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ON SOME FAMILIES OF LINEAR DIOPHANTINE GRAPHS

O. M. SALAMA[1,*], M.A. SEOUD[1], M. ANWAR[1], A. ELSONBATY[1], §

ABSTRACT. Diophantine labeling of graphs is an extension of the prime labeling of graphs. In this manuscript, we introduce some necessary conditions for determining whether a given graph admits Diophantine labeling or not, and if yes, we will find such a Diophantine labeling. We also study specific families of graphs, including the Complete graphs K_n, Wheel graphs W_n and $W_{n,n}$, Circulant graphs $C_n(j)$, Path graphs $P_n(j)$, Cartesian product graphs $C_3 \times C_m$, Normal Product graphs $P_n \circ P_n$, Corona graphs $G \odot H$, Double Fan graphs $g_n = P_n + \overline{K_2}$, Power graphs P_n^2 and P_n^3, to ascertain their Diophantine nature.

Keywords: Graph labeling, Prime labeling, Diophantine labeling, Families of Diophantine graphs.

AMS Subject Classification: 05C78

1. Introduction

In this work, we deal with finite, simple and undirected graph $G = (V, E)$, where $V = V(G)$ denotes the vertex set and $E = E(G)$ denotes the edge set. $|V| = n$ vertices and $|E| = m$ edges. The term $|V|$ is called the order of the graph G, while $|E|$ is called the size of the graph G. A set of vertices $S \subset V(G)$ is said to be independent if no two vertices u and v in S are adjacent in G. The maximum number of vertices of an independent set in G is called the vertex independence number of G or simply independence number, usually denoted by $\alpha(G)$ [1, 6, 9, 11]. The Cartesian product $G_1 \times G_2$ of G_1 with n vertices and G_2 with m vertices is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set

$$\{(u_1, v_1)(u_2, v_2) : (u_1 = u_2 \;\text{ and }\; v_1v_2 \in E(G_2)) \quad \text{or} \quad (v_1 = v_2 \;\text{ and }\; u_1u_2 \in E(G_1))\},$$

this means that we have n copies of G_2 and m copies of G_1 [8, 13]. The corona graph $G_1 \odot G_2$ of graphs G_1 and G_2 obtained by taking one copy of G_1(which has n_1 vertices) and n_1 copies of G_2(which has n_2 vertices), and then joining the i^{th} vertex of $G_1 \neq 0$ to every vertex in the i^{th} copy of G_2, $G_1 \odot G_2$ has $n_1.(n_2 + 1)$ [8, 13].

[1] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt.
e-mail: omarsalama@sci.asu.edu.eg; ORCID: https://orcid.org/0000-0001-9829-2846.
e-mail: M.Seoud@sci.asu.edu.eg; ORCID: https://orcid.org/0000-0003-3216-5873 .
e-mail: mohamedanwar@sci.asu.edu.eg; ORCID: https://orcid.org/0000-0003-0668-4071 .
e-mail: ahmedelsonbaty@sci.asu.edu.eg; ORCID: https://orcid.org/0000-0002-9001-4961.
* Corresponding author.
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Definition 1.1. *The normal product $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph with vertex-set $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$, where*

$$V(G_1) = \{u_1, u_1, \ldots, u_n\},\ V(G_2) = \{v_1, v_2, \ldots, v_m\},$$

and $(x_1, y_1), (x_2, y_2) \in V(G_1) \times V(G_2)$ are adjacent in $G_1 \circ G_2$ if and only if

(1) *$x_1 = x_2$ and y_1 is adjacent to y_2 in G_2*
(2) *$y_1 = y_2$ and x_1 is adjacent to x_2 in G_1*
(3) *x_1 is adjacent to x_2 in G_1 and y_1 is adjacent to y_2 in G_2.*

The k^{th} power of a graph G, denoted as G^k, is a graph with the same vertex set as G, but where two vertices are adjacent if their distance in G is at most k [1]. Powers of graphs are referred to using terminology similar to that of exponentiation of numbers : G^2 is called the square of G, G^3 is called the cube of G, etc. The k^{th} power of a graph G contains all the original edges of this graph G [1]. The Diameter of a graph G is the maximum distance between pairs of vertices in this graph G [1, 2, 11, 22]. A circulant graph $C_n(j)$ of order n, for a fixed j is a super graph of the cycle graph C_n, defined as follows:

$$V(C_n(j)) = V(C_n) = \{v_1, v_2, \ldots, v_n\},$$

and edge set

$$E(C_n(j)) = E(C_n) \cup \{v_i v_{i+j(mod n)} : i \in \{1, 2, 3, \ldots, n\}\}, j \leq \frac{n}{2}$$

[10]. A graph $P_n(j)$ of order n, for a fixed j is a super graph of the path graph P_n, defined as follows:

$$V(P_n(j)) = V(P_n) = \{v_1, v_2, \ldots, v_n\},$$

and edge set

$$E(P_n(j)) = E(P_n) \cup \{v_i v_{i+j(mod n)} : i \in \{1, 2, 3, \ldots, n\}\},\ j \leq \frac{n}{2}$$

[10]. The wheel graph W_n, denoted as $C_n + K_1$ has $n + 1$ vertices [5, 8].

Definition 1.2. [5, 8, 23]. *The double wheel graph is a graph consisting of two cycles of vertices connected to a common center (hub). For every $n \geq 4, |V(W_{n,n})| = 2n + 1$ and*

$$\alpha(W_{n,n}) = \begin{cases} n & \text{if } n \text{ even} \\ n - 1 & \text{if } n \text{ odd} \end{cases}$$

A triangular snake TS'_n is obtained from a path $P_n = \{u_1, u_2, u_3, \ldots, u_n\}$ by joining u_i and u_{i+1} to a new vertex v_i, where $1 \leq i \leq n - 1$, and joining v_i to v_{i+1}, $1 \leq i \leq n - 2$, TS'_n has $2n - 1$ vertices [6].

The terminology and notations in this manuscript follow from Harary [11] and Allan Bickel [1]. Graph labeling involves assigning real values to vertices and edges while satisfying certain conditions. The Literature of this field which began with the seminal paper of Rosa [14], includes thousands of papers covering hundred of methods of labelings.

Definition 1.3. [21] *Let $G = (V, E)$ be a simple graph of order n. The graph G is called a prime graph if there exists a bijective map $f : V \longrightarrow \{1, 2, \ldots, n\}$ such that $(f(u), f(v)) = 1$ for all $uv \in E$ .i.e, $f(u)$ and $f(v)$ are relatively prime, this bijective map f is referred to as a prime labeling of G.*

The concept of prime labeling was introduced by Roger Entringer and discussed in a paper by Tout [21]. Vertex prime labeling was also discussed in a paper by Deretsky [4]. Seoud and Youssef discussed necessary and sufficient conditions for prime labeling [19].

Many researchers have studied prime graphs and made significant contributions to the field. In 1994, Fu and Huang proved that the path P_n on n vertices is a prime graph [7]. In 1991, Deretsky et al. proved that the cycle C_n on n vertices is a prime graph [4]. In 1998, S. Lee et al. showed that the Wheel W_n is a prime graph if and only if n is even [12]. Around 1980 Roger Etringer conjectured that all trees have prime labeling, but this conjecture remains unsettled to this day.

Definition 1.4. [20] *Let $G = (V, E)$ be a simple graph of order n. The graph G is called Diophantine if there exists a bijective map $f : V \longrightarrow \{1, 2, \dots, n\}$ such that $(f(u), f(v)) \mid n$ for all $uv \in E$, where $(f(u), f(v))$ is the greatest common divisor of $f(u)$ and $f(v)$, this bijective map f is referred to as a Diophantine labeling of G.*

The notion of Diophantine labeling is clearly an extension of the concept of prime labeling. The complete graph K_4 is Diophantine but not prime and K_5 is not Diophantine.

Lemma 1.1. [8] *If G and H are two graphs of n and m vertices respectively, then*

$$\alpha(G \odot H) = n\alpha(H)$$

Lemma 1.2. [6] *For every $n \geq 2$, the order of the maximum independent set for the normal product graph $P_n \circ P_n$ is $\lceil \frac{n}{2} \rceil^2, |V(P_n \circ P_n)| = n^2$.*

The independence numbers of circulant graphs satisfy the following relations.

Lemma 1.3. [6, 10] *Let C_n be the cycle graph on n vertices, $C_n(j)$ be the circulant graph with one jump $j \leq \frac{n}{2}$, then,*

(1) $\forall n \geq 3, \alpha(C_n(2)) = \lfloor \frac{n}{3} \rfloor$,
(2) $\forall$ *even number* $j \leq \frac{n}{2}, \alpha(C_n(j)) < \lfloor \frac{n}{2} \rfloor$,
(3) $\forall$ *odd number* $j \leq \frac{n}{2}, \alpha(C_n(j)) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \lfloor \frac{n}{2} \rfloor - \lfloor \frac{j}{2} \rfloor, & \text{if } n \text{ is odd} \end{cases}$

The Gauss's Pi function $\pi(x)$ is defined as the number of primes not exceeding a non-negative real number x [3, 15]. In modular arithmetic, for integers a,b, and m where $m > 0$, we say that a is congruent to b modulo m if and only if m divides $a - b$, denoted as $a \equiv b \ (mod\ m)$ [3, 15]. The greatest common divisor of two integers a and b, denoted as (a, b), is the largest positive integer that divides both a and b [3, 15]. If $(a, b) = 1$ we say that a and b are relatively prime. (See [3]). In this paper, we follow the basic definitions and notations of number theory as presented in [3] and [15].

Theorem 1.1. *(Ramanujan's theorem)* [16, 18]
Let $\pi(x)$ denote the number of primes not exceeding x.
Then $\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \dots,$ *if* $x \geq 2, 11, 17, 29, 41, \dots$, *respectively.*

Definition 1.5. [17, 20] *For a given prime p, the $p-$adic valuation ν_p is a function from $\mathbb{N}$ to $\mathbb{N}$ defined for all $n \in \mathbb{N}$, $\nu_p(n) := t$, where $p^t | n$ and $p^{t+1} \nmid n$, $t > 0$ and $\nu_p(0) := \infty$. Moreover, a critical prime power number for n with respect to p which is the number $p^{\nu'_p(n)} = cr(p, n)$ is the least prime power of a prime p which does not divide n, where $\nu'_p(n) := \nu_p(n) + 1$.*

Theorem 1.2 (Necessary condition 1)**.** [20] *Suppose G is a simple graph with n vertices and m edges. If $\alpha(G) < \max\limits_{1<p<n/2} \left\lfloor \frac{n}{p^{\nu'_p(n)}} \right\rfloor$, where $v'_p(n) = \nu_p(n) + 1$, then G is not a Diophantine graph.*

Theorem 1.3 (Necessary Condition 2)**.** *Let G be a graph with n vertices. When n is an odd number, if $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, then G is not a Diophantine graph.*

Proof. Since the number of vertices of even labels in $\{1, 2, \ldots, n\}$ is $\lfloor \frac{n}{2} \rfloor$, where n is odd, and $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, at least two vertices of even labels, say x and y, must be adjacent, hence $(f(x), f(y))$ is even, which does not divide n, so the graph G is not Diophantine. □

Theorem 1.4 (Necessary condition 3)**.** *Let G be a graph with n vertices and n is even. If $n \equiv 2 \mod 6$ or $n \equiv 4 \mod 6$, $\alpha(G) < \lfloor \frac{n}{3} \rfloor$, then G is not Diophantine.*

Proof. It is a special case from Theorem 1.2, since 3 is the least prime power that does not divide such n. □

The following three lemmas can be proven using mathematical induction.

Lemma 1.4. $\max\{\lfloor a_1 \rfloor, \lfloor a_2 \rfloor, \ldots, \lfloor a_n \rfloor\} = \lfloor \max\{a_1, a_2, \ldots, a_n\} \rfloor$, *$a_i \in \mathbb{Q}^+$ for $1 \leq i \leq n$.*

Lemma 1.5. *Let $n \in \mathbb{N}$, $\frac{n}{a_i} \in \mathbb{Q}^+$ for $1 \leq i \leq m$, then* $\max\{\frac{n}{a_1}, \frac{n}{a_2}, \ldots, \frac{n}{a_m}\} = \frac{n}{\min\{a_1, a_2, \ldots, a_m\}}$.

Lemma 1.6. $\max_{2 \leq p \leq n/2} \left\lfloor \frac{n}{p^{\nu'_p(n)}} \right\rfloor = \left\lfloor \max_{2 \leq p \leq n/2} \left\{ \frac{n}{p^{\nu'_p(n)}} \right\} \right\rfloor = \left\lfloor \frac{n}{\min_{2 \leq p \leq n/2} \left\{ p^{\nu'_p(n)} \right\}} \right\rfloor$, $\forall n \geq 4,\ p \in \mathbb{P}$.

2. Main Results

Lemma 2.1. *For a positive integer n, $4\sqrt{n} \leq n$ if and only if $n \geq 16$.*

Proof. If $4\sqrt{n} \leq n$, then, by squaring, $16n \leq n^2$. Therefore, for $n \neq 0, 16 \leq n$. Conversely, if $n \geq 16$, then, by multiplying by $n \neq 0, n^2 \geq 16n$. Consequently $n \geq 4\sqrt{n}$. □

Lemma 2.2. *Let $a > 1$. There exists at most one prime $p > \sqrt{a}$ such that $p \mid a$.*

Proof. Let $a > 1$ be a composite number. If there exist two primes, say p_1 and p_2, such that $p_1, p_2 > \sqrt{a}$, we will show that either $p_1 \nmid a$ or $p_2 \nmid a$. Assume for contradiction that both p_1 and p_2 divide a, i.e., $p_1 \mid a$ and $p_2 \mid a$. Since $\gcd(p_1, p_2) = 1$ (because they are prime numbers), we have $p_1 \cdot p_2 \mid a$. Now, since $p_1, p_2 > \sqrt{a}$, we can conclude that $p_1 \cdot p_2 > \sqrt{a} \cdot \sqrt{a} = a$. However, this contradicts the assumption that $p_1 \cdot p_2 \mid a$. Therefore, if $p_1, p_2 > \sqrt{a}$, it follows that $p_1 \nmid a$ or $p_2 \nmid a$. In summary, we have proved that if $p_1, p_2 > \sqrt{a}$, then $p_1 \nmid a$ or $p_2 \nmid a$. □

Corollary 2.1. *Let $a > 1$ be a composite number. If there exist two primes, say p_1 and p_2, such that $p_1, p_2 > \sqrt{a}$, then either $p_1 \nmid a$ or $p_2 \nmid a$.*

Lemma 2.3. *For $n \geq 4$ The vertex independence number of the cubic power of the Path graph P_n^3 which is denoted by $\alpha(P_n^3)$ is equal to $\lceil \frac{n}{4} \rceil$.*

Proof. Using mathematical induction. □

Theorem 2.1. *The complete graph K_n is Diophantine if and only if $n \in \{1, 2, 3, 4, 6\}$.*

Proof. Let K_n be a Diophantine graph. We aim to prove that $n \in \{1, 2, 3, 4, 6\}$. Suppose, by contrapositive, that $n \notin \{1, 2, 3, 4, 6\}$. Our goal is to demonstrate that K_n is not a Diophantine graph. According to Ramanujan's Theorem 1.1, we are guaranteed the existence of at least two prime numbers, denoted as p_1 and p_2 by utilizing the equation $\pi(x) - \pi(\frac{x}{2}) \geq 2$ for $x \geq 11$. Furthermore, from corollary 2.1, we know that these primes p_1 and $p_2 > \sqrt{n}$, implying that $p_1 \nmid n$ or $p_2 \nmid n$. By selecting two primes within the range of $n/4$ and $n/2$, as per Ramanujan's Theorem 1.1, we obtain further insight. If $\sqrt{n} < \frac{n}{4}$, then $16 < n$ (from Lemma 2.1) and $n \neq 0$. Additionally, according to Ramanujan 1.1, if $n \geq 22$, then $\pi(n/2) - \pi(n/4) \geq 2$, ensuring the existence of two primes p_1 and p_2 between $n/4$ and $n/2$, with $2p_1, 2p_2 < n$ for all $n \geq 22$. This implies that $p_1, p_2 > \frac{n}{4} > \sqrt{n}$ for all $n \geq 22$ from corollary 2.1. Consequently, we have $p_1 \nmid n$ or $p_2 \nmid n$. Let p_1 be a prime number such that $p_1 \nmid n$ and $\frac{n}{4} < p_1 < \frac{n}{2} < n$, $f(u)$ and $f(v)$ be the labels of u and v, respectively in $V(K_n)$, such that $f(u) = p_1 < n$ and $f(v) = 2p_1 < n$. Therefore, $(f(u), f(v)) = (p_1, 2p_1) = p_1 \nmid n$ for all $n \geq 22$. Hence, for all $n \geq 22$, we conclude that K_n is not a Diophantine graph. Finally, it is easy to show that K_n is not a Diophantine graph for $n = 5$ and $7 \leq n \leq 21$. Conversely, it is evident that the proof also works in the other direction. ☐

Theorem 2.2. *The wheel W_n is a Diophantine graph for all $n > 1$.*

Proof. Let $V(W_n) = \{u_0; u_1, u_2, \ldots, u_n\}$, where u_0 is the central vertex for W_n. There exists a labeling function $f : V(W_n) \longrightarrow \{1, 2, 3, \ldots, n+1\}$, which satisfies the condition in Definition 1.4, the labeling function can be defined as follows:
$f(u_0) = 1, f(u_1) = 2, f(u_2) = 3, \ldots, f(u_{n-1}) = n, f(u_n) = n + 1$ (Figure 1).

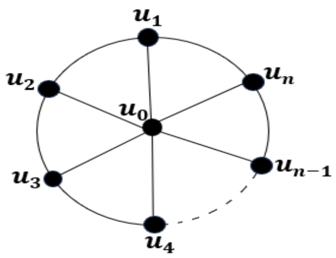

FIGURE 1. A Diophantine labeling for the wheel graph W_n, for all $n > 1$.

Since

$$(f(u_1), f(u_n)) = (2, n+1) = \begin{cases} 2 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases},$$

which divides the order of W_n. The remaining labels are assigned as consecutive numbers such that their greatest common divisor is equal to 1, hence, W_n is a Diophantine graph for all $n > 1$. ☐

Theorem 2.3. *The double wheel $W_{n,n}$ is a Diophantine graph if and only if n is even, $n \geq 4$.*

Proof. Let $V(W_{n,n}) = \{u_0, u_1, u_2, \ldots, u_n; v_1, v_2 \ldots, v_n\}$, where u_0 is the central vertex for $W_{n,n}$. Case 1. Let n be odd. $\alpha(W_{n,n}) = n - 1 < n = \left\lfloor \frac{2n+1}{2} \right\rfloor = \left\lfloor \frac{|V|}{2} \right\rfloor$. Therefore, from the necessary condition 2 (Theorem 1.3), the graph $W_{n,n}$ is not Diophantine.
Case 2. Let n be even and $n \geq 4$, then there exists a labeling function
$f : V(W_{n,n}) \longrightarrow \{1, 2, 3, \ldots, |V(W_{n,n})|\}$, which satisfies the condition in Definition 1.4, the labeling function can be defined as follows:
$f(u_0) = 1, f(u_1) = 2, f(u_2) = 3, \ldots, f(u_{n-1}) = n, f(u_n) = n + 1;\ f(v_1) = n + 2, f(v_2) = n + 3, \ldots, f(v_{n-1}) = 2n, f(v_n) = 2n + 1$ (Figure 2).

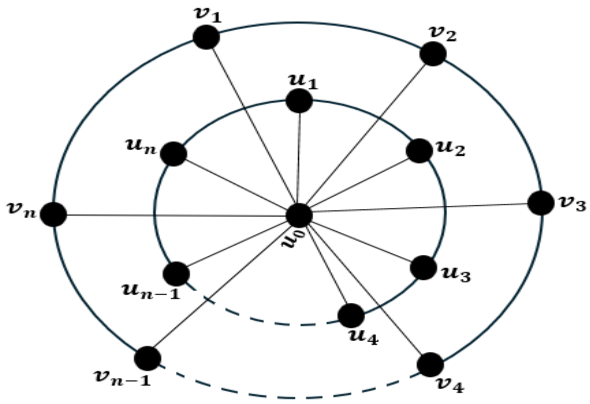

FIGURE 2. A Diophantine labeling for the double wheel graph $W_{n,n}$ for n is even and $n \geq 4$.

Since $(f(u_1), f(u_n)) = (2, n+1) = 1 \mid |V(W_{n,n})|$ whenever n is even,

$$(f(v_1), f(v_n)) = (n+2, 2n+1) = \begin{cases} 3 \text{ if } n \equiv 4 \mod 6 \\ 1 \text{ if } n \not\equiv 4 \mod 6 \end{cases}.$$

In case of $n \not\equiv 4 \mod 6$ and n is even, then we have either, $n \equiv 0 \mod 6$,i.e. $n = 6k$, $k \in \mathbb{N}$,therefore, $(2n+1, n+2) = (2(6k)+1, 6k+2) = (12k+1, 6k+2) = 1$. Or, $n \equiv 2 \mod 6$, i.e., $n = 2+6k', k' \in \mathbb{N}$, therefore, $(2n+1, n+2) = (5+12k', 4+6k') = 1$.

In case of $n \equiv 4 \mod 6$ and n is even, $(2n+1, n+2) = 3 \mid |V(W_{n,n})| = 2n+1$, since $n \equiv 4 \mod 6$, i.e. $n = 4+6k'', k'' \geq 0$. Thus, $|V(W_{n,n})| = 2n+1 = 2(4+6k'')+1 = 8+12k''+1 = 3(3+4k'') = 3t$, where $t = 3+4k'' \in \mathbb{N}$, therefore $(2n+1, n+2) = 3 \mid |V(W_{n,n})|$. The remaining labels are assigned as consecutive numbers such that their greatest common divisor is equal to 1, hence, $W_{n,n}$ is a Diophantine graph for all n is an even number, $n \geq 4$. □

Theorem 2.4. *The circulant graph $C_n(2)$ with jump $= 2$ and $n \geq 3$ is Diophantine if and only if n is even.*

Proof. Suppose, by contrapositive, that n is odd. From lemma 1.3, $\forall n \geq 3$, $\alpha(C_n(2)) = \lfloor \frac{n}{3} \rfloor$. Additionally, $\left\lfloor \frac{|V(C_n(2))|}{2} \right\rfloor = \lfloor \frac{n}{2} \rfloor$. $Therefore\ \ \alpha(C_n(2)) < \left\lfloor \frac{|V(C_n(2))|}{2} \right\rfloor$. Hence, from the necessary condition 2 (Theorem 1.3), $C_n(2)$ is not a Diophantine graph.
Conversely, let n be an even number. We want to prove that $C_n(2)$ is a Diophantine graph. Let $V(C_n(2)) = \{v_1, v_2, \ldots, v_n\}$ and

$$E(C_n(2)) = E(C_n) \cup \{v_i v_{i+2(mod n)} : i \in \{1, 2, 3, \ldots, n\}\}.$$

There exists a labeling function $f : V(C_n(2)) \longrightarrow \{1, 2, 3, \ldots, n\}$, which can be defined as follows:
$f(v_1) = 1, f(v_{2i+1}) = 2i+1, f(v_{2i}) = 2i$, for $1 \leq i \leq \frac{n-2}{2}$, $f(v_n) = n$ (Figure 3).

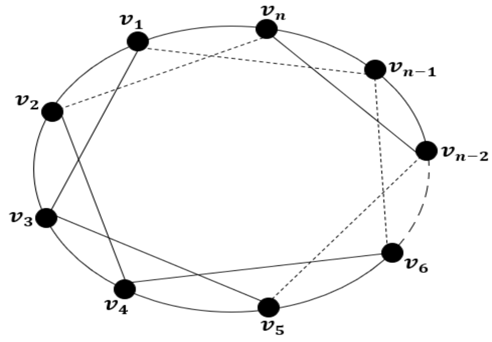

FIGURE 3. A Diophantine labeling for the circulant graph $C_n(2)$ for n is even and $n \geq 3$.

Since n is even, $(f(v_2), f(v_n)) = (2, n) = 2 \mid n$, $(f(v_{2i}), f(v_{2i+2})) = (2i, 2i+2) = 2 \mid n$, $1 \leq i \leq (n-2)/2$, $(f(v_{n-2}), f(v_n)) = (n-2, n) = 2 \mid n$. The remaining labels are assigned as consecutive numbers such that their greatest common divisor is equal to 1, hence $C_n(2)$ is a Diophantine graph. □

Theorem 2.5. *The cartesian product of the graph $C_3 \times C_m$ is Diophantine if and only if m is an even number, $m \geq 3$.*

Proof. Suppose, by contrapositive, let m be odd and $m \geq 3$. In fact $|V(C_3 \times C_m)| = 3m$, and m is odd. Therefore, $\alpha(C_3 \times C_m) = m < \left\lfloor \frac{3m}{2} \right\rfloor = \left\lfloor \frac{|V(C_3 \times C_m)|}{2} \right\rfloor$, hence from the necessary condition 2 (Theorem 1.3), the graph $C_3 \times C_m$ is not Diophantine for $m \geq 3$. Conversely, let m be even, we want to prove that $C_3 \times C_m$ is a Diophantine graph.
Let $V(C_3 \times C_m) = \{u_0, u_1, u_2, \ldots, u_{m-1}; v_0, v_1, v_2, \ldots, v_{m-1}; w_0, w_1, w_2, \ldots, w_{m-1}\}$, there exists a labeling function $f : V(C_3 \times C_m) \longrightarrow \{1, 2, 3, \ldots, 3m\}$, the labeling function can be defined as follows: $f(u_i) = 3i + 1, f(v_i) = 3i + 2, f(w_i) = 3i + 3$, for $0 \leq i \leq m - 1$ (Figure 4).

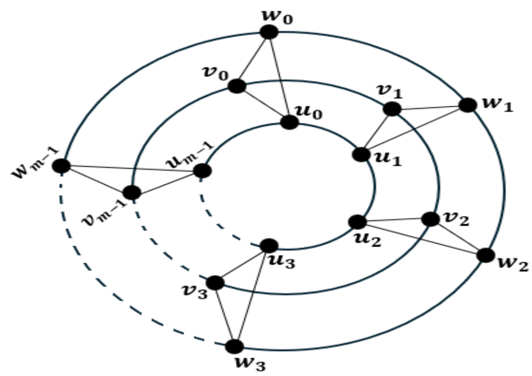

FIGURE 4. A Diophantine labeling for the graph $C_3 \times C_m$, m is even and $m \geq 3$.

Since $(f(w_i), f(w_{i+1})) = (3i + 3, 3i + 6) = 3 \mid 3m, 0 \leq i \leq m - 1,$

$$(f(u_i), f(w_i)) = (3i+1, 3i+3) = \begin{cases} 1 & \text{if } i \text{ even} \\ 2 & \text{if } i \text{ odd.} \end{cases}$$

The remaining labels are assigned as consecutive numbers such that their greatest common divisor is equal to 1, hence, $C_3 \times C_m$ is a Diophantine graph. □

Theorem 2.6. *If the normal product graph $P_n \circ P_n$ is Diophantine, then $n \in \{1, 2, 3\}$ or $n = 6k$ for $k \geq 1$.*

Proof. Suppose, by contrapositive, $n \notin \{1, 2, 3\}$ and $n \neq 6k$ for $k \geq 1$. We have two cases: Case 1. If n is odd and $n \neq 1, 3$, i.e. $n = 2k' + 1,\ k' > 1$. From lemma 1.2, $\alpha(P_n \circ P_n) = \lceil \frac{n}{2} \rceil^2, n = 2k'+1,\ k' > 1$, so, $\alpha(P_{2k'+1} \circ P_{2k'+1}) = (k')^2 + 2k' + 1,\ k' > 1$, $|V(P_n \circ P_n)| = n^2$, $|V(P_{2k'+1} \circ P_{2k'+1})| = 4(k')^2 + 4(k') + 1,\ k' > 1$, $\left\lfloor \frac{|V(P_{2k'+1} \circ P_{2k'+1})|}{2} \right\rfloor = 2(k')^2 + 2k', k' > 1$, $\alpha(P_{2k'+1} \circ P_{2k'+1}) = (k')^2 + 2k' + 1 < 2(k')^2 + 2k' = \left\lfloor \frac{|V(P_{2k'+1} \circ P_{2k'+1})|}{2} \right\rfloor,\ k' > 1$, hence from the necessary condition 2 (Theorem 1.3), the graph $P_n \circ P_n$ is not Diophantine in this case.
Case (2) : If n is even, $n \neq 2$ and $n \neq 6k$ for any k ,i.e. $n \not\equiv 0 \mod 6$, then we have two subcases: Subcase (i) If $n \equiv 2 \mod 6$, *i.e.* $n = 2 + 6.k'', k'' \in \mathbb{N}$, it follows that :
$\alpha(P_{2+6k''} \circ P_{2+6k''}) = \lceil \frac{2+6k''}{2} \rceil^2 = 1 + 6k'' + 9(k'')^2, k'' \in \mathbb{N}$. In this case, from lemma 1.6, where $|V(P_n \circ P_n)| = n^2$, we have that

$$\max_{2 \leq p \leq \frac{(2+6k'')^2}{2}} \left\lfloor \frac{(2+6k'')^2}{p^{v'_p(2+6k'')^2}} \right\rfloor = \left\lfloor \frac{(2+6k'')^2}{3} \right\rfloor, k'' \geq 1,$$

since 3 is the least prime power that does not divide such n, $\left\lfloor \frac{(2+6k'')^2}{3} \right\rfloor = 12(k'')^2 + 8k'' + 1$, $k'' \geq 1$, therefore $\alpha(P_{2+6k''} \circ P_{2+6k''}) = 9(k'')^2 + 6k'' + 1 < 12k''^2 + 8k'' + 1 = \left\lfloor \frac{(2+6k'')^2}{3} \right\rfloor = \max_{2 \leq p \leq \frac{(2+6k'')^2}{2}} \left\lfloor \frac{(2+6k'')^2}{p^{v'_p(2+6k'')^2}} \right\rfloor$, hence, we have that $P_n \circ P_n$ is not a Diophantine graph, based on the necessary conditions 1 and 3 (Theorems 1.2 and 1.4). Likewise subcase (ii) If $n \equiv 4$ *mod* 6, we similarly conclude from the necessary conditions 1 and 3 (Theorems 1.2 and 1.4) that $P_n \circ P_n$ is not a Diophantine graph also in this case. We proved that for $n \notin \{1, 2, 3\}$ and $n \neq 6k,\ k \geq 1$, then $P_n \circ P_n$ is not a Diophantine graph. □

Remark 2.1. *It is clear to see that the graph $P_n \circ P_n$ is Diophantine if $n \in \{1, 2, 3\}$ and $n = 6k$, for $k = 1$ and 2.*

Conjecture 2.7. *The normal product graph $P_{6k} \circ P_{6k}$ for $k \geq 3$ is Diophantine.*

Theorem 2.8. *If G and H are two graphs with n and m vertices respectively and $\alpha(H) \leq \frac{m}{2}$ such that $n \geq 3$ is odd and $m \geq 2$ is even, then the corona $G \odot H$ is not a Diophantine graph. Moreover, this upper bound $\frac{m}{2}$ of $\alpha(H)$ is the best possible.*

Proof. Since $\alpha(G \odot H) = n\alpha(H)$ (lemma 1.1), $|V(G \odot H)| = n(1+m)$, $\alpha(G \odot H) = n\alpha(H) \leq n\frac{m}{2}$, $\left\lfloor \frac{|V(G \odot H)|}{2} \right\rfloor = \left\lfloor \frac{n(m+1)}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + n\frac{m}{2}$, therefore $\alpha(G \odot H) \leq n\frac{m}{2} < \left\lfloor \frac{n}{2} \right\rfloor + n\frac{m}{2} = \left\lfloor \frac{|V(G \odot H)|}{2} \right\rfloor$, from the necessary condition 2 (Theorem 1.3), $G \odot H$ is not a Diophantine graph. Moreover, $\frac{m}{2}$ is the best possible upper bound to guarantee that $G \odot H$ is a non -Diophantine graph. Since $\alpha(G \odot H) = n\alpha(H) < \left\lfloor \frac{n}{2} \right\rfloor + n\frac{m}{2}$, dividing both sides of the inequality by $n \geq 3$ gives $\alpha(H) < \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor + \frac{m}{2}$. Since $\frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor < 1$, we have $\alpha(H) \leq \frac{m}{2}$. If $\alpha(H) = \frac{m}{2} + 1$, then we cannot judge the corona graph $G \odot H$ is not

Diophantine. However, there are cases where this theorem does not apply. If $m \geq 2$ is even, $\alpha(H) = \frac{m}{2} + 1$, and $n \geq 3$ is odd, then we cannot apply this theorem (See example 2.1).

Example 2.1. *For instance, the graph $TS_3' \odot S_4$ shown in Figure 5 is Diophantine.*

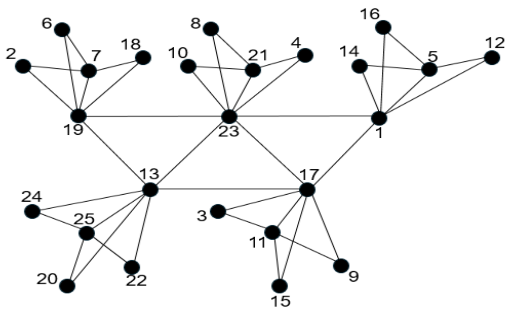

FIGURE 5. A Diophantine labeling for the graph $TS_3' \odot S_4$.

We cannot guarantee the validity of the theorem in the cases: n and m are even, n and m are odd, and n is even and m is odd. The following graphs in example 2.2 ensures the last cases respectively.

Example 2.2. *The following graphs $P_4 \odot P_2$, $P_3 \odot K_3$ and $P_2 \odot C_3$ respectively are Diophantine (Figure 6).*

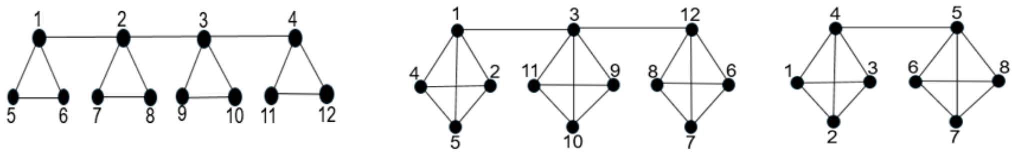

FIGURE 6. Diophantine labeling graphs.

□

Corollary 2.2. *There are several families of graphs satisfying the conditions of theorem [2.8] which are not Diophantine, such as the following below:*

(1)$TS_n' \odot K_m$ (2)$TS_n' \odot K_{m,m}$ (3)$P_n \odot P_m$ (4)$H_n^* \odot C_m$
(5)$TS_n' \odot P_m$ (6)$\overline{K_n} \odot C_m$ (7)$H_n^* \odot K_m$ (8)$H_n^* \odot K_{m,m}$
(9)$TS_n' \odot C_m$ (10)$\overline{K_n} \odot P_m$ (11)$H_n^* \odot P_m$ (12)$H_n^* \odot \overline{K}_m$,

Where H_n^* is the flower graph.

Theorem 2.9. *The double fan graph $g_n = p_n + \overline{K_2}$ is Diophantine for all $n \geq 3$.*

Proof. $|V(g_n)| = n + 2$. Let $V(g_n) = \{v_1, v_2; u_1, u_2, \ldots, u_n\}$. In fact, for all $n \geq 3$, there exists a labeling function $f : V(g_n) \to \{1, 2, \ldots, n+2\}$ which we can defined it as follows:

$f(v_1) = 1$, $f(v_2) = n + 2$, $f(u_i) = i + 1$, $1 \leq i \leq n$ (Figure 7).

This labeling guarantees that each two adjacent labeled vertices in the graph g_n have greatest common divisor which divides the order of g_n, hence, the graph g_n is Diophantine. □

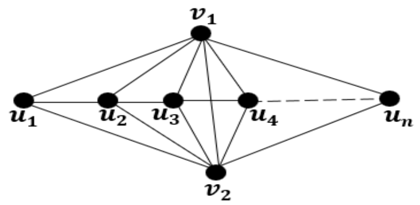

FIGURE 7. A Diophantine labeling for the double fan graph for all $n \geq 3$.

Theorem 2.10. *The square of the path graph P_n^2 is Diophantine if and only if $n \in \{1, 3, 5, 7\}$ or n is even.*

Proof. Suppose, by contrapositive, that $n \notin \{1, 3, 5, 7\}$ and n is odd, we have three cases: $n \equiv 1 \; mod \; 6$ or $n \equiv 3 \; mod \; 6$ or $n \equiv 5 \; mod \; 6$.
Case 1: If $n \neq 1, \; 7$ and $n \equiv 1 \; mod \; 6$, i.e. $n = 1 + 6k$, where $k \geq 2$, we know that $\alpha\left(P_n^2\right) = \left\lceil \frac{n}{3} \right\rceil$ and $\left|V\left(P_n^2\right)\right| = n$, therefore, $\alpha\left(P_{1+6k}^2\right) = \left\lceil \frac{1+6k}{3} \right\rceil = 2k + 1$, $\left|V\left(P_{1+6k}^2\right)\right| = 1 + 6k$ *for* $k \geq 2$. *Thus*, $\alpha\left(P_{1+6k}^2\right) = 2k + 1 < 3k = \left\lfloor \frac{6k+1}{2} \right\rfloor = \left\lfloor \frac{\left|V\left(P_{1+6k}^2\right)\right|}{2} \right\rfloor$, *for* $k \geq 2$, hence, from the necessary condition 2 (Theorem 1.3), we conclude that the graph P_{1+6k}^2 is not Diophantine for $k \geq 2$.

Similarly, in case 2, when $n \equiv 3 \; mod \; 6$, and $n \neq 3$, and in case 3, when $n \equiv 5 \; mod \; 6$ and $n \neq 5$, from the last three cases, we can show that the graph P_n^2 is not Diophantine.

Conversely, it is obvious that the graph P_n^2 is Diophantine for $n \in \{1, 3, 5, 7\}$, and also for n is even, $n \geq 2$, as follows : let $V\left(P_n^2\right) = \{v_1, v_2, \ldots, v_n\}$. There exists a labeling function $f : V\left(P_n^2\right) \to \{1, 2, 3, \ldots, n\}$, which can be defined as follows: $f(v_i) = i$, for $1 \leq i \leq n$ (Figure 8).

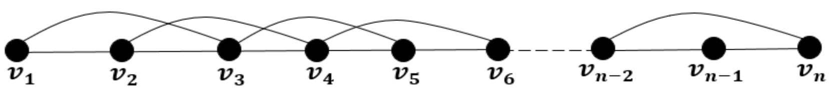

FIGURE 8. A Diophantine labeling of the graph P_n^2 for n is even, $n \geq 2$.

The labels $(f(v_i), f(v_{i+2})) = (i, i+2) = 2 \mid n$ for i *even* and equal to $1 \mid n$ otherwise. Similarly, $(f(v_{n-2}), f(v_n)) = (n-2, n) = 2 \mid n$. The remaining labels are assigned as consecutive numbers such that their greatest common divisor is equal to 1, hence, P_n^2 is a Diophantine graph.

□

Corollary 2.3. *The graph P_n^2 is labeled isomorphic to the path graph $P_n(2)$.*

Figure 9 shows P_6^2 is labeled isomorphic to $P_6(2)$.

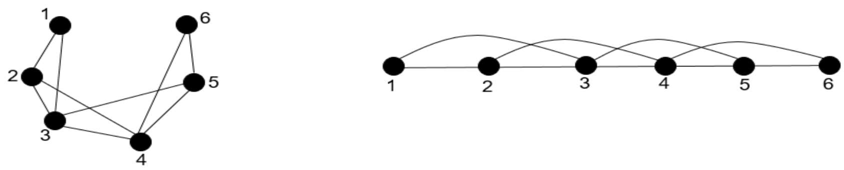

FIGURE 9. A Diophantine labeling for P_6^2 *and* $P_6(2)$.

Corollary 2.4. *The graph C_n^2 is labeled isomorphic to the circulant graph $C_n(2)$.*

It follows from Theorem 2.4 that the square graph C_n^2 is a Diophantine if and only if n is even.

Corollary 2.5. *If the diameter in a graph G is k, then the k^{th} power of the cycle graph C_n will be a complete graph K_n, i.e., $C_n^k \cong K_n$ is labeled isomorphic, for $1 \le k \le \lfloor \frac{n}{2} \rfloor$, hence, the graph C_n^k is Diophantine if and only if $n \in \{1,2,3,4,6\}$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$.*

Proof. Consequently, since $C_n^k \cong K_n$, it follows from theorem 2.1 that C_n^k is a Diophantine graph if and only if $n \in \{1,2,3,4,6\}$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$. □

Theorem 2.11. *For $n \ge 4$, the cube of the path graph P_n^3 is Diophantine if and only if $n \in \{4,5,8,10,14\}$ or $n \equiv 0 \ (mod\ 6)$.*

Proof. Suppose, by contrapositive, for $n \ge 4$, $n \notin \{4,5,8,10,14\}$ and $n \not\equiv 0\,(mod\ 6)$, then we have the following cases:

Case 1: If $n > 5$ is odd, then we have $n \equiv 1\,(mod\ 6)$, i.e., $n = 1 + 6k$ for $k \ge 1$. From lemma (2.3), $\alpha(P_n^3) = \lceil \frac{n}{4} \rceil = \lceil \frac{1+6k}{4} \rceil = \lceil \frac{1}{4} + \frac{3}{2}k \rceil$ for $k \ge 1$. $\left\lfloor \frac{|V\ (P_n^3)|}{2} \right\rfloor = \lfloor \frac{n}{2} \rfloor = \lfloor \frac{1+6k}{2} \rfloor = 3k$ for $k \ge 1$, therefore, $\alpha(P_{1+6k}^3) = \lceil \frac{1}{4} + \frac{3}{2}k \rceil < 3k = \left\lfloor \frac{|V\ (P_{1+6k}^3)|}{2} \right\rfloor$ for $k \ge 1$, hence, from the necessary condition 2 (Theorem 1.3), we have that the graph P_n^3 in this case is not Diophantine. Similarly, the other cases, when $n \equiv 3\,(mod\ 6)$ and $n \equiv 5\,(mod\ 6)$.

Case 2: If n is even and $n \notin \{4,8,10,14\}$ and $n \not\equiv 0\,(mod\ 6)$, then we have: $n \equiv 2\,(mod\ 6)$, i.e., $n = 2 + 6.L$ for $L \ge 3$. From lemma 2.3 and lemma 1.6, $\alpha(P_n^3) = \lceil \frac{n}{4} \rceil = \lceil \frac{2+6L}{4} \rceil = \lceil \frac{1}{2} + \frac{3}{2}L \rceil$ for $L \ge 3$, since 3 is the least prime power that does not divide such n, so $\max_{1<p<n/2} \left\lfloor \frac{n}{p^{\nu_p'(n)}} \right\rfloor = \lfloor \frac{n}{3} \rfloor = \lfloor \frac{2+6L}{3} \rfloor = \lfloor \frac{2}{3} + 2L \rfloor = 2L$ for $L \ge 3$, therefore, $\alpha({P^3}_{2+6L}) = \lceil \frac{1}{2} + \frac{3}{2}L \rceil < 2L = \lfloor \frac{2}{3} + 2L \rfloor = \lfloor \frac{n}{3} \rfloor = \max_{1<p<n/2} \left\lfloor \frac{n}{p^{\nu_p'(n)}} \right\rfloor$ for $L \ge 3$, hence, from the necessary conditions 1 and 3 (Theorems 1.2 and 1.4), we have that the graph P_n^3 in this case is not Diophantine. Similarly, in the case, when $n \equiv 4\,(mod\ 6)$. From the last five cases we proved the first direction of the theorem.

Conversely, it is obvious that P_n^3 is Diophantine for $n \in \{4,5,8,10,14\}$. Now, consider $n \equiv 0\ mod\ 6$, let $V\left(P_n^3\right) = \{v_1, v_2, \dots, v_n\}$ be the vertices of P_n^3, where $n = 6k\ for\ k \ge 1$. There exists a labeling function $f : V\left(P_n^3\right) \to \{1,2,3,\dots,n\}$, which can be defined as follows:

$f(v_i) = i$, for $1 \le i \le n$ (Figure 10).

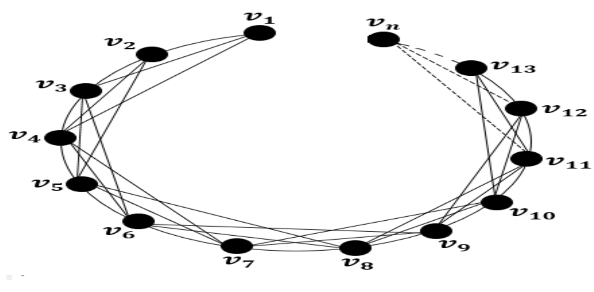

FIGURE 10. A Diophantine labeling for the graph P_n^3, when $n \equiv 0\ mod\ 6$.

Since

$$(f(v_i), f(v_{i+2})) = (i, i+2) = \begin{cases} 1 & if\ i \not\equiv 0 \mod 2 \\ 2 & if\ i \equiv 0 \mod 2 \end{cases}$$

$$(f(v_i), f(v_{i+3})) = (i, i+3) = \begin{cases} 1 & if\ i \not\equiv 0 \mod 3 \\ 3 & if\ i \equiv 0 \mod 3 \end{cases}$$

Which divides the order of P_n^3, hence, the graph P_n^3, $n = 6.k$ for $k \geq 1$ is Diophantine. □

Remark 2.2. *For* $n = 1, 2$ *and* 3 *(it's obvious that the graph* P_n^3 *is Diophantine).*

3. Conclusion

This manuscript examines various graph families, including complete graphs, wheel graphs, circulant graphs, and Cartesian products, to determine their Diophantine nature and establishing necessary and sufficient conditions for Diophantine labeling. We extend the concept of prime labeling, concluding that all prime graphs are also Diophantine. Furthermore, we proved that complete graphs K_n are Diophantine for specific values of n, while wheel graphs W_n are Diophantine for all $n > 1$. Additionally, we identify conditions for more complex structures, such as normal product graphs and cubes of path graphs, exhibit Diophantine properties. For our future research, we may focus on generalizing these conditions and identifying new Diophantine graph families.

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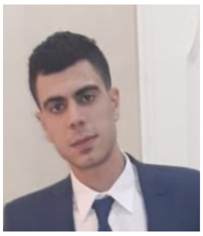

Omar Mohamed Salama graduated from Ain Shams University. He is currently pursuing his M.Sc. in the Department of Mathematics at the Faculty of Science, Ain Shams University and his research interests are in graph labeling, number theory. Presently, he is working as a teaching assistant in the Department of Mathematics at the Faculty of Science, Ain Shams University.

Mohammed Abdel Azim Seoud for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.15, N.7.

Mohamed Anwar for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.15, N.7.

Ahmed Elsonbaty is an Associate Professor of Pure Mathematics at the Faculty of Science, Ain Shams University. He received his Ph.D. in Pure Mathematics from the University of Bath in 2004. His current research interests include number theory and its applications in graph labeling and combinatorial problems.