

WIENER AND HARARY INDICES OF MYCIELSKIAN GRAPHS

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ABSTRACT. Let $G = (V(G), E(G))$ be a graph, where $V = \{v_1, v_2, \dots, v_n\}$. Let $V' = \{v'_1, v'_2, \dots, v'_n\}$ be the twin of the vertex set $V(G)$. The Mycielskian graph $\mathbb{M}(G)$ of G is defined as the graph whose vertex set is $V(G) \cup V'(G) \cup \{w\}$ and the edge set is $E(G) \cup \{v_i v'_j : v_i v_j \in E(G)\} \cup \{v'_i w \in V'(G)\}$. The vertex v'_i is the twin of the vertex v_i (or v_i is twin of the vertex v'_i) and the vertex w is the root of $\mathbb{M}(G)$. The closed Mycielskian graph $\mathbb{M}[G]$ of G is defined as the graph whose vertex set is $V(G) \cup V'(G) \cup \{w\}$ and the edge set is $E(G) \cup \{v_i v'_j : v_i v_j \in E(G)\} \cup \{v_i v'_i : i = 1, 2, \dots, n\} \cup \{v'_i w \in V'(G)\}$. The vertex v'_i is the twin of the vertex v_i (or v_i is twin of the vertex v'_i) and the vertex w is the root of $\mathbb{M}[G]$. In this paper, we study the Wiener and Harary indices of the Mycielskian and closed Mycielskian graphs.

Keywords: closed splitting graph, shadow graph, closed shadow graph, Mycielskian graph, closed Mycielskian graph.

AMS Subject Classification: 05C10, 05C12

1. INTRODUCTION

Graphs are fundamental structures in mathematics and computer science, representing networks of interconnected points (vertices) and lines (edges). The vertex set and the edge set are represented as $V(G)$ and $E(G)$ respectively. Two vertices are *adjacent* if they are connected by an edge. Two edges are *incident* if they share a common vertex.

The *adjacency matrix*, $A(G)$, with the vertex set $\{v_1, v_2, \dots, v_n\}$ is the $n \times n$ binary matrix $A(G) = [a_{ij}]$ where $a_{ij} = 1$ if the vertices v_i, v_j are adjacent, and $a_{ij} = 0$ otherwise $\forall i, j = 1, 2, \dots, n$.

The *distance matrix*, $D(G)$, of a graph G is the matrix $[d_{ij}]$ with $d_{ij} = d(v_i, v_j)$, where $d(v_i, v_j)$ is the length of the shortest distance between v_i and v_j .

Topological index is a numerical quantity mathematically derived from the graph structure. The first topological index to ever introduced was Wiener Index, by Harry Wiener [15]. It was developed to find the structure of the paraffin molecule and its boiling point

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and was first called “path number” but later on was studied as gross status, the distance of a graph and transmission in pure mathematics. It is helpful for comparing chemical structures since it is a graph invariant, meaning it remains unaltered under graph isomorphism. Many academics in the fields of mathematics and chemistry have studied this graph invariant (see [5–9, 11, 16]).

Definition 1.1. The Wiener Index, $W(G)$, is defined as sum of distance between all unordered pairs of vertices in a connected graph G , i.e.,

$$W(G) = \sum_{\{v_i, v_j\} \subseteq V(G)} d(v_i, v_j) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n d(v_i, v_j).$$

The Harary index was introduced independently by Plavšić [13] and then by Ivanciuc [10]. It provides insights into the connectivity and structural properties of graphs, particularly in molecular and network analysis. Recent studies have explored generalized forms of the Harary index, enhancing its applicability to various graph families and operations (see [2–4]).

Definition 1.2. The Harary index, $H(G)$, of a graph is defined as the reciprocal sum of the distances between all unordered pairs of vertices in a connected graph G , i.e.,

$$H(G) = \sum_{\{v_i, v_j\} \subseteq V(G)} \frac{1}{d(v_i, v_j)} = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{d(v_i, v_j)}.$$

Let G be a graph and $V'(G)$ be the copy of the vertex set $V(G)$, i.e. $V'(G) = \{v' : v \in V(G)\}$.

Definition 1.3. The splitting graph, $\Lambda(G)$, of a graph G is a graph with vertex set $V(\Lambda(G)) = \{V(G) \cup V'(G)\}$ and edge set $E(\Lambda(G)) = E(G) \cup \{xy' : xy \in E(G)\}$, introduced by Sampathkumar and Walikar [14].

Definition 1.4. The closed splitting graph, $\Lambda[G]$, of a graph G is a graph whose vertex set $V(\Lambda[G]) = \{V(G) \cup V'(G)\}$ and edge set $E(\Lambda[G]) = E(G) \cup \{xx' : x \in V(G)\} \cup \{xy' : xy \in E(G)\}$, mentioned in [7].

Definition 1.5. The shadow graph, $D_2(G)$, of a graph G is a graph whose vertex set $V(D_2(G)) = \{V(G) \cup V'(G)\}$ and edge set $E(D_2(G)) = E(G) \cup \{x'y' : xy \in E(G)\} \cup \{xy' : xy \in E(G)\}$, see in [1].

Definition 1.6. The closed shadow graph, $D_2[G]$, of a graph G is a graph whose vertex set $V(D_2[G]) = \{V(G) \cup V'(G)\}$ and edge set $E(D_2[G]) = E(G) \cup \{x'y' : xy \in E(G)\} \cup \{xy' : xy \in E(G)\} \cup \{xx' : x \in V(G)\}$, mentioned in [7, 9].

Figure 1 depicts a graph G with its splitting graph $H = \Lambda(G)$, closed splitting graph $H' = \Lambda[G]$, shadow graph $H'' = D_2(G)$ and closed shadow graph $H''' = D_2[G]$.

Definition 1.7. Let $G = (V(G), E(G))$ be a graph, where $V = \{v_1, v_2, \dots, v_n\}$. Let $V' = \{v'_1, v'_2, \dots, v'_n\}$ be the twin of the vertex set $V(G)$. The Mycielskian graph $\mathbb{M}(G)$ of G is defined as the graph whose vertex set is $V(G) \cup V'(G) \cup \{w\}$ and the edge set is $E(G) \cup \{v_i v'_j : v_i v_j \in E(G)\} \cup \{v'_i w \in V'(G)\}$. The vertex v'_i is the twin of the vertex v_i (or v_i is twin of the vertex v'_i) and the vertex w is the root of $\mathbb{M}(G)$.

Mycielski [12] introduced a graph operator for finding triangle-free graphs, whose chromatic number is arbitrarily large.

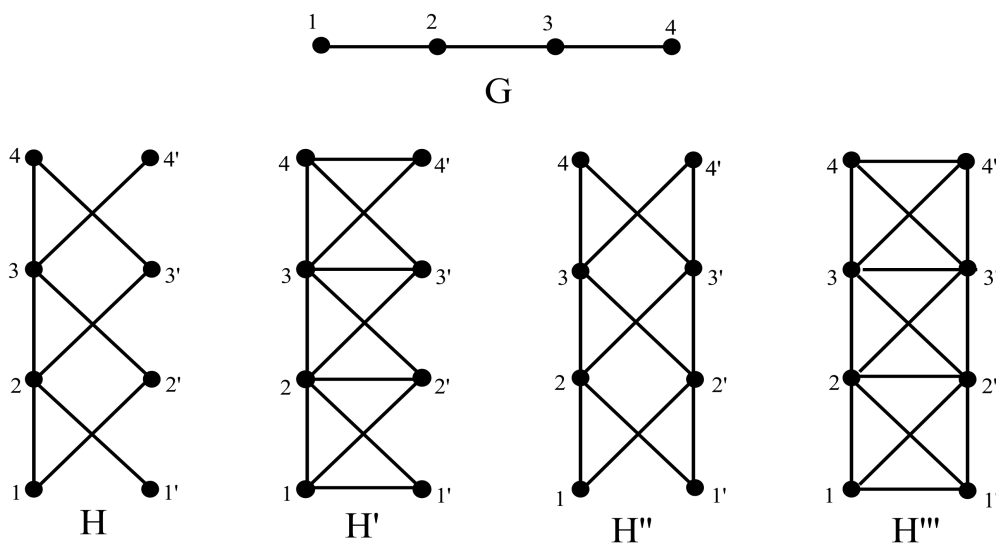


FIGURE 1. A graph G with its splitting graph $H = \Lambda(G)$, closed splitting graph $H' = \Lambda[G]$, shadow graph $H'' = D_2(G)$ and closed shadow graph $H''' = D_2[G]$.

Definition 1.8. Let $G = (V(G), E(G))$ be a graph, where $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $V'(G) = \{v'_1, v'_2, \dots, v'_n\}$ be the twin of the vertex set $V(G)$. The closed Mycielskian graph $\mathbb{M}[G]$ of G is defined as the graph whose vertex set is $V(G) \cup V'(G) \cup \{w\}$ and the edge set is $E(G) \cup \{v_i v'_j : v_i v_j \in E(G)\} \cup \{v_i v'_i : i = 1, 2, \dots, n\} \cup \{v'_i w \in V'(G)\}$. The vertex v'_i is the twin of the vertex v_i (or v_i is twin of the vertex v'_i) and the vertex w is the root of $\mathbb{M}[G]$.

Figure 2 depicts a graph G with its Mycielskian graph $M = \mathbb{M}(G)$ and closed Mycielskian graph $M' = \mathbb{M}[G]$.

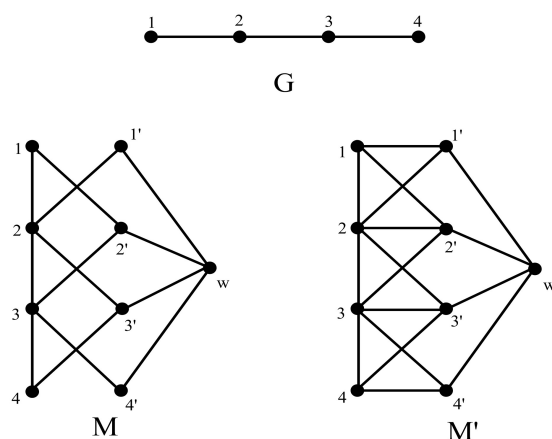


FIGURE 2. A graph G with its Mycielskian graph $M = \mathbb{M}(G)$ and closed Mycielskian graph $M' = \mathbb{M}[G]$.

2. WIENER INDEX

Egan et al. [4] derived the Wiener index of splitting graph as follows:

Lemma 2.1. [4] *Let G be a connected triangle-free graph on n vertices and m edges. The Wiener index of $\Lambda(G)$ is given by*

$$W(\Lambda(G)) = 4W(G) + 2n + 2m.$$

Lemma 2.2. [4] *For a connected graph $G = (V(G), E(G))$ on $n \geq 2$ vertices and $m \geq 1$ edges such that any pair of adjacent vertices have at least one common neighbour, the Wiener index of $\Lambda(G)$ is given by*

$$W(\Lambda(G)) = 4W(G) + 2n + m.$$

Lemma 2.3. *Let G be a connected graph with n vertices and m edges. The closed splitting graph $\Lambda[G]$ of a graph G has Wiener index given by*

$$W(\Lambda[G]) = 4W(G) + n + m.$$

Definition 2.1. *The matrix of all-ones or the all-ones matrix has all entry equal to one, denoted by J .*

Theorem 2.1. *Let G be a connected graph of n vertices and m edges. The Wiener index of Mycielskian graph $\mathbb{M}(G)$ of G is given by*

$$W(\mathbb{M}(G)) = 3W(G) + n^2 + 4n.$$

Proof. The distance matrix of $\mathbb{M}(G)$ as a 3×3 block matrix entries is written as

$$D(\mathbb{M}(G)) = \begin{bmatrix} D(G) & D(G) + 2I_n & 2C \\ D(G) + 2I_n & 2J_n - 2I_n & C \\ 2B & B & O \end{bmatrix}$$

where $D(G)$ is the distance matrix of graph G , I_n is the identity matrix of size n , J_n is the all-ones matrix of size n , B is the row matrix of order n with $b_{ij} = 1 \forall i, j$, C is the column matrix of order n with $c_{ij} = 1 \forall i, j$ and O is the zero matrix of order 1.

The Wiener index of $\mathbb{M}(G)$ can be expressed using distance matrix $D(\mathbb{M}(G))$ as follows:

$$\begin{aligned} W(\mathbb{M}(G)) &= \frac{1}{2} \sum D(\mathbb{M}(G)) \\ &= \frac{1}{2} \left[\sum D(G) + \sum (D(G) + 2I_n) + \sum 2C + \sum (D(G) + 2I_n) \right] \\ &\quad + \frac{1}{2} \left[\sum (2J_n - 2I_n) + \sum C + \sum 2B + \sum B \right] \\ &= \frac{1}{2} \left[\sum 3D(G) + \sum 2I_n + \sum 2J_n + \sum 3B + \sum 3C \right] \\ &= \frac{1}{2} [3(2W(G)) + 2n + 2n^2 + 3n + 3n] \\ &= 3W(G) + n^2 + 4n. \end{aligned}$$

Hence the theorem. □

Theorem 2.2. *Let G be a connected graph of n vertices and m edges. The Wiener index of closed Mycielskian graph $\mathbb{M}[G]$ of G is given by*

$$W(\mathbb{M}[G]) = 3W(G) + n^2 + 3n.$$

Proof. The distance matrix of $\mathbb{M}[G]$ as a 3×3 block matrix entries is written as

$$D(\mathbb{M}[G]) = \begin{bmatrix} D(G) & D(G) + I_n & 2C \\ D(G) + I_n & 2J_n - 2I_n & C \\ 2B & B & O \end{bmatrix}$$

where $D(G)$ is the distance matrix of graph G , I_n is the identity matrix of size n , J_n is the all-ones matrix of size n , B is the row matrix of order n with $b_{ij} = 1 \forall i, j$, C is the column matrix of order n with $c_{ij} = 1 \forall i, j$ and O is the zero matrix of order 1.

The Wiener index of $\mathbb{M}[G]$ can be expressed using distance matrix $D(\mathbb{M}[G])$ as follows:

$$\begin{aligned} W(\mathbb{M}[G]) &= \frac{1}{2} \sum D(\mathbb{M}[G]) \\ &= \frac{1}{2} \left[\sum D(G) + \sum (D(G) + I_n) + \sum 2C + \sum (D(G) + I_n) \right] \\ &\quad + \frac{1}{2} \left[\sum (2J_n - 2I_n) + \sum C + \sum 2B + \sum B \right] \\ &= \frac{1}{2} \left[\sum 3D(G) + \sum 2J_n + \sum 3B + \sum 3C \right] \\ &= \frac{1}{2} [3(2W(G)) + 2n^2 + 3n + 3n] \\ &= 3W(G) + n^2 + 3n. \end{aligned}$$

Hence the theorem. $\square$

3. HARARY INDEX

Egan et al. [4] derived the Harary index of splitting graph as follows:

Lemma 3.1. [4] *Let G be a connected triangle-free graph on n vertices and m edges. The Harary index of $\Lambda(G)$ is given by*

$$H(\Lambda(G)) = 4W(G) + \frac{n}{2} - \frac{2}{3}m.$$

Lemma 3.2. [4] *For a connected graph $G = (V(G), E(G))$ on $n \geq 2$ vertices and $m \geq 1$ edges such that any pair of adjacent vertices have at least one common neighbour, the Harary index of $\Lambda(G)$ is given by*

$$H(\Lambda(G)) = 4W(G) + \frac{n - m}{2}.$$

In order to compute the Harary index, we can consider the distance matrix of the graphs as we considered in the case of Wiener index. The Harary index can be calculated by taking the reciprocals of the non-zero entries in its distance matrix.

For instance, let $A = [a_{ij}]$ be any matrix and $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = \frac{1}{a_{ij}}$ if $a_{ij} \neq 0$ and 0 otherwise. If the entries in A are either 0 or 1, then $\sum \bar{A} = \sum A$. Also for any non-zero c , $\sum c\bar{A} = c \sum \bar{A}$ (mentioned in [4]).

Lemma 3.3. [4] *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ square matrices with real entries such that at least one of a_{ij} or b_{ij} is zero for all $0 \leq i, j, \leq n$, then*

$$\sum \overline{A+B} = \sum \bar{A} + \sum \bar{B}.$$

Lemma 3.4. [4] *Let $D(G)$ and $A(G)$ be the distance matrix and adjacency matrix of a graph G on n vertices and I_n be the identity matrix of size n . Let c be any non-zero real number.*

$$(1) \sum \overline{cA(G)} = \frac{1}{c} \sum A(G)$$

$$(2) \sum \overline{D(G) + cA(G)} = \sum \overline{D(G) - A(G)} + \sum \overline{(c+1)A(G)}$$

$$(3) \sum \overline{D(G) + cI_n} = \sum \overline{D(G)} + \sum \overline{cI_n}$$

Lemma 3.5. Let G be a connected graph with n vertices and m edges. Consider $\Lambda[G]$, then we have,

- (1) $d(v_i, v'_i) = 1$ for $i = 1, 2, \dots, n$;
- (2) $d(v_i, v_j) = d(v'_i, v'_j)$ for $1 \leq i, j \leq n, i \neq j$;
- (3) $d(v'_i, v'_j) = 2$ for vertices v_i and v_j

Lemma 3.6. The distance matrix of the closed splitting graph $\Lambda[G]$ is a 2×2 block matrix, using Lemma 3.5, given by

$$D(\Lambda[G]) = \begin{bmatrix} D(G) & D(G) + I_n \\ D(G) + I_n & D(G) + A(G) \end{bmatrix}$$

where $D(G)$ and $A(G)$ are the distance matrix and adjacency matrix of graph G respectively and I_n is the identity matrix of order n .

Lemma 3.7. Let G be a connected graph with n vertices and m edges. Consider $D_2(G)$, then we have,

- (1) $d(v_i, v_j) = d(v'_i, v'_j)$ for $1 \leq i, j \leq n, i \neq j$;
- (2) $d(v_i, v_j) = d(v'_i, v'_j)$ for vertices v_i and v_j

Lemma 3.8. The distance matrix of the shadow graph $D_2(G)$ is a 2×2 block matrix, using Lemma 3.7, given by

$$D(D_2(G)) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ D(G) + 2I_n & D(G) \end{bmatrix}$$

where $D(G)$ is the distance matrix of graph G respectively and I_n is the identity matrix of order n .

Lemma 3.9. Let G be a connected graph with n vertices and m edges. Consider $D_2[G]$, then we have,

- (1) $d(v_i, v'_i) = 1$ for $i = 1, 2, \dots, n$;
- (2) $d(v_i, v_j) = d(v'_i, v'_j)$ for $1 \leq i, j \leq n, i \neq j$;
- (3) $d(v_i, v_j) = d(v'_i, v'_j)$ for vertices v_i and v_j

Lemma 3.10. The distance matrix of the closed shadow graph $D_2[G]$ is a 2×2 block matrix, using Lemma 3.9, given by

$$D(D_2[G]) = \begin{bmatrix} D(G) & D(G) + I_n \\ D(G) + I_n & D(G) \end{bmatrix}$$

where $D(G)$ is the distance matrix of graph G respectively and I_n is the identity matrix of order n .

Theorem 3.1. Let G be a connected graph with n vertices and m edges. The Harary index of closed splitting graph $\Lambda[G]$ of a graph G is given by

$$H(\Lambda[G]) = 4H(G) + n - \frac{m}{2}.$$

Proof. From Lemma 3.6, the distance matrix of $\Lambda[G]$ is

$$D(\Lambda[G]) = \begin{bmatrix} D(G) & D(G) + I_n \\ D(G) + I_n & D(G) + A(G) \end{bmatrix}$$

The Harary index of $\Lambda[G]$ is given as:

$$\begin{aligned} H(\Lambda[G]) &= \frac{1}{2} \sum \overline{D(\Lambda[G])} \\ &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G) + I_n} + \sum \overline{D(G) + I_n} + \sum \overline{D(G) + A(G)} \right] \\ &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G)} + \sum \overline{I_n} + \sum \overline{D(G)} + \sum \overline{I_n} \right] \\ &\quad + \frac{1}{2} \left[\sum \overline{D(G) - A(G)} + \sum \overline{2A(G)} \right] \quad [By Lemma 3.4] \\ &= \frac{1}{2} \left[4 \sum \overline{D(G)} + 2 \sum \overline{I_n} + \sum \overline{2A(G)} - \sum \overline{A(G)} \right] \\ &= 4 \left[\frac{1}{2} \sum \overline{D(G)} \right] + \sum \overline{I_n} + \left[\frac{1}{2} \sum \overline{2A(G)} \right] - \left[\frac{1}{2} \sum \overline{A(G)} \right] \\ &= 4H(G) + n + \frac{m}{2} - m \quad \left[\sum A(G) = 2m \right] \\ &= 4H(G) + n - \frac{m}{2}. \end{aligned}$$

Hence the theorem. $\square$

Theorem 3.2. Let G be a connected graph with n vertices and m edges. The Harary index of shadow graph $D_2(G)$ of a graph G is given by

$$H(D_2(G)) = 4H(G) + \frac{n}{2}.$$

Proof. From Lemma 3.8, the distance matrix of $D_2(G)$ is

$$D(D_2(G)) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ D(G) + 2I_n & D(G) \end{bmatrix}$$

The Harary index of $D_2(G)$ is given as:

$$\begin{aligned} H(D_2(G)) &= \frac{1}{2} \sum \overline{D(D_2(G))} \\ &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G) + 2I_n} + \sum \overline{D(G) + 2I_n} + \sum \overline{D(G)} \right] \\ &= \frac{1}{2} \left[2 \sum \overline{D(G)} + 2 \sum \overline{D(G) + 2I_n} \right] \\ &= \frac{1}{2} \left[2 \sum \overline{D(G)} + 2 \sum \overline{D(G)} + 2 \sum \overline{2I_n} \right] \quad [By Lemma 3.4] \\ &= \frac{1}{2} \left[4 \sum \overline{D(G)} + 2 \sum \overline{2I_n} \right] \\ &= \frac{1}{2} \left[4 (2 H(G)) + 2 \left(\frac{n}{2} \right) \right] \\ &= 4H(G) + \frac{n}{2}. \end{aligned}$$

Hence the theorem. $\square$

Theorem 3.3. *Let G be a connected graph with n vertices and m edges. The Harary index of closed shadow graph $D_2[G]$ of a graph G is given by*

$$H(D_2[G]) = 4H(G) + n.$$

Proof. From Lemma 3.10, the distance matrix of $D_2[G]$ is

$$D(D_2[G]) = \begin{bmatrix} D(G) & D(G) + I_n \\ D(G) + I_n & D(G) \end{bmatrix}$$

The Harary index of $D_2[G]$ is given as:

$$\begin{aligned} H(D_2[G]) &= \frac{1}{2} \sum \overline{D(D_2[G])} \\ &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G) + I_n} + \sum \overline{D(G) + I_n} + \sum \overline{D(G)} \right] \\ &= \frac{1}{2} \left[2 \sum \overline{D(G)} + 2 \sum \overline{D(G) + I_n} \right] \\ &= \frac{1}{2} \left[2 \sum \overline{D(G)} + 2 \sum \overline{D(G)} + 2 \sum \overline{I_n} \right] \quad [By Lemma 3.4] \\ &= \frac{1}{2} \left[4 \sum \overline{D(G)} + 2 \sum \overline{I_n} \right] \\ &= \frac{1}{2} [4(2H(G)) + 2n] \\ &= 4H(G) + n. \end{aligned}$$

Hence the theorem. □

Theorem 3.4. *Let G be a connected graph of n vertices and m edges. The Harary index of Mycielskian graph $\mathbb{M}(G)$ of G is given by*

$$H(\mathbb{M}(G)) = 3H(G) + \frac{1}{4}n^2 + \frac{7}{4}n.$$

Proof. The distance matrix of $\mathbb{M}(G)$ as a 3×3 block matrix entries is written as

$$D(\mathbb{M}(G)) = \begin{bmatrix} D(G) & D(G) + 2I_n & 2C \\ D(G) + 2I_n & 2J_n - 2I_n & C \\ 2B & B & O \end{bmatrix}$$

where $D(G)$ is the distance matrix of graph G , I_n is the identity matrix of size n , J_n is the all-ones matrix of size n , B is the row matrix of order n with $b_{ij} = 1 \forall i, j$, C is the column matrix of order n with $c_{ij} = 1 \forall i, j$ and O is zero matrix of order 1.

The Harary index of $\mathbb{M}(G)$ can be expressed using distance matrix $D(\mathbb{M}(G))$ as follows:

$$\begin{aligned}
 H(\mathbb{M}(G)) &= \frac{1}{2} \sum \overline{D(\mathbb{M}(G))} \\
 &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G) + 2I_n} + \sum \overline{2C} + \sum \overline{D(G) + 2I_n} \right] \\
 &\quad + \frac{1}{2} \left[\sum \overline{2J_n - 2I_n} + \sum \overline{C} + \sum \overline{2B} + \sum \overline{B} \right] \\
 &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G)} + \sum \overline{2I_n} + \sum \overline{2C} + \sum \overline{D(G)} + \sum \overline{2I_n} \right] \\
 &\quad - \frac{1}{2} \left[\sum \overline{2J_n} + \sum \overline{2I_n} + \sum \overline{C} + \sum \overline{2B} + \sum \overline{B} \right] \quad [By Lemma 3.4] \\
 &= \frac{1}{2} \left[3 \sum \overline{D(G)} + \sum \overline{2I_n} + \sum \overline{2C} + \sum \overline{2J_n} + \sum \overline{C} + \sum \overline{2B} + \sum \overline{B} \right] \\
 &= \frac{1}{2} \left[3(2H(G)) + \frac{1}{2}n + \frac{1}{2}n + \frac{1}{2}n^2 + n + \frac{1}{2}n + n \right] \\
 &= 3H(G) + \frac{1}{4}n^2 + \frac{7}{4}n.
 \end{aligned}$$

Hence the theorem. □

Theorem 3.5. *Let G be a connected graph of n vertices and m edges. The Harary index of closed Mycielskian graph $\mathbb{M}[G]$ of G is given by*

$$H(\mathbb{M}[G]) = 3H(G) + \frac{1}{4}n^2 + \frac{9}{4}n.$$

Proof. The distance matrix of $\mathbb{M}[G]$ as a 3×3 block matrix entries is written as

$$D(\mathbb{M}[G]) = \begin{bmatrix} D(G) & D(G) + I_n & 2C \\ D(G) + I_n & 2J_n - 2I_n & C \\ 2B & B & O \end{bmatrix}$$

where $D(G)$ is the distance matrix of graph G , I_n is the identity matrix of size n , J_n is the all-ones matrix of size n , B is the row matrix of order n with $b_{ij} = 1 \forall i, j$, C is the column matrix of order n with $c_{ij} = 1 \forall i, j$ and O is zero matrix of order 1.

The Harary index of $\mathbb{M}[G]$ can be expressed using distance matrix $D(\mathbb{M}[G])$ as follows:

$$\begin{aligned}
 H(\mathbb{M}[G]) &= \frac{1}{2} \sum \overline{D(\mathbb{M}[G])} \\
 &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G) + I_n} + \sum \overline{2C} + \sum \overline{D(G) + I_n} + \sum \overline{2J_n - 2I_n} \right] \\
 &\quad + \frac{1}{2} \left[\sum \overline{C} + \sum \overline{2B} + \sum \overline{B} \right] \\
 &= \frac{1}{2} \left[\sum \overline{D(G)} + \sum \overline{D(G)} + \sum \overline{I_n} + \sum \overline{2C} + \sum \overline{D(G)} + \sum \overline{I_n} \right] \\
 &\quad - \frac{1}{2} \left[\sum \overline{2J_n} + \sum \overline{2I_n} + \sum \overline{C} + \sum \overline{2B} + \sum \overline{B} \right] \quad [By Lemma 3.4] \\
 &= \frac{1}{2} \left[3 \sum \overline{D(G)} + 2 \sum \overline{I_n} + \sum \overline{2C} + \sum \overline{2J_n} - \sum \overline{2I_n} \right] \\
 &\quad + \frac{1}{2} \left[\sum \overline{C} + \sum \overline{2B} + \sum \overline{B} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[3(2H(G)) + 2n + \frac{1}{2}n + \frac{1}{2}n^2 - \frac{1}{2}n + n + \frac{1}{2}n + n \right] \\
&= 3H(G) + \frac{1}{4}n^2 + \frac{9}{4}n.
\end{aligned}$$

Hence the theorem. $\square$

4. CONCLUSIONS

In this paper, we have given some results on the Wiener and Harary indices of the Mycielskian and closed Mycielskian graphs.

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