(G) \rightarrow \{1, 2, ..., 2(2n+t)\}$ as follows:
If $n \geq 3$ and $t \geq 2$ then we Consider the following cases:
Case(i) If n is odd and t is even
Step 1: Label the vertex v_1 by $2n+2t-1$ and assign $2n+1-2i$ to the vertices v_i of cycle for each $2 \leq i \leq n$. The vertices w_j labelled by $2n+2t+1-2j$ for $2 \leq j \leq t$.
Step 2: Designate label t for edge v_1w_t. Label the vertices and edges with precise information in the subsequent sub-cases. Two subcases $n > t$ and $n < t$ are considered.
Subcase(a) $n > t$
Label the remaining vertex w_1 and the edges in the following manner:

$$\begin{aligned}
\psi(w_1) &= n+t \\
\psi(v_iv_{i+1}) &= \begin{cases} n+i; & i = 1, 3, ..., n-2 \\ 3n-1+i; & i = 2, 4, ..., n-1 \end{cases} \\
\psi(v_1v_n) &= 2n \\
\psi(w_jw_{j+1}) &= \begin{cases} n+t+j+2; & j = 1, 3, ..., t-1 \\ j; & j = 2, 4, ..., t-2 \end{cases}
\end{aligned}$$

The odd vertex N-magic property of the $(9,8)$-kite graph is illustrated in Figure 1.

Subcase(b) $n < t$
Put the following labels on the remaining vertex w_1 and the edges:

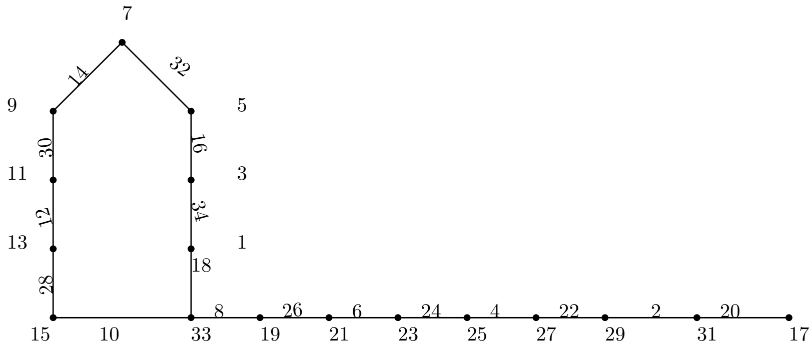


FIGURE 1. Odd Vertex N-magic for (9,8)-kite graph

$$
\begin{aligned}
\psi(w_1) &= n+t-2 \\
\psi(v_iv_{i+1}) &= \begin{cases} n+2+i; & i=1,3,...,n-2 \\ 3n+3+i; & i=2,4,...,n-1 \end{cases} \\
\psi(v_1v_n) &= n+t+1 \\
\psi(w_jw_{j+1}) &= \begin{cases} 2t+j+1; & j=1,3,...,t-1 \\ j; & j=2,4,...,t-2 \end{cases}
\end{aligned}
$$

Step 3: The magic constants are $2n+2t+3$,$6n-1$, and $5n+3t$ if $n>t$.
The magic constants are $2t+2n+1$, $6n+5$, and $4n+4t+3$ if $n<t$.

Case(ii) If n is even and t is odd

Step 1: Assign the vertices w_j labelled by $2n+2t+1-2j$ for each $1 \leq j \leq t$.

Step 2: The edges v_1w_t labelled by $t+1$ and $w_{t-1}w_t$ by $n+2t$, respectively. The sub-cases below provide the descriptions for the remaining vertex and edge labels.

We consider the following two subcases $n>t$ and $n<t$.

Subcase(a) $n>t$

Indicate the remaining vertices v_i for each $i=1,2,...,n$ and its edges with the following labels:

$$
\begin{aligned}
\psi(v_i) &= n+t+2-2i; & 1 \leq i \leq n \\
\psi(v_iv_{i+1}) &= \begin{cases} 3n+i-1; & i=1,3,...,n-1 \\ n+i; & i=2,4,...,n-2 \end{cases} \\
\psi(v_1v_n) &= 2n \\
\psi(w_jw_{j+1}) &= \begin{cases} j+1; & j=1,3,...,t-2 \\ 2t+j+2; & j=2,4,...,t-3 \end{cases}
\end{aligned}
$$

Subcase(b) $n<t$

Adhere to these designations for the remaining vertices v_i for each $i=1,2,...,n$ and its edges:

$$\begin{aligned}
\psi(v_i) &= n+t-2i; && 1 \leq i \leq n\\
\psi(v_iv_{i+1}) &= \begin{cases} 3n+i+3; & i=1,3,...,n-1\\ n+i+3; & i=2,4,...,n-2 \end{cases}\\
\psi(v_1v_n) &= 2(n+1)\\
\psi(w_jw_{j+1}) &= \begin{cases} j+1; & j=1,3,...,t-2\\ 2t+j; & j=2,4,...,t-3 \end{cases}
\end{aligned}$$

Step 3: The magic constants are $2n+2t+1$, $5n+t$, and $6n+2t+1$ if $n>t$.
The magic constants are $2n+2t+1$, $5n+t+4$ and $6n+2t+5$ if $n<t$.
Case (iii) $n=t$**=odd**
Step 1: Assign the labels $n+t-2i+1$ to the vertices v_i for each $1 \leq i \leq n$. The vertices w_j are labelled by $2n+2t-j-2$ for each $1 \leq j \leq t$.
Step 2: All edges of the graph are labeled as follows:

$$\begin{aligned}
\psi(v_1w_t) &= t+1\\
\psi(v_iv_{i+1}) &= \begin{cases} n+i+2; & i=1,3,...,n-2\\ 2n+t+i+1; & i=2,4,...,n-1 \end{cases}\\
\psi(v_1v_n) &= n+t+2\\
\psi(w_jw_{j+1}) &= \begin{cases} j+1; & j=1,3,...,t-2\\ n+t+j+2; & j=2,4,...,t-3 \& t \neq 3 \end{cases}\\
\psi(w_{n-1}w_n) &= 3t+1
\end{aligned}$$

Step 3: The magic constants are $2n+2t-1$, $4n+2t+3$ and $3n+3t+5$.
Case(iv) $n=t$**=even**
The $(8,8)$-kite graph with an odd vertex N-Magic is shown in figure 2.

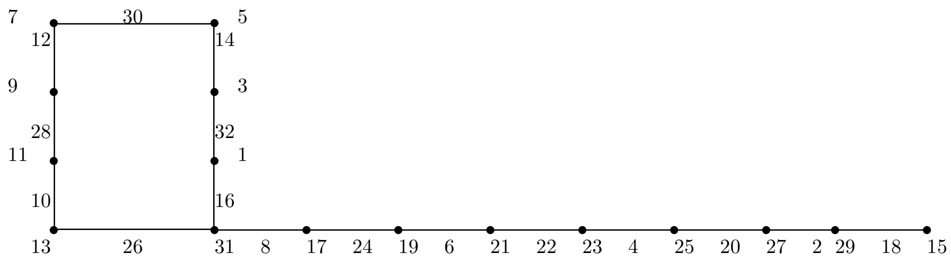


FIGURE 2. Odd Vertex N-magic for (8,8)-kite graph

Step 1: Designate vertices v_1 and w_1 as $2n+2t-1$ and $n+t-1$, respectively. The remaining vertices v_i are represented by $n+t-2i+1$ for each $2 \leq i \leq n$ and w_j by $2n+2t-2j+1$ for each $2 \leq j \leq t$.
Step 2: Allocate the labels $2n$ to the edge v_1v_n and the edge v_1w_t by t. The edge $w_{n-1}w_n$

is labelled as $3t$. The discourse below delves deeper into the remaining edges.

$$\psi(v_iv_{i+1}) = \begin{cases} 3n+i+1; & i=1,3,...,n-1 \\ n+i; & i=2,4,...,n-2 \end{cases}$$

$$\psi(w_jw_{j+1}) = \begin{cases} 2n+j+1; & j=1,3,...,t-3 \\ j; & j=2,4,...,t-2 \end{cases}$$

Step 3: The magic constants are $3n+t+1$, $5n+t+1$ and $7n+3t+1$. □

Theorem 2.6. *For any odd positive integer $n \geq 3$, the bistar $B_{n,n}$ is an odd VNMT graph.*

Let the vertex and edge sets be defined as follows:
$V(B_{n,n}) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_{2n}, u, v\}$ and $E(B_{n,n}) = \{uu_i; 1 \leq i \leq n\} \cup uv \cup \{vv_i, 1 \leq i \leq n\}$. Define the bijective function $\psi : V(B_{n,n}) \cup E(B_{n,n}) \to \{1, 2, ..., 2(2n)+3\}$ as follows:
Step 1: Allocate the label $4n+1$ to the vertex u and $4n+3$ to vertex v. The rest of the vertices are designated as follows: $\psi(u_i) = \{1, 7, 9, 15, 17, ..., 4n-3\}$ and $\psi(v_i) = \{3, 5, 11, 13, 19, ..., 4n-1\}$.
Step 2: Allocate the labels for the edges by $\psi(uv) = 2$, $\psi(uu_i) = \{4n+2, 4n-4, 4n-6, 4n-12, ...30, 22, 16, 14, 8, 6\}$ and $\psi(vv_i) = \{4n, 4n-2, 4n-8, 4n-10, 4n-16, 20, 18, 12, 10, 4\}$.
Step 3: The method to obtain the sum of the weighted vertices is outlined below.$k_1 = 4n+3$ Now the weighted sum of the pendant vertices u_i and v_i is $k_1 = 4n+3$ for each $1 \leq i \leq n$. The weight of central vertices is obtained by the sum of that label $4n+3$ and its endpoints.

Theorem 2.7. *Let G be any graph by attaching the terminus of degree 2 of a $P_n \odot K_1$ graph to a pendant vertex. Then obtain an odd vertex N-magic graph with three different magic constants.*

Proof. In a graph G, a pendant vertex u is connected to the $P_n \odot K_1$ graph's degree 2 endpoint, and the set of vertices and edges is $\{u_1, u_2....u_n\} \cup \{v_1, v_2, v_3, ..., v_n\} \cup u$ and edges $\{u_1v_1, u_2v_2, ..., u_nv_n\} \cup \{u_iu_{i+1} \ : 1 \leq i \leq n-1\}1 \cup uu_1$.

Define a total bijective map $\psi : V \cup E \to \{1, 2, ..., 4n+1\}$ whose labelings are as follows:
Step 1: The pendant vertex u must be labeled by $2n+1$ and the vertex v_i by $2i-1$ for $i = 1, 2, ..., n$ respectively. The remaining vertices $\psi(u_1) = 2n+3$, u_2 are labeled by $4n+1$ and the remaining consecutive vertices u_i are labeled by subtracting the value two from $4n+1$ for each $i = 3, ..., n$.
Step 2: Compute the remaining edge labels by the following scheme:

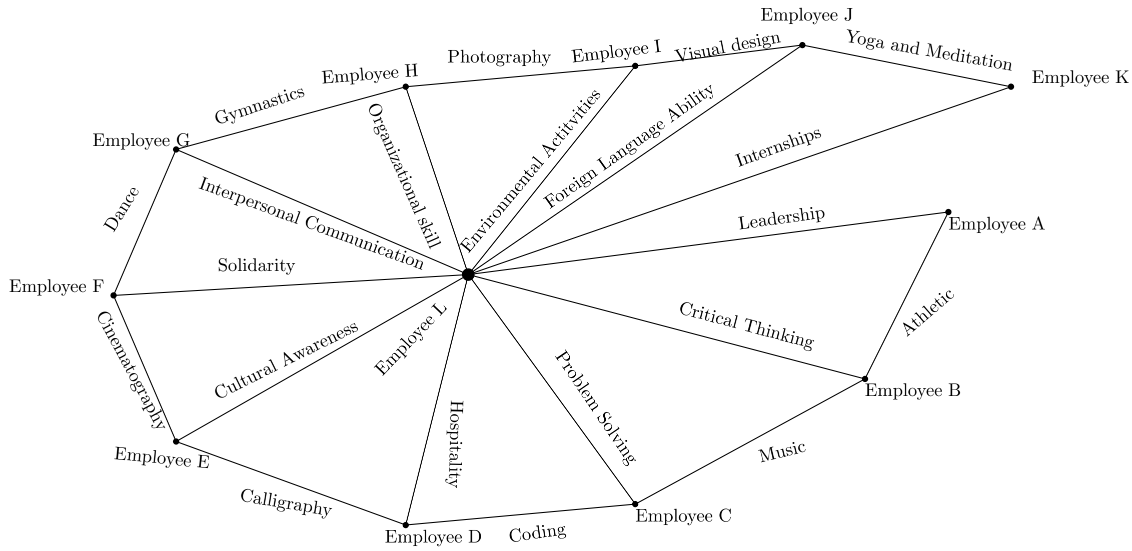
$$\begin{aligned} \psi(uu_1) &= 2n \\ \psi(u_iu_{i+1}) &= 2i; \qquad i = 1, 2, ..., n-1 \\ \psi(u_iv_i) &= 4n+2-2i \qquad i = 1, 2, ..., n \end{aligned}$$

Step 3: The weighted sum of pendant vertices u and v_i for each $1 \leq i \leq n$, $k_1 = 4n+1$. The weighted sum of vertex u_n, $k_2 = 6n+5$. The weighted sum of the vertex u_i for each $1 \leq i \leq n-1$, $k_3 = 8n+5$. The magic constants are $4n+1$ and $6n+5$ and $8n+5$. □

3. Applications of Odd Vertex N-Magic Labeling in Real-world Problems

3.1. **Analyzing the Impact of Extracurricular Activities on Skill Development and Recruitment Competencies in Modern Companies:** The acquisition of discipline-specific abilities is typically the focus of universities and their credentials. Companies prioritize candidates with a blend of specialized skills and deep knowledge in their field.

The fact that they are aware of this leads them to view recreational activities as a crucial means of developing these abilities. The emphasis has been on the effect of extracurricular activities on proficiency and development work because of the need to understand how to assist individuals in acquiring these skills. This motivation is strongly related to the approach to development, which holds that involvement in extracurricular endeavors helps students develop the abilities and knowledge that enhance academic accomplishment. After recruiting skilled employees, management assesses the data of the selected team. The analysis considers the potential for overqualification, influencing the network weights in the process.



Graph model shows a 12 registered people database in the following example. Every employee has the specified professional skills and extracurricular activities. Professional skills include leadership, critical thinking, problem-solving, Hospitality, cultural awareness, solidarity, interpersonal communication, organizational skills, environmental activities, foreign language ability, and internships. Extracurricular activities include athletics, music, coding, calligraphy, cinematography, dance, gymnastics, photography, visual design, yoga, and meditation on. The data hosted corresponds to the employers from the company if exactly one employee has all professional skills and two employees have exactly one professional skill and extra skill. The remaining employees had three activities. Employees with the same number of activities have the same capacity. Determine the capacity of each employee and specify them in ascending order using N-magic constants under vertex N-magic total labeling.

To construct the graph, Graph G represents data where nodes identify specific individuals, and the edges depict the links between them. If a weighted graph can be written as $Wgt(v)$ for $v \in V(G)$, indicating the degree of association between nodes and their incident edges. The edges might be weighted in a certain way to reflect the quantifiable relationship between the nodes, such as the level of user intimacy. Let us consider employees $A, B, C, D, E, F, G, H, I, J, K$, and L, labeled 1 to 12. Employee L shares a connection with all other employees regarding professional expertise. Employee L is the exception regarding extracurricular activities among employees. The capacity of an employee is attained by adding the label of that employee and linking it to that employee. Moreover, we have to label employees A and K as having the same capacity, and each employee, except for A, K, and L, has the same capacity. Moreover, employee

L has different classes defined by different professional skills.
Interconnections between employees and extracurricular activities:
Employees A and B possess similar athletic talents.
Employees B and C have a musical experience.
Both employees C and D imparted the coding.
Employees D and E imparted Calligraphy.
The cinematography abilities of employees E and F are noteworthy.
The employees had a dance experience in common: F and G.
Employees G and H have gymnastics skills.
Employees H and I imparted photography skills.
The employees at I and J have designing skills.
Yoga anMeditationon were bestowed upon employees J and K.
Interconnection between employees and Professional Skills:
Employee A and Employee L has Leadership as an extra curricular activity.
Employee B and employee L have Critical Thinking.
Employee C and employee L have problem-solving skills.
Employee D and employee L have Hospitality.
Employee E and employee L have Cultural Awareness.
Employee, F and employee L, have Solidarity.
Employee G and employee L have Interpersonal Communications.
Employee H and employee L have Organizational skills.
Employee I and employee L have Environmental studies.
Employee J and employee L have foreign language abilities.

TABLE 1. Interconnection between professional skills and extracurricular activity with employee

Employee A	**Athletic**	**Leadership**
1	33	14
Employee K	**Yoga**	**Internship**
11	24	13

Concerning the capacities of the employees, Here, Employee A matches Employee K in the answers of the capacities; this is to say, both employees have the same capacity 48 as shown in table 1. The remaining employees, except A, K, and L, had 82 as capacity in table 2. The employee L has an interconnection among all professional skills. Thus, we determine the capacity of employee L by adding label 12 to the labels of all professional skills to obtain 210.

In our criteria, we found the hierarchical order of the capacities k_i in ascending form is $k_1 < k_2 < k_3$.

3.2. **Applications of Odd Vertex Labeling in Electric Switching: Load Balancing in Power Grids.** Odd vertex N-magic labeling enhances load balancing in electric switching systems, ensuring that all components (switches, transformers, and substations) receive equal workloads. Load balancing is important in power grids because it ensures that electricity is efficiently distributed and transformer overloading is avoided. The odd vertex label gives a systematic mathematical method of labeling substations, transformers, and power stations uniquely with odd numbers so that electrical loads are evenly distributed. All of the nodes in the power grid graph are given odd and unique integer labels. The edges of the graph represent power transmission lines. This ensures that the

TABLE 2. Interconnection between one professional skill and two extracurricular activities with employee

B	**Critical Thinking** 15	**Music** 32	**Athletic** 33
C	**Problem solving** 16	**Music** 32	**Coding** 31
D	**Hospitality** 17	**coding** 31	**Calligraphy** 30
E	**Cultural Awareness** 18	**Cinematography** 29	**Calligraphy** 30
F	**Solidarity** 19	**Dance** 28	**Cinematography** 29
G	**Gymnastics** 27	**Interpersonal Communications** 20	**Dance** 28
H	**Photography** 26	**Organizational skills** 21	**Gymnastics** 27
I	**Visual design** 25	**Environmental studies** 22	**photography** 26
J	**Yoga and meditation** 24	**Foreign Language ability** 23	**Visual design** 25

sum of the labels at each vertex always equals the sum of the values at each edge. It allows a minimum energy loss through power grids that enhance fault tolerance and optimize power rerouting for failure situations. Moreover, modern smart grids leverage odd vertex N-magic labeling to dynamically adjust power distribution based on real-time demand, improving grid stability and integrating renewable energy sources more effectively. This structured labeling method not only enhances energy efficiency but also supports the scalability of power networks, making it an essential tool in electrical switching and grid optimization.

In graph theory, an example of odd vertex N-magic labeling is the assignment of unique odd numbers to every node in a network, such as substations, transformers, or power stations. Transmission lines or power flow pathways are represented by edges to ensure an even distribution of loads. The sum of vertex labels and incident edge values remains constant to prevent overloading. In the representation of power grids, odd-labeled nodes or vertices can represent substations, power stations, or transformers. Electrical connection transmission lines between nodes are called edges. Odd vertex N-magic labeling also helps in achieving appropriate load balancing for maximum optimization of a power flow and less energy loss in transmission.

The use of odd-numbered labels for substations encourages an even distribution of power loads, preventing overloads and underutilization. This ensures fault tolerance since it is easier to identify other power paths if there is a failure in any substation. This helps to avoid cascading grid failures. Balanced labeling minimizes transmission losses by optimizing the resistance paths. It also ensures that energy is used efficiently. In smart grids, AI algorithms use this kind of labeling to dynamically distribute power, making it easier to add renewable energy sources.

Example: A power grid composed of 12 servers S_i if $i = 1, 2, ..., 10$, C_1, and C_2, where edges represent electrical connections. The strategy is to distribute workloads fairly among servers, utilizing three distinct methods for computation.
Graph Representation

Vertices (Servers) $\rightarrow S_i$
Edges (Connections/Tasks) $\rightarrow S_iC_1$, S_iC_2 and C_1C_2
Magic Sums $\rightarrow$ 23, 89

A power grid with four substations and transmission lines connecting them is the subject of this study. Substations are assigned unique odd integer labels, and the calculation

Vertex labels to C_1	Edge labels to C_1	Vertex labels to C_2	Edge labels to C_2
$S_1 \rightarrow 1$	$S_1C_1 \rightarrow 22$	$S_6 \rightarrow 3$	$S_6C_2 \rightarrow 20$
$S_2 \rightarrow 7$	$S_2C_1 \rightarrow 16$	$S_7 \rightarrow 5$	$S_7C_2 \rightarrow 18$
$S_3 \rightarrow 9$	$S_3C_1 \rightarrow 14$	$S_8 \rightarrow 11$	$S_8C_2 \rightarrow 12$
$S_4 \rightarrow 15$	$S_4C_1 \rightarrow 8$	$S_9 \rightarrow 13$	$S_9C_2 \rightarrow 10$
$S_5 \rightarrow 17$	$S_5C_1 \rightarrow 6$	$S_{10} \rightarrow 19$	$S_{10}C_2 \rightarrow 4$
$C_1 \rightarrow 21$	$C_1C_2 \rightarrow 2$	$C_2 \rightarrow 23$	

Table 3. Load Balancing in Power Grids

of these labels involves the sum of the substation's value and the value of its adjacent vertex. $Wg_t(S_i) = S_i + S_iC_1$ for $i = 1, ..., 5$ & $Wg_t(S_i) = S_i + S_iC_2$ for $i = 6, ..., 10$ and $Wg_t(C_1) = C_1 + C_1C_2 + S_iC_1$ if $i = 1, 2, ..., 5$ & $Wg_t(C_2) = C_2 + C_1C_2 + S_iC_2$ if $i = 6, ..., 10$.

The total load at each substation, denoted as S_i, remains constant, with a total load of (23). The values of C_1 and C_2 ensure balanced power distribution, resulting in a total load of 89 in table 3. Substations are designed to handle loads without exceeding capacity, minimizing overload and energy waste.

The application of load balancing in graph labeling statements serves multiple purposes: it reduces the frequency of power overloads, enhances grid stability, maximizes energy efficiency, facilitates the integration of renewable energy sources, and improves system scalability. The method ensures a balanced load distribution between substations, decreases the frequency of power outages, reduces transmission losses, and promotes sustainable energy solutions. It also enables the expansion of the grid without compromising the integrity of load balancing operations. This method is crucial in transforming electricity networks into intelligent and sustainable ones.

4. Conclusion and Future Discussions

In summary, this research has outlined an extensive study of odd-vertex N-Magic total labeling in different specialized graphs such as path graphs, star graphs, sun graphs, banana trees, and kite graphs. I have shown through detailed theorems and proofs that these graphs must have certain conditions and labeling schemes for them to have odd vertex N-magic features. The results capture the beauty and intricacy of graph labeling with a fine balance of vertices, edges, and magic constants.

Secondly, the use of the labeling methods extends beyond mathematical theory to real-life applications such as the analysis of the effect of extracurricular activities on skill development and recruitment skills in contemporary firms. By expressing real-world data through Odd Vertex N-Magic Total Labeling, organizations can better understand skill distribution and competency gaps and maximize their recruitment process.

Here, I describe the categories of non-regular graphs that have been investigated and are known to satisfy odd VNMT labeling. In this manuscript, the odd vertex N-magic total graphs are a novel notion of vertex N-magic-type graphs and, based on N-magic with distinct degrees, determine the odd vertex labels relevant to the property. Finally, I explored the real-life application of odd VNMT labeling. An application discusses the relevance of these labeling techniques in employee skill analysis within a company, using a database of registered individuals with professional skills and extracurricular activities. A graph model where nodes represent employees and edges represent shared activities or skills, with weights assigned based on the association level.

Further research could delve into diverse graph types, refine labeling methods, and explore the practical applications of vertex magic labeling in network analysis, resource allocation, and social network modeling. In the future, experts can do extensive studies on odd VNMT labeling of highly irregular graphs using odd N-magic constants. I conclude by proposing an issue.

Conjecture: Determine all even orders for $n > 4$ such that every P_n admits an odd VNMT labeling.

References

[1] Cichacz, S., Froncek, D., Singgih, I., (2017), Vertex-magic total labelings of 2-regular graphs, Discrete Mathematics, 340, pp. 3117-3124.

[2] Gray, I. D., MacDougall, J. A., (2010), Vertex-magic labeling of non-regular graphs, Australas. J. Combin., 46, pp. 173-183.

[3] Gray, I. D., MacDougall, J. A., (2012), Vertex-magic labeling of regular graphs: Disjoint unions and assemblages, Discrete Appl. Math., 160(7-8), pp. 1114-1125.

[4] Gray, I. D., MacDougall, J. A., Mcsorley, J. P., Wallis, W. D., (2003), Vertex-magic labeling of trees and forests, Discrete Mathematics, 261(1-3), pp. 285-298.

[5] Gallian, J. A., (2020), A dynamic survey of graph labeling, Electron. J. Combin., 6(23).

[6] Holden, J., McQuillan, D., McQuillan, J. M., (2009), A conjecture on strong magic labelings of 2-regular graphs, Discrete Mathematics, 309(12), pp. 4130-4136.

[7] MacDougall, J. A., Miller, M., Wallis, W. D., (2002), Vertex-magic total labelings of wheels and related graphs, Utilitas Mathematica, 62, pp. 175-183.

[8] Marr, A. M., Wallis, W. D., (2013), Magic Graphs, 2nd ed., Birkhäuser/Springer, New York.

[9] Marimuthu, G., Kumar, G. (2016), Vertex N-magic total labeling of graphs, J. Graph Label., 2(2), pp. 123-133.

[10] Nishanthini, R., Jeyabalan, R., Balasundar, S., Kumar, G. (2021), Consecutive z-index vertex magic labeling graphs, J. Intell. Fuzzy Syst., 41, pp. 219-230.

[11] Nishanthini, R., Jeyabalan, R. (2024), Sharp bounds on vertex N-magic total labeling of graphs, Mathematics and Statistics, 12(3), pp. 234-239. DOI: 10.13189/ms.2024.120303.

[12] Nishanthini, R., Jeyabalan, R. (2025), Strong VNMT Labelings on Graph Families and Their Applications in Disease Transmission Networks, Mathematics in Engineering, Science and Aerospace (MESA) Vol. 16 No. 3, pp. 743-752.

[13] Rahim, M. T., Tomescu, I., Slamin, (2007), On vertex-magic total labeling of some wheel-related graphs, Utilitas Mathematica, 73, pp. 97-104.

[14] Sedláček, J., (1963), Problem 27. Theory of graphs and its applications, Proc. Symp. Smolenice, Praha, pp. 163-164.

[15] Tezer, M., (2018), Vertex magic total labeling of selected trees, AIP Conf. Proc., DOI: 10.1063/1.5049028020034.

[16] Wang, T. M., Zhang, G. H., (2017), On vertex magic total labeling of disjoint union of sun graphs, Utilitas Mathematica, 103, pp. 289-298.

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