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# THE $k$-BINOMIAL AND CATALAN TRANSFORMS OF THE $k$-LEONARDO NUMBERS

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ABSTRACT. In this paper, we investigate the binomial transform and the Catalan transform of the $k$-Leonardo numbers and examine the new integer sequences. We apply the $k$-binomial, rising $k$-binomial, and falling $k$-binomial transforms to the $k$-Leonardo sequences. Further, we study the associated generating and exponential generating functions for these transforms. Finally, we implement the Hankel transform on the Catalan transforms of the $k$-Leonardo numbers to obtain the determinant of Hankel sub-matrices.

Keywords: Leonardo numbers, $k$-binomial transform, Catalan transform, Generating function, Hankel transform.

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## 1. Introduction

For positive integer $N$ and $n \geq 0$, the generalized Leonardo numbers are defined [10] by a second order non-homogeneous difference equation as

$$\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n + N, \quad \text{with } \mathcal{L}_0 = \mathcal{L}_1 = 1. \tag{1}$$

For the case $N = 1$, we get the standard Leonardo numbers $\mathcal{L}e_n$ [4].

The closed form formula for Leonardo numbers $\mathcal{L}e_n$ is given by

$$\mathcal{L}e_n = 2\Big(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\Big) - 1,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

Study on the Leonardo numbers, their generalizations and applications become more attentive after the study of Catarino and Borges [4] who studied a recurrence relation for the Leonardo numbers $\{\mathcal{L}e_n\}_{n\geq 0}$ and investigated some properties of them in connection with Fibonacci and Lucas sequence. Alp and Koçer [1] in their study of Leonardo numbers obtained some remarkable properties of them and extended the study to the matrix

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representation and in [2] presented its hybrid form. Prasad et al. [16] extended the study to octonions algebra. Some recent developments on the generalized Leonardo numbers are due to Kuhapatanakul and Chobsorn [10], Kumari et al. [11], Özimamoğlu [13], Soykan [20], etc.

The Leonardo sequence is closely associated with the Fibonacci sequence as $\mathcal{L}e_n = 2F_{n+1} - 1$ and in general as $\mathcal{L}_n = (N+1)F_{n+1} - N$. A generalization of Leonardo numbers referred as the Leonardo polynomials is recently studied by Prasad and Kumari [17] which preserve the above Leonardo-Fibonacci relation for polynomial version. Replacing the real variable $x$ with real number $k$ in [17] gives the $k$-Leonardo numbers which extends the relation for $k$-Fibonacci numbers. Generalization and many algebraic properties of the $k$-Leonardo numbers are discussed in [18]. By virtue of [18], the $k$-Leonardo numbers are defined as follows:

**Definition 1.1.** *Let $k \in \mathbb{R}^+$, then the $k$-Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ is defined as*

$$\mathcal{L}_{k,n+2} = k\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k \quad \text{with } \mathcal{L}_{k,0} = 1, \mathcal{L}_{k,1} = 2k - 1.$$

The first few $k$-Leonardo numbers are shown in the following table.

| $n$ | $\mathcal{L}_{k,n}$ |
|---|---|
| 0 | 1 |
| 1 | $2k-1$ |
| 2 | $2k^2+1$ |
| 3 | $2k^3+4k-1$ |
| 4 | $2k^4+6k^2+1$ |
| 5 | $2k^5+8k^3+6k-1$ |
| 6 | $2k^6+10k^4+12k^2+1$ |
| 7 | $2k^7+12k^5+20k^3+8k-1$ |
| $\dots$ | $\dots$ |

In particular for $k = 1$, the $k$-Leonardo sequence $\{\mathcal{L}_{k,n}\}$ gives the standard Leonardo sequence $\{1, 1, 3, 5, 9, 15, 25, ...\}$: A001595. Also for $k = 2$ it becomes $\{1, 3, 9, 23, 87, 139, 337, ...\}$ which is enlisted as A133654 on OEIS and for $k = 3$, $\{1, 5, 19, 65, 217, ...\}$.

## 1.1. **Catalan Transformation.**

In literature, there are various transforms that performs on number sequences. For instance the binomial Transform (BT) [22, 6], Catalan Transform (CT) [8, 3], Hankel Transform (HT) [19, 12], Discrete Fourier Transform (DFT), etc. The overall estimation of BT and CT is based on integers, not based on the floating numbers, make them faster and more reliable method of transform unlikely DFT, DCT, etc.

The Catalan numbers ($C_n$) [A000108] become famous due to the French-Belgian mathematician Eugene Charles Catalan and in context of binomial coefficients, they are defined by

$$C_n = \frac{1}{1+n}\binom{2n}{n} = \frac{2n!}{(1+n)!(n)!}.$$

The first few terms of Catalan sequence are $1, 1, 2, 5, 14, 42, 132, 429$, etc. The recurrence relation for Catalan numbers $C_n$ can be easily obtained from

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}.$$

By the virtue of P. Barry [3], the Catalan transform $\{TA_n\}$ of a number sequence $\{A_n\}$ is a sequence transform given as follows:

$$TA_n = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r} A_r,\ n \geq 1,\ \text{where } TA_0 \text{ is given.} \tag{2}$$

Also, for the Catalan numbers, the ordinary generating function $c(x)$ is given as

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{3}$$

Some recent works on Catalan transform for a number sequence can be seen in [15, 23, 8, 3, 21].

## 2. Transforms on k-Leonardo numbers

Let $\alpha = (k+\sqrt{k^2+4})/2$ and $\beta = (k-\sqrt{k^2+4})/2$ be two roots of the characteristic equation $x^2 - kx - 1 = 0$. Then the Binet's formula of the $k$-Fibonacci numbers is given by

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{k^2+4}}. \tag{4}$$

**Theorem 2.1** (Binet formula for $\mathcal{L}_{k,n}$)**.** *For $n \geq 0$, we have*

$$\mathcal{L}_{k,n} = \frac{2(\alpha^{n+1} - \beta^{n+1}) - \sqrt{k^2+4}}{\sqrt{k^2+4}}.$$

*Proof.* The proof is easy using the Leonardo-Fibonacci relation $\mathcal{L}_{k,n} = 2F_{k,n+1} - 1$. □

**Theorem 2.2.** *For $n \geq 0$, the following recurrence holds*

$$\mathcal{L}_{k,n+3} = (k+1)\mathcal{L}_{k,n+2} - (k-1)\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n}, \tag{5}$$

*where $\mathcal{L}_{k,0} = 1,\ \mathcal{L}_{k,1} = 2k-1$ and $\mathcal{L}_{k,2} = 2k^2+1$.*

*Proof.* From Definition 1.1, we write

$$\begin{aligned}\mathcal{L}_{k,n+3} - \mathcal{L}_{k,n+2} &= (k\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+1} + k) - (k\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k).\\ &= k\mathcal{L}_{k,n+2} + (1-k)\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n}\end{aligned}$$

On simplifying, it gives $\mathcal{L}_{k,n+3} = (1+k)\mathcal{L}_{k,n+2} + (1-k)\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n}$.
This completes the proof. □

### 2.1. Binomial transform.

In this section, we discuss the binomial transform and the Catalan transform to the $k$-Leonardo sequences (5) and obtain the new integer sequences. First, we implement the binomial transform and investigate new sequences.

Recently, Falcon and Plaza[7] obtained the $k$-Fibonacci sequence by studying the two geometrical transforms and later in [6] they studied the binomial transforms for these sequences. S. Falcon investigated the binomial transforms[9] for generalized $k$-Fibonacci sequences and Catalan transform[8] for the $k$-Fibonacci Numbers. Some recent works in this direction with known number sequences can be seen in [23, 8, 14, 5].

In the study of number sequences, the binomial transform of an integer sequence $\{a_n\}$ associates to the integer sequence $\{b_n\}$ as follows

$$b_n = \sum_{r=0}^{n} \binom{n}{r} a_r$$

which is an invertible transformation and inverse transformation is presented as

$$a_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} b_r.$$

Let us define the binomial transform of $k$-Leonardo sequences in classical form as $B_k = \{B\mathcal{L}_{k,n}\}$, where

$$B\mathcal{L}_{k,n} = \sum_{r=0}^{n} \binom{n}{r} \mathcal{L}_{k,r}.$$

Thus the terms of binomial transform after applying on the $k$-Leonardo sequence are as follow:

$$\begin{aligned} B_{k,n} =& \{1,\ 2k, 2k^2+4k,\ 2k^3+6k^2+10k,\ 2k^4+8k^3+18k^2+24k, \\ & 2k^5+10k^4+28k^3+50k^2+56k,\ 2k^6+12k^5+40k^4+88k^3+132k^2+128k, ...\}. \end{aligned}$$

The first few terms of the above binomial transforms are shown below:

| $n \rightarrow$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|
| $B_{1,n}$ | 1 | 2 | 6 | 18 | 52 | 146 | 402 | ... |
| $B_{2,n}$ | 1 | 4 | 16 | 60 | 216 | 760 | 2640 | ... |
| $B_{3,n}$ | 1 | 6 | 30 | 138 | 612 | 2670 | 11562 | ... |
| $B_{4,n}$ | 1 | 8 | 48 | 264 | 1408 | 7424 | 38976 | ... |
| $B_{5,n}$ | 1 | 10 | 70 | 450 | 2820 | 70530 | 108690 | ... |

TABLE 1. Sequences of binomial transforms for k=1,2,3,4,5

Observe that the binomial transform of the Leonardo sequences ($k = 1$) is convolution of OEIS sequences A129519 and A218482 and indexed as A318570.

2.2. **The k-binomial, rising k-binomial and falling k-binomial transforms.** In a similar fashion to the binomial transform, M. Spivey and L. Steil [22] proposed three variants of the binomial transform with two inputs, an integer sequence $A_n$ and a fixed quantity (scalar) $k$, namely the $k$-binomial, rising and falling $k$-binomial Transform. In [6], Falcon and Plaza studied several $k$-binomial transforms for $k$-Fibonacci sequences and presented their triangle. Here, we investigate these transforms for $k$-Leonardo sequences.

The $k$-binomial transform $W_k$ for the $k$-Leonardo sequence $\{\mathcal{L}_{k,n}\}$ is the sequence $W_k = W(\mathcal{L}_{k,n}, k) = \{w_{k,n}\}_{n\geq 0}$, where $w_{k,n}$ is defined as

$$w_{k,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (k)^n \mathcal{L}_{k,i} & \text{for } k \neq 0 \text{ or } n \neq 0, \\ \mathcal{L}_{k,0} & \text{for } k = 0, n = 0. \end{cases}$$

The rising $k$-binomial transform $R_k$ and falling $k$-binomial transform $F_k$ for $\{\mathcal{L}_{k,n}\}$ are the sequence $R_k = R(\mathcal{L}_{k,n}, k) = \{r_{k,n}\}_{n\geq 0}$ and $F_k = F(\mathcal{L}_{k,n}, k) = \{f_{k,n}\}_{n\geq 0}$, respectively, where $r_{k,n}$ and $f_{k,n}$ are defined as

$$r_{k,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (k)^i \mathcal{L}_{k,i} & \text{for } k \neq 0, \\ \mathcal{L}_{k,0} & \text{for } k = 0, \end{cases} \text{ and } f_{k,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (k)^{n-i} \mathcal{L}_{k,i} & \text{for } k \neq 0, \\ \mathcal{L}_{k,0} & \text{for } k = 0. \end{cases}$$

If $k = 1$, all three $k$-binomial transforms overlap with the binomial transform $B_1$ and are the unique sequence indexed in OEIS (A318570).

On going through the above formula and performing the necessary operations, some of the terms of $k$-binomial transform $W(\mathcal{L}_{k,n},k)$ are:

$$W_k = \{1,\ 2k^2, 2k^4+4k^3,\ k^3(2k^3+6k^2+10k),\ k^4(2k^4+8k^3+18k^2+24k),$$
$$k^5(2k^5+10k^4+28k^3+50k^2+56k),\ k^6(2k^6+12k^5+40k^4+88k^3+132k^2+128k), ...\}.$$

Thus, first few $k$-binomial transforms of $k$-Leonardo sequence are:

$$\begin{aligned} W_1 &= \{1,2,6,18,52,146,402,...\} : A318570 \\ W_2 &= \{1,8,64,480,3456,24320,168960,...\} \\ W_3 &= \{1,18,270,3726,49572,648810,8428698,...\} \\ W_4 &= \{1,32,768,16896,360448,7602176,159645696,...\}. \end{aligned}$$

Similarly, performing the necessary operations with the formula of the rising $k$-binomial transform, we obtain the following sequences in $k$:

$$R_k = \{1,\ 2k^2-k+1,\ 2k^4+5k^2-2k+1,\ 2k^6+10k^4-k^3+9k^2-3k+1,$$
$$2k^8+14k^6+29k^4-4k^3+14k^2-4k+1,\ 2k^{10}+18k^8+56k^6-k^5+65k^4-10k^3+20k^2-5k+1,$$
$$2k^{12}+22k^{10}+90k^8+167k^6-6k^5+125k^4-20k^3+27k^2-6k+1, ...\}.$$

Hence, first few rising $k$-binomial transforms of $k$-Leonardo sequence are:

$$\begin{aligned} R_1 &= \{1,2,6,18,52,146,402,...\} : A318570 \\ R_2 &= \{1,7,49,311,1889,11239,66193,...\} \\ R_3 &= \{1,16,202,2314,25684,281938,3082546,...\} \\ R_4 &= \{1,29,585,10821,195793,3521453,63230361,...\}. \end{aligned}$$

Finally, performing with the formula of falling $k$-binomial transform gives the following sequences in $k$

$$\begin{aligned} F_k = &\{1, 3k-1, 7k^2-2k+1, 15k^3-3k^2+7k-1, 31k^4-4k^3+28k^2-4k+1, \\ &63k^5-5k^4+88k^3-10k^2+11k-1, 127k^6-6k^5+243k^4-20k^3+63k^2-6k+1, ...\}. \end{aligned}$$

The first few falling $k$-binomial transforms of $k$-Leonardo sequence are:

$$\begin{aligned} F_1 &= \{1,2,6,18,52,146,402,...\} : A318570 \\ F_2 &= \{1,5,25,121,569,2621,11905,...\} \\ F_3 &= \{1,8,58,398,2644,17222,110818,...\} \\ F_4 &= \{1,11,105,939,8113,68747,575961,...\}. \end{aligned}$$

2.3. **Catalan transform.** Now following the Eqn. (2), we define the Catalan transform of the $k$-Leonardo sequence $\{\mathcal{L}_{k,n}\}$ as

$$T\mathcal{L}_{k,n} = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r}\mathcal{L}_{k,r} \quad \text{for } n \geq 1, \text{ with } T\mathcal{L}_{k,0} = 0. \tag{6}$$

Thus, some terms of the Catalan transform for sequence $\{\mathcal{L}_{k,n}\}$ are listed below:

$$\begin{aligned}
T\mathcal{L}_{k,1} &= \sum_{r=0}^{1} \frac{r}{2-r}\binom{2-r}{1-r}\mathcal{L}_{k,r} = 0\mathcal{L}_{k,0} + 1\mathcal{L}_{k,1} = \mathcal{L}_{k,1} = 2k-1,\\
T\mathcal{L}_{k,2} &= \sum_{r=0}^{2} \frac{r}{4-r}\binom{4-r}{2-r}\mathcal{L}_{k,r} = \frac{1}{3}\binom{3}{1}\mathcal{L}_{k,1} + \frac{2}{2}\binom{4}{0}\mathcal{L}_{k,2} = \mathcal{L}_{k,1} + \mathcal{L}_{k,2} = 2k^2+2k,\\
T\mathcal{L}_{k,3} &= \sum_{r=0}^{3} \frac{r}{6-r}\binom{6-r}{3-r}\mathcal{L}_{k,r} = \frac{1}{5}\binom{5}{2}\mathcal{L}_{k,1} + \frac{2}{4}\binom{4}{1}\mathcal{L}_{k,2} + \frac{3}{3}\binom{3}{0}\mathcal{L}_{k,3}\\
&= 2\mathcal{L}_{k,1} + 2\mathcal{L}_{k,2} + \mathcal{L}_{k,3} = 2k^3+4k^2+8k-1,\\
T\mathcal{L}_{k,4} &= \sum_{r=0}^{4} \frac{r}{8-r}\binom{8-r}{4-r}\mathcal{L}_{k,r} = \frac{1}{7}\binom{7}{3}\mathcal{L}_{k,1} + \frac{2}{6}\binom{6}{2}\mathcal{L}_{k,2} + \frac{3}{5}\binom{5}{1}\mathcal{L}_{k,3} + \frac{4}{4}\binom{4}{0}\mathcal{L}_{k,4}\\
&= 5\mathcal{L}_{k,1} + 5\mathcal{L}_{k,2} + 3\mathcal{L}_{k,3} + \mathcal{L}_{k,4} = 2k^4+6k^3+16k^2+22k-2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
T\mathcal{L}_{k,5} &= \sum_{r=0}^{5} \frac{r}{10-r}\binom{10-r}{5-r}\mathcal{L}_{k,r} = 14\mathcal{L}_{k,1} + 14\mathcal{L}_{k,2} + 9\mathcal{L}_{k,3} + 4\mathcal{L}_{k,4} + \mathcal{L}_{k,5}\\
&= 2k^5+8k^4+26k^3+52k^2+70k-6,\\
T\mathcal{L}_{k,6} &= \sum_{r=0}^{6} \frac{r}{12-r}\binom{12-r}{6-r}\mathcal{L}_{k,r} = 42\mathcal{L}_{k,1} + 42\mathcal{L}_{k,2} + 28\mathcal{L}_{k,3} + 14\mathcal{L}_{k,4} + 5\mathcal{L}_{k,5} + \mathcal{L}_{k,6}\\
&= 2k^6+10k^5+38k^4+96k^3+180k^2+226k-18,\\
T\mathcal{L}_{k,7} &= 132\mathcal{L}_{k,1} + 132\mathcal{L}_{k,2} + 90\mathcal{L}_{k,3} + 48\mathcal{L}_{k,4} + 20\mathcal{L}_{k,5} + 6\mathcal{L}_{k,6} + \mathcal{L}_{k,7}\\
&= 2k^7+12k^6+52k^5+156k^4+360k^3+624k^2+752k-57.
\end{aligned}$$

Let us define $C\mathcal{L}_{k,n} = [T\mathcal{L}_{k,1}, T\mathcal{L}_{k,2}, T\mathcal{L}_{k,3}, ...]$ and $X = [\mathcal{L}_{k,1}, \mathcal{L}_{k,2}, \mathcal{L}_{k,3}, ...]$ then above can be represented as $C\mathcal{L}_k^T = LX^T$, where $L = [a_{ij}]_{i,j\geq 1}$ is the lower triangular matrix given as follow:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots\\ 1 & 1 & 0 & 0 & 0 & 0 & \dots\\ 2 & 2 & 1 & 0 & 0 & 0 & \dots\\ 5 & 5 & 3 & 1 & 0 & 0 & \dots\\ 14 & 14 & 9 & 4 & 1 & 0 & \dots\\ 42 & 42 & 28 & 14 & 5 & 1 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

Elements of the matrix L satisfy the following relation:

(1) for $i > 1$, the entries of the first and second columns are equal and these are the Catalan numbers.
(2) for $i, j \geq 1$, $a_{i,j} = \sum_{r=j-1}^{i-1} a_{i-1,r}$.

Moreover, the lower triangular matrix $L_{n,n-i}$ is popularly known as Catalan triangle whose entries satisfy the formula $a_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1}$, thus

$$a_{n,n-i} = \frac{(i+1)(2n-i)!}{(n+1)!(n-1)!}, \quad 0 \leq i \leq n.$$

Hence, the associated Catalan triangle with the coefficients of the Catalan transform of the $k$-Leonardo sequence is presented in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $T\mathcal{L}_0$ | 0 | | | | | | | |
| $T\mathcal{L}_1$ | 2 | -1 | | | | | | |
| $T\mathcal{L}_2$ | 2 | 2 | 0 | | | | | |
| $T\mathcal{L}_3$ | 2 | 4 | 8 | -1 | | | | |
| $T\mathcal{L}_4$ | 2 | 6 | 16 | 22 | -2 | | | |
| $T\mathcal{L}_5$ | 2 | 8 | 26 | 52 | 70 | -6 | | |
| $T\mathcal{L}_6$ | 2 | 10 | 38 | 96 | 180 | 226 | -18 | |
| $T\mathcal{L}_7$ | 2 | 12 | 52 | 156 | 360 | 624 | 752 | -57 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

TABLE 2. Catalan triangle of the $k$-Leonardo sequence

Observe that the right most diagonal sequence $\{-1, 0, -1, -2, -6, -18, -57, ...\}$ is the negative of Fine's sequence (A000957).

Finally, we obtain some terms of the series of Catalan transform (for $k = 1, 2, 3, ...$) of the $k$-Leonardo sequence, given as follow:

| $n \to$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|
| $C\mathcal{L}_{1,n}$ | 0 | 1 | 4 | 13 | 44 | 152 | 534 | ... |
| $C\mathcal{L}_{2,n}$ | 0 | 3 | 12 | 47 | 186 | 742 | 2978 | ... |
| $C\mathcal{L}_{3,n}$ | 0 | 5 | 24 | 113 | 532 | 2508 | 11838 | ... |
| $C\mathcal{L}_{4,n}$ | 0 | 7 | 40 | 223 | 1238 | 6866 | 38070 | ... |
| $C\mathcal{L}_{5,n}$ | 0 | 9 | 60 | 389 | 2508 | 16144 | 103862 | ... |
| $C\mathcal{L}_{6,n}$ | 0 | 11 | 84 | 623 | 4594 | 33822 | 248874 | ... |

TABLE 3. Sequences of Catalan transforms for k=1,2,3,4,5,6

## 3. Generating functions

In this section, we discuss some generating functions of the $k$-Leonardo numbers then give the generating functions of corresponding binomial, $k$-binomial and Catalan transforms. From [18], we should note that the ordinary generating function $G(t)$ for the $k$-Leonardo sequence is given by

$$G(t) = \frac{1 + (k-2)t + t^2}{1 - (k+1)t + (k-1)t^2 + t^3}. \tag{8}$$

**Theorem 3.1.** *For the $k$-Leonardo sequence, the exponential generating function $E_k(t)$ is given by*

$$E_k(t) = \frac{2(\alpha e^{\alpha t} - \beta e^{\beta t})}{\sqrt{k^2+4}} - e^t,$$

*where $\alpha = (k + \sqrt{k^2+4})/2$ and $\beta = (k - \sqrt{k^2+4})/2$.*

*Proof.* Let $E_k(t) = \sum_{n=0}^{\infty} \mathcal{L}_{k,n} \frac{t^n}{n!}$ then on using $\mathcal{L}_{k,n} = 2F_{k,n+1} - 1$, we have

$$E_k(t) = 2\sum_{n=0}^{\infty} F_{k,n+1}\frac{t^n}{n!} - \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$= \frac{2(\alpha e^{\alpha t} - \beta e^{\beta t})}{\sqrt{k^2+4}} - e^t \quad \text{(using generating function of } F_{k,n})$$

as required. □

**Corollary 3.1.** *The generating function $g(t)$ and exponential generating function $E(t)$ for the Leonardo sequence are given, respectively, by*

$$g(t) = \frac{1-t+t^2}{1-2t+t^3} \quad \textit{and} \quad E(t) = \frac{2(\alpha e^{\alpha t} - \beta e^{\beta t})}{\sqrt{5}} - e^t,$$

*where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.*

By the virtue of [3], we have the result that if $A(t)$ is a generating function for any sequence $A_n$ then the generating function $B(t)$ of the binomial transform of $A_n$ is given by

$$B(t) = \frac{1}{1-t} A\Big(\frac{t}{1-t}\Big). \tag{9}$$

and if $c(t)$ is the ordinary generating function (see (3)) of the sequence of Catalan numbers then the generating function of the Catalan transform of the sequence $A_n$ is given by $A(tc(t))$. Thus the following theorem.

**Theorem 3.2.** *The generating function $B_k(t)$ and $G_k(t)$ for the binomial transform and Catalan transform, respectively, of the $k$-Leonardo sequence, are given by*

$$B_k(t) = \frac{1-(4-k)t+(4-k)t^2}{1-(4+k)t+(4+3k)t^2-2kt^3}$$

*and*

$$G_k(t) = \frac{1+k-2t-(k-1)\sqrt{1-4t}}{1-(2k+1)t+(1+t)\sqrt{1-4t}}.$$

*Proof.* Consider the generating function of the $k$-Leonardo sequence from Theorem 8. So the generating function $B_k(t)$ of the corresponding binomial transform is given by

$$\begin{aligned}
B_k(t) &= \frac{1}{1-t} G\Big(\frac{t}{1-t}\Big) \\
&= \frac{1}{1-t}\left(\frac{1+(k-2)\frac{t}{1-t}+\left(\frac{t}{1-t}\right)^2}{1-(k+1)\frac{t}{1-t}+(k-1)\left(\frac{t}{1-t}\right)^2+\left(\frac{t}{1-t}\right)^3}\right) \\
&= \frac{1}{1-t}\left(\frac{(1-t)\big[(1-t)^2+(k-2)t(1-t)+t^2\big]}{(1-t)^3-(k+1)t(1-t)^2+(k-1)t^2(1-t)+t^3}\right) \\
&= \frac{(1-t)^2+(k-2)t(1-t)+t^2}{(1-t)^3-(k+1)t(1-t)^2+(k-1)t^2(1-t)+t^3}.
\end{aligned}$$

On some elementary calculations, it gives

$$B_k(t) = \frac{1-(4-k)t+(4-k)t^2}{1-(4+k)t+(4+3k)t^2-2kt^3}.$$

Now, for the Catalan transform of the $k$-Leonardo sequence, the generating function $G_k(t)$ is given by

$$\begin{aligned}
G_k(t) &= G(tc(t)) \\
&= \frac{1+(k-2)tc(t)+(tc(t))^2}{1-(k+1)tc(t)+(k-1)(tc(t))^2+(tc(t))^3} \\
&= \frac{1+(k-2)t\Big(\frac{1-\sqrt{1-4t}}{2t}\Big)+t^2\Big(\frac{1-\sqrt{1-4t}}{2t}\Big)^2}{1-(k+1)t\Big(\frac{1-\sqrt{1-4t}}{2t}\Big)+(k-1)t^2\Big(\frac{1-\sqrt{1-4t}}{2t}\Big)^2+t^3\Big(\frac{1-\sqrt{1-4t}}{2t}\Big)^3} \\
&= \frac{8+4(k-2)(1-\sqrt{1-4t})+2(1-\sqrt{1-4t})^2}{8-4(k+1)(1-\sqrt{1-4t})+2(k-1)(1-\sqrt{1-4t})^2+(1-\sqrt{1-4t})^3} \\
&= \frac{4[1+k-2t-(k-1)\sqrt{1-4t}]}{8\sqrt{1-4t}-4t(2k-2)+4[1-3t-(1-t)\sqrt{1-4t}]}.
\end{aligned}$$

On simplifying the above equation, we get

$$G_k(t) = \frac{1+k-2t-(k-1)\sqrt{1-4t}}{1-(2k+1)t+(1+t)\sqrt{1-4t}}$$

as required. □

The generating function of the binomial transform and the Catalan transform, respectively, of the Leonardo sequence, is obtained by substituting $k=1$ in the Theorem 3.2, i.e. given by

$$B_1(t) = \frac{1-3t+3t^2}{1-5t+7t^2-2t^3} \quad \text{and} \quad G_1(t) = \frac{2-2t}{1-3t+(1+t)\sqrt{1-4t}}.$$

**Theorem 3.3.** *The generating function $w_k(t)$ of the $k$-binomial transform of the $k$-Leonardo sequence is*

$$w_k(t) = \frac{1-k(4-k)t+k^2(4-k)t^2}{1-k(4+k)t+k^2(4+3k)t^2-2k^4t^3}.$$

*Proof.* The result can be easily proved using the fact that if $G(t)$ is the generating function for $\{\mathcal{L}_{k,n}\}$ then for the associated $k$-binomial transform, the generating function is given by[22]

$$w_k(t) = \frac{1}{1-kt}G\Big(\frac{kt}{1-kt}\Big).$$

□

**Theorem 3.4.** *For the $k$-binomial transform $W(\mathcal{L}_{k,n},k)$, rising $k$-binomial transform $R(\mathcal{L}_{k,n},k)$ and falling $k$-binomial transform $F(\mathcal{L}_{k,n},k)$, the exponential generating functions are given, respectively, by*

$$EW_k(t) = \frac{2e^{kt}(\alpha e^{\alpha kt}-\beta e^{\beta kt})}{\sqrt{k^2+4}} - e^{2kt}, \quad ER_k(t) = \frac{2e^{t}(\alpha e^{\alpha kt}-\beta e^{\beta kt})}{\sqrt{k^2+4}} - e^{(k+1)t}$$
$$\text{and} \quad EF_k(t) = \frac{2e^{kt}(\alpha e^{\alpha t}-\beta e^{\beta t})}{\sqrt{k^2+4}} - e^{(k+1)t}.$$

*Proof.* The argument for the results follows from the Theorem 5.1 of [22] which states that if $E(t)$ is the exponential generating function of a sequence $A_n$ then the exponential generating function for sequences $W(A_n,k)$, $R(A_n,k)$ and $F(A_n,k)$ are given by $e^{kt}E(kt)$, $e^tE(kt)$ and $e^{kt}E(t)$, respectively. □

## 4. The Hankel transform

In this section, we implement the Hankel transform to the Catalan transform of the $k$-Leonardo sequences. We should note that for a sequences $X = \{h_0, h_1, h_2, h_3, ...\}$ of real numbers, the Hankel transform [12, 19] of $X$ is a sequence of Hankel determinants $\{|H_1|, |H_2|, |H_3|, ...\}$, where the Hankel matrix $H = [a_{ij}]$ is defined as follows:

$$H = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & \dots \\ h_1 & h_2 & h_3 & h_4 & h_5 & \dots \\ h_2 & h_3 & h_4 & h_5 & h_6 & \dots \\ h_3 & h_4 & h_5 & h_6 & h_7 & \dots \\ h_4 & h_5 & h_6 & h_7 & h_8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \text{where } a_{ij} = h_{i+j-2};\ i, j \geq 1.$$

The left-upper sub-matrix of size $n \times n$ of $H$ is the Hankel matrix $H_n$ of order $n$ and the corresponding sequence of determinants $(\det(H_n) = |H_n|_{n\geq 1})$ gives the Hankel transforms.

Note that for Catalan number sequence, the Hankel transform is $\{1, 1, 1, ...\}$ [12] and for the sum of consecutive generalized Catalan numbers, the Hankel transform is the bisection of Fibonacci numbers [19].

Proceeding by taking into account the entries of Catalan transform of the $k$-Leonardo sequence from the preceding section, we get:

$$\begin{aligned} HT\mathcal{L}_{k,1} &= |T\mathcal{L}_{k,1}| = 2k-1, \\ HT\mathcal{L}_{k,2} &= \begin{vmatrix} T\mathcal{L}_{k,1} & T\mathcal{L}_{k,2} \\ T\mathcal{L}_{k,2} & T\mathcal{L}_{k,3} \end{vmatrix} = \begin{vmatrix} 2k-1 & 2(k^2+k) \\ 2(k^2+k) & 2k^3+4k^2+8k-1 \end{vmatrix} \\ &= -2k^3+8k^2-10k+1, \\ HT\mathcal{L}_{k,3} &= \begin{vmatrix} T\mathcal{L}_{k,1} & T\mathcal{L}_{k,2} & T\mathcal{L}_{k,3} \\ T\mathcal{L}_{k,2} & T\mathcal{L}_{k,3} & T\mathcal{L}_{k,4} \\ T\mathcal{L}_{k,3} & T\mathcal{L}_{k,4} & T\mathcal{L}_{k,5} \end{vmatrix} \\ &= \begin{vmatrix} 2k-1 & 2(k^2+k) & 2k^3+4k^2+8k-1 \\ 2(k^2+k) & 2k^3+4k^2+8k-1 & 2k^4+6k^3+16k^2+22k-2 \\ 2k^3+4k^2+8k-1 & \begin{matrix} 2k^4+6k^3+16k^2 \\ +22k-2 \end{matrix} & \begin{matrix} 2k^5+8k^4+26k^3+52k^2 \\ +70k-6 \end{matrix} \end{vmatrix} \\ &= 2k^5-16k^4+48k^3-64k^2+18k-1, \\ HT\mathcal{L}_{k,4} &= \begin{vmatrix} T\mathcal{L}_{k,1} & T\mathcal{L}_{k,2} & T\mathcal{L}_{k,3} & T\mathcal{L}_{k,4} \\ T\mathcal{L}_{k,2} & T\mathcal{L}_{k,3} & T\mathcal{L}_{k,4} & T\mathcal{L}_{k,5} \\ T\mathcal{L}_{k,3} & T\mathcal{L}_{k,4} & T\mathcal{L}_{k,5} & T\mathcal{L}_{k,6} \\ T\mathcal{L}_{k,3} & T\mathcal{L}_{k,5} & T\mathcal{L}_{k,6} & T\mathcal{L}_{k,7} \end{vmatrix} \\ &= -2k^7+24k^6-118k^5+300k^4-376k^3+184k^2-26k+1. \end{aligned}$$

We can continue in this fashion to get the entries for the Hankel transform for the Catalan transform of the $k$-Leonardo sequence as follow:

$$\begin{aligned} H\mathcal{L}_k &= \{HT\mathcal{L}_{k,1}, HT\mathcal{L}_{k,2}, HT\mathcal{L}_{k,3}, ...\} \\ &= \{2k-1,\ -2k^3+8k^2-10k+1,\ 2k^5-16k^4+48k^3-64k^2+18k-1, \\ &\qquad -2k^7+24k^6-118k^5+300k^4-376k^3+184k^2-26k+1, ...\}. \end{aligned}$$

In particular, the Hankel transform associated with the Catalan transform of the Leonardo sequence ($k = 1$) is

$$H\mathcal{L}_k = \{1, -3, -13, -13, ...\}.$$

## 5. Conclusion

In summary, the binomial, $k$-binomial, rising $k$-binomial and falling $k$-binomial transforms, and the Catalan transform are applied to the $k$-Leonardo sequence, and several new integer sequences obtained, some of them are matched with sequences listed on OEIS. Here, we discussed the generating and exponential generating functions of the binomial and Catalan transforms associated with the $k$-Leonardo sequences. In addition, we obtained the $k$-binomial transform, the Catalan transform, and the Catalan triangle for some values of $k$. Finally, the Hankel transform associated with the Catalan transform of the $k$-Leonardo sequence is manifested.

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