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PMC-LABELING OF PRISM ALLIED GRAPH, PENDULUM GRAPH AND COG WHEEL GRAPH

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Abstract. In this paper, we investigate the PMC-labeling behavior of some graphs like pendulum graph, hexagonal circular ladder, cog wheel graph, triangular wheel graph, double crown-wheel graph, crown-triangular wheel graph, $H_{n,n}$, $P_3 \times P_n$, prism allied graph and antiprism allied graph.

Keywords: pendulum graph, hexagonal circular ladder, cog wheel graph, triangular wheel graph, prism allied graph.

AMS Subject Classification: 05C38, 05C78

1. Introduction

In this study, we now consider only a undirected, finite and simple graphs. The sets $V(G)$ and $E(G)$, respectively denote the vertex set and edge set of the graph G. We refer to Harary [4] for all other terms and fundamental notations in graph theory. The concept of graph labeling was first presented by Rosa in 1967 [14]. Edge odd graceful labeling in some wheel-related graphs has been investigated by Aljohani and Daoud [1]. Ibrahim et al. [5] have worked on edge H-irregularity strength of hexagonal and octagonal grid graphs. L(2,1) labeling of lollipop and pendulum graphs were discussed in [7]. Susanti et al. [16] proved that double sun flower graph, triangular winged prism graphs and rectangular winged prism graphs are all edge odd graceful. Seoud et al. have worked on difference graphs [17] and mean graphs [18]. Chit presented the idea of cordial graphs [2]. Prime cordial and 3-equitable prime cordial graphs were studied in [19]. Sharma and Bhat [15]have worked on vertex edge resolvability for the web graph and prism related graph. Kumari and Mehra [6] have worked on vertex product cordial labeling. Prajapati and Gajjar [13] proved that the generalized prism graph is Prime cordial. Edge product cordial labeling of some graphs were examined in [12]. The rainbow connection number of origami graphs and pizza graphs have investigated by Nabila and Salman [8]. Gallian updates his dynamic survey on graph labeling on a regular basis [3]. The process of

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PMC-labeling graphs was first presented in [9], and PMC-labeling different graph families was covered in [10, 11]. This study computes the PMC-labeling behavior of some graphs like pendulum graph, hexagonal circular ladder, cog wheel graph, triangular wheel graph, double crown-wheel graph, crown-triangular wheel graph, $H_{n,n}$, $P_3 \times P_n$, prism allied graph and antiprism allied graph.

2. Preliminaries

Definition 2.1. [7] *The pendulum graph $S_{n,k}$ is a graph that is created by joining each leaf of the star graph $K_{1,k}$ with a cycle graph C_n.*

Definition 2.2. [15] *The heptagonal circular ladder HCL_n is the graph by adding new vertices between the vertices of degree three in the pentagonal circular ladder PCL_n.*

Definition 2.3. [1] *The cog wheel graph CW_n is a graph obtained by combining a wheel graph $W_n = C_n + K_1$, where $C_n = v_1v_2v_3 \dots v_nv_1$ and $K_1 = \{u_0\}$ with a set of vertices $\{u_1, u_2, u_3, \dots, u_n\}$ such that a vertex u_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$. More over, the vertex u_n is adjacent to vertices v_1 and v_n.*

Definition 2.4. [1] *The triangular wheel graph TW_n is a graph obtained by combining the wheel graph $W_n = C_n + K_1$, where $C_n = v_1v_2v_3 \dots v_nv_1$ and $K_1 = \{u_0\}$ with a set of vertices $\{u_1, u_2, u_3, \dots, u_n\}$ such that vertex u_i is adjacent to vertices v_i and v_{i+1}, and vertex u_n is adjacent to vertices v_1 and v_n. Additionally, each vertex u_i is adjacent to the vertex u_0.*

Definition 2.5. [1] *The double crown-wheel graph DCW_n is a graph obtained by combining a wheel graph $W_n = C_n + K_1$, where $C_n = v_1v_2v_3 \dots v_nv_1$ and $K_1 = \{u_0\}$ with two sets of vertices. The first set is $\{w_1, w_2, w_3, \dots, w_n\}$, where each vertex w_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$, and the vertex w_n is adjacent to vertices v_1 and v_n. The second set is $\{u_1, u_2, u_3, \dots, u_n\}$, where each vertex u_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$, and the vertex u_n is adjacent to vertices v_1 and v_n.*

Definition 2.6. [1] *The crown-triangular wheel graph CTW_n is a graph obtained by combining the triangular wheel graphs TW_n, its vertex set $V(TW_n) = \{u_0, v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$ with the set $\{w_1, w_2, w_3, \dots, w_n\}$, where each vertex w_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$, and the vertex w_n is adjacent to vertices v_1 and v_n.*

Definition 2.7. [3] *The direct product of G and H is the graph denoted $G \times H$ whose vertex set is $V(G) \times V(H)$ and for which vertices (u, v) and (u', v') are adjacent precisely if $uu' \in E(G)$ and $vv' \in E(H)$. Then $V(G \times H) = \{(u, v) \mid u \in V(G) and \quad v \in V(H)\}$, $E(G \times H) = \{(u, v), (u', v') \mid uu' \in E(G) and \quad vv' \in E(H)\}$.*

Definition 2.8. [3] *The graph $H_{n,n}$ has the vertex set $V(H_{n,n}) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and the edge set $E(H_{n,n}) = \{u_iv_j \mid 1 \leq i \leq n, n-i+1 \leq j \leq n\}$.*

Definition 2.9. [15] *The prism allied graph DA_n has vertex set of cardinality $4n$ and edge set of cardinality $6n$, indicated by $V(DA_n)$ and $E(DA_n)$ respectively, where $V(DA_n) = \{u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(DA_n) = \{u_iv_i, x_iy_i, v_ix_i \mid 1 \leq i \leq n\} \cup \{v_ix_{i+1}, v_nx_1, u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1 \mid 1 \leq i \leq n-1\}$.*

Definition 2.10. [15] *The antiprism allied graph AA_n has vertex set of cardinality $4n$ and edge set of cardinality $7n$, indicated by $V(AA_n)$ and $E(AA_n)$ respectively, where $V(AA_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(AA_n) = \{v_iw_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1}, v_nv_1, u_iu_{i+1}, u_nu_1, v_iu_{i+1}, v_nu_1 \mid 1 \leq i \leq n-1\}$*

3. PMC-Labeling

Let a graph $G = (V, E)$ be a graph with p vertices and q edges. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling(PMC-labeling) is called a pair mean cordial graph(PMC-graph). The following figure 1 is the simple example of the PMC-graph.

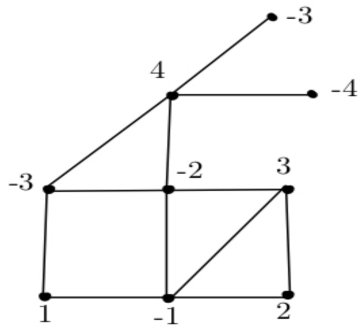

Figure 1. PMC-graph.

4. Main Results

Theorem 4.1. *The heptagonal circular ladder HCL_n is a PMC-graph for all values of $n \geq 3$.*

Proof. Consider the heptagonal circular ladder HCL_n. Denote by $V(HCL_n) = \{u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(HCL_n) = \{u_ix_i, x_iv_i, v_iy_i \mid 1 \leq i \leq n\} \cup \{x_iv_{i+1}, x_nv_1, u_iu_{i+1}, u_nu_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of the heptagonal circular ladder HCL_n. Then, it has $4n$ vertices and $5n$ edges. Define $\lambda(v_1) = -n$, $\lambda(u_1) = -n-1$, $\lambda(x_1) = n+2$, and $\lambda(x_n) = 1$. We now consider two cases:
Case (i) : n is odd
Let us assign the labels $2, 3, \ldots, n+1$ and $-1, -2, \ldots, -n+1$ to the corresponding vertices $y_1, y_2, \ldots, y_n$ and $v_2, v_3, \ldots, v_n$. So, assign the labels $-n-2, -n-3, \ldots, \frac{-3n-1}{2}$ and $n+3, n+4, \ldots, \frac{3n+3}{2}$ to the corresponding vertices $u_2, u_4, \ldots, u_{n-1}$ and $u_3, u_5, \ldots, u_n$. Then, assign the labels $\frac{-3n-3}{2}, \frac{-3n-5}{2}, \ldots, -2n$ and $\frac{3n+5}{2}, \frac{3n+7}{2}, \ldots, 2n$ to the corresponding vertices $x_2, x_4, \ldots, x_{n-1}$ and $x_3, x_5, \ldots, x_{n-2}$.
Case (ii) : n is even
Also, assign the labels to the vertices $y_i, v_i, 1 \leq i \leq n$ as in case (i). Thus, assign the labels $-n-2, -n-3, \ldots, \frac{-3n-2}{2}$ and $n+3, n+4, \ldots, \frac{3n+2}{2}$ to the corresponding vertices $u_2, u_4, \ldots, u_n$ and $u_3, u_5, \ldots, u_{n-1}$. Now, assign the labels $\frac{-3n-4}{2}, \frac{-3n-6}{2}, \ldots, -2n$

and $\frac{3n+6}{2}, \frac{3n+8}{2}, \ldots, 2n$ to the corresponding vertices $x_2, x_4, \ldots, x_{n-2}$ and $x_3, x_5, \ldots, x_{n-1}$. Further, the edges $v_{i+1}y_i$, $v_{i+1}y_{i+1}$, $1 \leq i \leq n-1$, y_nv_1, v_1x_1, x_1u_1 and $u_{2i}u_{2i+1}$ for $1 \leq i \leq \frac{n-1}{2}$ when n is odd, $u_{2i}u_{2i+1}$ for $1 \leq i \leq \frac{n-2}{2}$ when n is even, are labeled with 1 and all other edges are labeled by the integers other than 1.

TABLE 1. The following table 1 establish that the PMC-labeling of the heptagonal circular ladder HCL_n for all values of $n \geq 3$.

Values of n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
n is odd	$\frac{5n+1}{2}$	$\frac{5n-1}{2}$
n is even	$\frac{5n}{2}$	$\frac{5n}{2}$

□

Example 4.1. *Figure 2 illustrates the PMC-labeling of the heptagonal circular ladder HCL_7.*

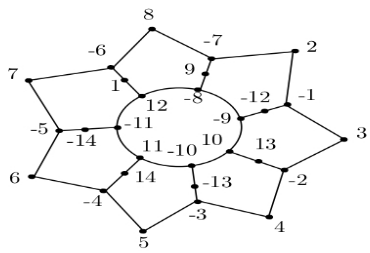

FIGURE 2. PMC-labeling of the heptagonal circular ladder HCL_7.

Theorem 4.2. *The pendulum graph $S_{n,k}$ is a PMC-graph for all values of $n \geq 3$ and $k \geq 1$.*

Proof. Let us now consider the pendulum graph $S_{n,k}$. Denote by $V(S_{n,k}) = \{v_0, v_{i,j} \mid 1 \leq i \leq n \text{and} 1 \leq j \leq k\}$ and $E(S_{n,k}) = \{v_0v_{1,j}, v_{i,j}v_{i+1,j}, v_{n,j}v_{1,j} \mid 1 \leq i \leq n-1 \text{and} 1 \leq j \leq k\}$ respectively, the vertex and edge sets of the pendulum graph $S_{n,k}$. Then, it has $nk+1$ vertices and $(n+1)k$ edges. We now consider two cases:
Case (i) : n is even
Define $\lambda(u_0) = -1$.
Subcase (i) : k is even

$$\lambda(v_{i,j}) = \begin{cases} \frac{(j-1)n+i+3}{2} & 1 \leq i \leq n,\ 1 \leq j \leq k-1,\ i \ \&\ j \text{ are odd} \\ \frac{(-j+1)n-i}{2} & 1 \leq i \leq n,\ 1 \leq j \leq k-1,\ i \text{ is even } \&\ j \text{ is odd}, \end{cases}$$

$$\lambda(v_{i,j}) = \begin{cases} \frac{(j-1)n+2i+2}{2} & i = 1, 2, \ldots, \frac{n}{2},\ 1 \leq j \leq k-1,\ j \text{ is even} \\ \frac{(-j-1)n+2i-2}{2} & i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n,\ 1 \leq j \leq k-1,\ j \text{ is even}, \end{cases}$$

$$\lambda(v_{i,k}) = \begin{cases} \frac{(k-1)n+i+3}{2} & i = 1 \\ \frac{(-k+1)n-2i+2}{2} & i = 2, 3, \ldots, \frac{n+2}{2} \\ \frac{(k+1)n-2i+4}{2} & i = \frac{n+4}{2}, \frac{n+6}{2}, \ldots, n-1 \\ 1 & i = n. \end{cases}$$

Subcase (ii) : k is odd
Let us assign the labels to the vertices $v_{i,j}, 1 \leq i \leq n,\ 1 \leq j \leq k-1$ as in case (i) of subcase (i). Then,

$$\lambda(v_{i,k}) = \begin{cases} \frac{(-k+1)n-i-1}{2} & i = 1, 3, \ldots, n-1 \\ \frac{(k-1)n+i+2}{2} & i = 2, 4, \ldots, n-2 \\ 1 & i = n. \end{cases}$$

In both subcases, the edges $v_0v_{1,1}$, $v_{i,j}v_{i+1,j}$, for $1 \leq i \leq n$, $1 \leq j \leq k-1$, j is odd, $v_{n,j}v_{1,j}$, $v_{\frac{n}{2},j}v_{\frac{n+2}{2},j}$, $1 \leq j \leq k-1$, j is even, and $v_{1,k}v_{2,k}$, when k is even, $v_{2i-1,k}v_{2i,k}$, when k is odd for $2 \leq i \leq \frac{n-2}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1.
Case (ii) : n is odd
Subcase (i) : k is even
Define $\lambda(u_0) = -1$.

$$\lambda(v_{i,j}) = \begin{cases} \frac{(j-1)n+i+3}{2} & 1 \leq i \leq n,\ 1 \leq j \leq k-1,\ i \text{ \& } j \text{ are odd} \\ \frac{(-j+1)n-i}{2} & 1 \leq i \leq n,\ 1 \leq j \leq k-1,\ i \text{ is even \& } j \text{ is odd}, \end{cases}$$

$$\lambda(v_{i,j}) = \begin{cases} \frac{(-j+1)n-2i+1}{2} & i = 1, 2, \ldots, \frac{n+1}{2},\ 1 \leq j \leq k-1,\ j \text{ is even} \\ \frac{(j+1)n-2i+5}{2} & i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n,\ 1 \leq j \leq k-1,\ j \text{ is even}, \end{cases}$$

$$\lambda(v_{i,k}) = \begin{cases} \frac{(-k+1)n-2i+1}{2} & i = 1 \\ \frac{(k-1)n+2i+1}{2} & i = 2, 3, \ldots, \frac{n-1}{2} \\ \frac{(-k-1)n+2i-1}{2} & i = \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1 \\ 1 & i = n. \end{cases}$$

Subcase (ii) : k is odd
Define $\lambda(u_0) = 1$. Further, assign the labels to the vertices $v_{i,j}, 1 \leq i \leq n,\ 1 \leq j \leq k-1$ as in case (i) of subcase (i). Thus,

$$\lambda(v_{i,k}) = \begin{cases} \frac{(k-1)n+i+3}{2} & i = 1, 3 \\ \frac{(-k+1)n-i}{2} & i = 2, 4 \\ \frac{(-k+1)n-i-1}{2} & i = 5, 7, \ldots, n \\ \frac{(k-1)n+i+2}{2} & i = 6, 8, \ldots, n-1. \end{cases}$$

In both subcases, the edges $v_{i,j}v_{i+1,j}$, for $1 \leq i \leq n$, $1 \leq j \leq k-1$, j is odd, $v_{n,j}v_{1,j}$, $v_{\frac{n+1}{2},j}v_{\frac{n+3}{2},j}$, $1 \leq j \leq k-1$, j is even, and $v_{1,k}v_{2,k}$, $v_0v_{1,1}$ when k is even, $v_{2i-1,k}v_{2i,k}$, $v_{2,k}v_{3,k}$ when k is odd for $2 \leq i \leq \frac{n-1}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1.

□

Table 2. The following table 2 establish that the PMC-labeling of the pendulum graph $S_{n,k}$ for all values of $n \geq 3$ and $k \geq 1$.

n and k	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
n and k are even	$\frac{(n+1)k}{2}$	$\frac{(n+1)k}{2}$
n is even and k is odd	$\frac{(n+1)k-1}{2}$	$\frac{(n+1)k+1}{2}$
n is odd k is even	$\frac{(n+1)k}{2}$	$\frac{(n+1)k}{2}$
n and k are odd	$\frac{(n+1)k}{2}$	$\frac{(n+1)k}{2}$

Example 4.2. *Figure* 3 *illustrates the PMC-labeling of the pendulum graph* $S_{5,4}$.

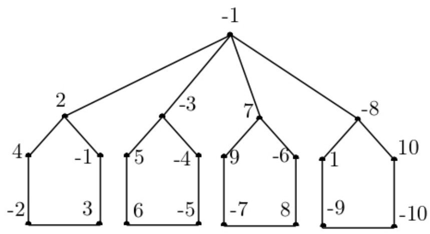

FIGURE 3. PMC-labeling of the pendulum graph $S_{5,4}$.

Theorem 4.3. *The cog wheel graph CW_n is not PMC-graph for every values of $n \geq 3$.*

Proof. Let us now consider the cog wheel graph CW_n. Denote by $V(CW_n) = \{u_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(CW_n) = \{u_0v_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{u_iv_{i+1}, v_iv_{i+1}, u_nv_1, v_nv_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of the cog wheel graph CW_n . Eventually CW_n holds $2n+1$ vertices and $4n$ edges. If possible, let λ be a PMC-graph. The possible outcomes are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$ if the edge uv receives the label one. Therefore, $2n-1$ is the maximum number of edges that can be labeled as 1. That is, $\bar{\mathbb{S}}_{\lambda_1} \leq 2n-1$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n+1$. Thus, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n+1-(2n-1) = 2 > 1$, a contradiction. □

Theorem 4.4. *The triangular wheel graph TW_n is not PMC-graph for every values of $n \geq 3$.*

Proof. Let us now denote by $V(TW_n) = \{u_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(TW_n) = \{u_0u_i, u_0v_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{u_iv_{i+1}, v_iv_{i+1}, u_1v_n, v_nv_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of the triangular wheel graph TW_n. Eventually, TW_n holds $2n+1$ vertices and $5n$ edges. Suppose now that TW_n is a PMC-graph. The possible outcomes are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$ if the edge uv receives the label one. Thereby, $2n-1$ is the maximum number of edges that can be labeled as 1. That's $\bar{\mathbb{S}}_{\lambda_1} \leq 2n-1$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 3n+1$. Therefore, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3n+1-(2n-1) = n+2 \geq 5 > 1$, a contradiction. □

Theorem 4.5. *The double crown-wheel graph DCW_n is not PMC-graph for every values of $n \geq 3$.*

Proof. The vertex and edge sets of the double crown-wheel graph DCW_n are denoted by $V(DCW_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(DCW_n) = \{u_0v_i, u_iv_i, w_iv_i \mid 1 \leq i \leq n\} \cup \{u_iv_{i+1}, w_iv_{i+1}, v_iv_{i+1}, u_nv_1, w_nv_1, v_nv_1 \mid 1 \leq i \leq n-1\}$ respectively. Eventually DCW_n holds $3n+1$ vertices and $6n$ edges. If possible, let DCW_n be a PMC-graph.
Case (i) : n is odd
Therefore, $2n$ is the maximum number of edges that can be labeled as 1. That's $\bar{\mathbb{S}}_{\lambda_1} \leq 2n$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 4n$. Thus, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4n-2n = 2n \geq 6 > 1$, a contradiction.
Case (ii) : n is even
Therefore, $2n+2$ is the maximum number of edges that can be labeled as 1. That is, $\bar{\mathbb{S}}_{\lambda_1} \leq 2n+2$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 4n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4n-2-(2n+2) = 2n-4 \geq 4 > 1$, a contradiction. □

Theorem 4.6. *The crown- triangular wheel graph CTW_n is not PMC-graph for every values of $n \geq 3$.*

Proof. Consider the crown- triangular wheel graph CTW_n. Denote by $V(CTW_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(CTW_n) = \{u_0v_i, u_0u_i, u_iv_i, w_iv_i \mid 1 \leq i \leq n\} \cup \{u_iv_{i+1}, w_iv_{i+1}, v_iv_{i+1}, u_nv_1, w_nv_1, v_nv_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of crown- triangular wheel graph CTW_n. Eventually CTW_n holds $3n+1$ vertices and $7n$ edges. If possible, let CTW_n be a PMC-graph.
Case $(i) : n = 3$
Then, $\bar{\mathbb{S}}_{\lambda_1} \leq 7$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 14$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 14 - 7 = 7 > 1$, a contradiction.
Case $(ii) : n$ is even
Hence, $2n+4$ is the maximum number of edges that can be labeled as 1. That is, $\bar{\mathbb{S}}_{\lambda_1} \leq 2n+4$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 5n-4$. Then, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 5n-4-(2n+4) = 3n-8 \geq 4 > 1$, a contradiction.
Case $(iii) : n$ is odd
Thus, $2n+2$ is the maximum number of edges that can be labeled as 1. That's $\bar{\mathbb{S}}_{\lambda_1} \leq 2n+2$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq 5n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 5n-2-(2n+2) = 3n-4 \geq 11 > 1$, a contradiction. □

Theorem 4.7. *The graph $H_{n,n}$ is a PMC-graph iff $1 \leq n \leq 5$.*

Proof. Now, consider the graph $H_{n,n}$. Denoting by $V(H_{n,n}) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and $E(H_{n,n}) = \{u_iv_j \mid 1 \leq i \leq n, n-i+1 \leq j \leq n\}$ respectively, the vertex and edge sets of the graph $H_{n,n}$. Then, $H_{n,n}$ has $2n$ vertices and $\frac{n(n+1)}{2}$ edges. Note that $H_{1,1} \simeq P_2$, the path P_2 is a PMC-graph [9]. We now consider two cases:
Case (i) : For $2 \leq n \leq 5$
Here, assign the labels $-1, -2, \ldots, -n$ to the corresponding vertices $u_1, u_2, \ldots, u_n$. Then, assign the labels $2, 3, \ldots, n$ to the corresponding vertices $v_1, v_2, \ldots, v_n$. Fix label 1 with v_n. More over, the edges u_iv_i, for $1 \leq i \leq n-1$, and u_iv_{i+1}, for $1 \leq i \leq n-2$ are labeled with 1 and all other edges are labeled by the integers other than 1. Hence, $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n(n+1)}{4} = \bar{\mathbb{S}}_{\lambda_1}$ if n is even and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n(n+1)+2}{4}$, $\bar{\mathbb{S}}_{\lambda_1} = \frac{n(n+1)-2}{4}$ if n is odd.
Case (ii) : For $n \geq 6$
Suppose now that $H_{n,n}$ is a PMC-graph. The possible outcomes are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$ if the edge uv receives the label one. Then $2n-3$ is the maximum number of edges that can be labeled as 1. That's $\bar{\mathbb{S}}_{\lambda_1} \leq 2n-3$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} = q - \bar{\mathbb{S}}_{\lambda_1} \geq q - (2n-3) = \frac{n^2-3n+6}{2}$. Therefore, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq \frac{n^2-3n+6}{2} - (2n-3) = \frac{n^2-7n+12}{2} \geq 3 > 1$, a contradiction.

□

Example 4.3. *Figure 4 illustrates the PMC-labeling of the graph $H_{4,4}$.*

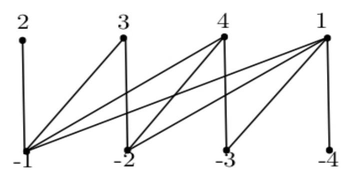

FIGURE 4. PMC-labeling of the graph $H_{4,4}$.

Theorem 4.8. *The graph $P_3 \times P_n$ is a PMC-graph for all values of $n \geq 1$.*

Proof. Consider the graph $P_3 \times P_n$. Denoting by $V(P_3 \times P_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(P_3 \times P_n) = \{u_iv_{i+1}, u_iw_{i+1}, u_{i+1}v_i, u_{i+1}w_i \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of the graph $P_3 \times P_n$. Then, $P_3 \times P_n$ has $3n$ vertices and $4n-4$ edges. We now consider two cases:
Case (i) : For odd n
Here, assign the labels $-1, -4, \ldots, \frac{-3n+1}{2}$ and $4, 7, \ldots, \frac{3n-1}{2}$ to the corresponding vertices $u_1, u_3, \ldots, u_n$ and $u_2, u_4, \ldots, u_{n-1}$. Then, assign the labels $-3, -6, \ldots, \frac{-3n+3}{2}$ and $3, 6, \ldots, \frac{3n-3}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-2}$ and $v_2, v_4, \ldots, v_{n-1}$. Fix label 1 with v_n. Next, assign the labels $-2, -5, \ldots, \frac{-3n+5}{2}$ and $2, 5, \ldots, \frac{3n-5}{2}$ to the corresponding vertices $w_1, w_3, \ldots, w_{n-2}$ and $w_2, w_4, \ldots, w_{n-1}$. Fix label 1 with w_n. Hence, the edges $u_{i+1}v_i$, $u_{i+1}w_i$, u_iv_{i+1} and u_iw_{i+1} for $1 \leq i \leq \frac{n-1}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1.
Case (ii) : For even n
Now, assign the labels $-1, -4, \ldots, \frac{-3n+4}{2}$ and $4, 7, \ldots, \frac{3n-4}{2}$ to the corresponding vertices $u_1, u_3, \ldots, u_{n-1}$ and $u_2, u_4, \ldots, u_{n-2}$. Fix label 1 with u_n. Then, assign the labels $-3, -6, \ldots, \frac{-3n}{2}$ and $3, 6, \ldots, \frac{3n}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{n-1}$ and $v_2, v_4, \ldots, v_n$. More over, assign the labels $-2, -5, \ldots, \frac{-3n+2}{2}$ and $2, 5, \ldots, \frac{3n-2}{2}$ to the corresponding vertices $w_1, w_3, \ldots, w_{n-1}$ and $w_2, w_4, \ldots, w_n$. Thus, the edges $u_{i+1}v_i$, $u_{i+1}w_i$, for $1 \leq i \leq \frac{n-2}{2}$ and u_iv_{i+1}, u_iw_{i+1} for $1 \leq i \leq \frac{n}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1. In both cases, $\bar{\mathbb{S}}_{\lambda_1^c} = 2n-2 = \bar{\mathbb{S}}_{\lambda_1}$. □

Example 4.4. *Figure 5 illustrates the PMC-labeling of the graph $P_3 \times P_4$.*

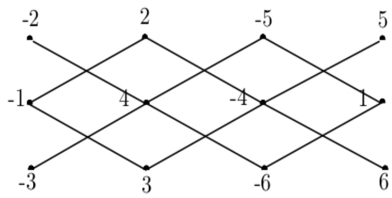

FIGURE 5. PMC-labeling of the graph $P_3 \times P_4$.

Theorem 4.9. *The prism allied graph DA_n is a PMC-graph for all values of $n \geq 3$.*

Proof. Consider prism allied graph DA_n. Denoting by $V(DA_n) = \{u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(DA_n) = \{u_iv_i, x_iy_i, v_ix_i \mid 1 \leq i \leq n\} \cup \{v_ix_{i+1}, v_nx_1, u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge sets of the prism allied graph DA_n. Then, it has $4n$ vertices and $6n$ edges. We now consider two cases:
Case (i) : n is odd
Let us assign the labels $-1, -2, \ldots, -n$ and $3, 4, \ldots, n+2$ to the corresponding vertices $x_1, x_2, \ldots, x_n$ and $v_1, v_2, \ldots, v_n$. Then, assign the labels $-n-1, -n-2, \ldots, \frac{-3n-1}{2}$ and $n+3, n+4, \ldots, \frac{3n+3}{2}$ to the corresponding vertices $u_1, u_3, \ldots, u_n$ and $u_2, u_4, \ldots, u_{n-1}$. Define $\lambda(y_1) = 2$. So, assign the labels $\frac{-3n-3}{2}, \frac{-3n-5}{2}, \ldots, -2n$ and $\frac{3n+5}{2}, \frac{3n+7}{2}, \ldots, 2n$ to the corresponding vertices $y_2, y_4, \ldots, y_{n-1}$ and $y_3, y_5, \ldots, y_{n-2}$. Finally, fix the label 1 with y_n.
Case (ii) : n is even

Now, assign the labels to the vertices $x_i, v_i, 1 \leq i \leq n$ as in case (i). Thus, assign the labels $-n-1, -n-2, \ldots, \frac{-3n}{2}$ and $n+3, n+4, \ldots, \frac{3n+4}{2}$ to the corresponding vertices $u_1, u_3, \ldots, u_{n-1}$ and $u_2, u_4, \ldots, u_n$. Define $\lambda(y_1) = 2$ and $\lambda(y_2) = \frac{-3n-2}{2}$. Also, assign the labels $\frac{-3n-4}{2}, \frac{-3n-6}{2}, \ldots, -2n$ and $\frac{3n+6}{2}, \frac{3n+8}{2}, \ldots, 2n$ to the corresponding vertices $y_3, y_5, \ldots, y_{n-1}$ and $y_4, y_6, \ldots, y_{n-2}$. Finally, fix the label 1 with y_n. In both cases, the edges $v_{i+1}x_i$, u_iu_{i+1}, $v_{i+1}x_{i+1}$, for $1 \leq i \leq n-1$, x_1y_1, v_1u_1 and x_nv_1, are labeled with 1 and all other edges are labeled by the integers other than 1. Thus, $\bar{\mathbb{S}}_{\lambda_1^c} = 3n = \bar{\mathbb{S}}_{\lambda_1}$. $\square$

Example 4.5. *Figure 6 illustrates the PMC-labeling of the prism allied graph DA_5.*

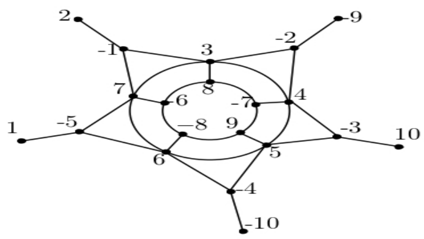

FIGURE 6. PMC-labeling of the prism allied graph DA_5.

Theorem 4.10. *The antiprism allied graph AA_n is not PMC-graph for all values of $n \geq 3$.*

Proof. Let us now consider antiprism allied graph AA_n. The vertex and edge sets of the antiprism allied graph AA_n, respectively denoted by $V(AA_n) = \{u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(AA_n) = \{x_iy_i, u_iv_i, v_ix_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1}, v_nv_1, u_iu_{i+1}, u_nu_1, v_iu_{i+1}, v_nu_1, x_iv_{i+1}, x_nv_1 \mid 1 \leq i \leq n-1\}$. Then, it has $4n$ vertices and $7n$ edges. Suppose now that the antiprism allied graph AA_n is a PMC-graph. Then, $3n$ is the maximum number of edges that can be labeled as 1. That is, $\bar{\mathbb{S}}_{\lambda_1} \leq 3n$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - (3n) = 4n$. Thus, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4n - (3n) = n \geq 3 > 1$, a contradiction. $\square$

5. CONCLUSION

In this paper, we have examined the PMC-labeling behavior of some special graphs like pendulum graph, hexagonal circular ladder, cog wheel graph, triangular wheel graph, double crown-wheel graph, crown-triangular wheel graph, $H_{n,n}$, $P_3 \times P_n$, prism allied graph and antiprism allied graph. Usually in the graph labeling vertices are assigned by positive integers. But in the labeling method that have studied in this paper, both positive and negative integers are used as label for vertices. Investigation of whether this labeling techniques is present or not on other graph families is an open problem for future work.

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R. Ponraj for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.15, N.2.

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