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REALIZATION ALGORITHM FOR DEFINING FRACTIONAL ORDER IN OSCILLATING SYSTEMS WITH LIQUID DAMPER

N.S. HAJIYEVA[1], I.V. ALIYEVA[2], F.A. ALIEV[3*], §

Abstract. In the paper the problem of defining the fractional order in oscillating systems with liquid damper. Firstly, the equation of the object is reduced to the Volterra integral equation of the second kind with respect to the second order derivative of the phase coordinate. Based on the statistical data the quadratic functional has been constructed. Using the method of successive approximations the obtained Volterra integral equation has been solved and its solution has the form of the Neumann series. By means of the least squares method, we ensure that the theoretical results coincide with the statistical data, and as a result, a more effective fractional order is determined. Then, an effective algorithm is proposed. Since some steps of this algorithm need explanation, the issue of the implementation of the algorithm is considered.

Keywords: fractional order differential equations, the least squares method, statistical data.
AMS Subject Classification: 49J15, 49J35.

1. Introduction

It is known that in recent years, the most widespread method of exploitation of oil wells [18,19,21,28] is the method of exploitation with rod pumping units [10,14,22,23], after the fountain method. Recently, much attention has been paid to the use of fractional order differential equations [1, 2, 5–9, 13, 15, 16, 20, 24, 26, 36–38] in various problems of mechanics and physics [27, 29, 30, 35]. Determining the fractional order [11] in these equations is one of the most important issues. In fact, this problem has been considered previously [3]. However, to solve this problem, the fractional order linear differential equation is reduced to the Volterra-type integral equation of the second kind with respect to the phase

[1] Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
[2] Baku State University, Baku, Azerbaijan
[3] Institute of Information Technology, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
e-mail: nazile.m@mail.ru; ORCID: https://orcid.org/0000-0002-9227-9007.
e-mail: irade.alieva14@gmail.com; ORCID: https://orcid.org/0009-0005-4694-8571.
e-mail: f_aliev@yahoo.com; ORCID: https://orcid.org/0000-0001-5402-8920.
[*] Corresponding author.
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coordinate, and the method of dividing the interval into two parts is used to determine the fractional linear equation.

But in the current work for determining this order the computational algorithm has been proposed reducing the fractional order linear differential equation to the Volterra-type integral equation of the second kind with respect to the second order derivative of phase coordinate [11] and some steps of this algorithm have been implemented. A table is introduced to determine the fractional order based on a simple example using the least squares method, and the effective fractional order is taken as the value at which the first variation of the functional approaches zero with an accuracy of 10^{-8}.

2. Problem Statement

Suppose the motion of an object is described by a system of fractional linear differential equations as follows [11]:

$$m\ddot{y}\left(x\right)+aD^{\alpha}y\left(x\right)+by\left(x\right)=f\left(x\right),x\geq x_0>0, \tag{1}$$

$$y\left(x_0\right)=0,\dot{y}\left(x_0\right)=y_1, \tag{2}$$

$\alpha\in(1,2), y\left(x\right)$- the desired function, m, a, b, y_1, x_0-given parameters,$f\left(x\right)$ - external force.

Let us compose the following quadratic functional to find α :

$$J(\alpha)=\min_{\alpha}\left(y\left(l\right)-\sum_{j=1}^{s}\frac{y_j}{s}\right)^2, \tag{3}$$

where $y_j,\ \ j=\overline{1,s}$ statistical data for finding α, $y\left(l\right)$ — the value of the solution to the problem (1)-(2) at the point l.

To solve the problem (1)-(3), we first reduce the problem (1)-(2) to the Volterra integral equation [12, 17, 25, 31, 32] of the second kind with respect to the phase coordinate $y(x)$:

$$\ddot{y}\left(x\right)+\int_{x_0}^{x}K\left(x-t\right)\ddot{y}\left(t\right)dt=F\left(x\right), \tag{4}$$

where

$$K\left(x-t\right)=\frac{a}{m}\frac{\left(x-t\right)^{1-\alpha}}{(1-\alpha)!}+\frac{b}{m}\left(x-t\right)\equiv K_0(x-t), \tag{5}$$

$$F(x)=\frac{f\left(x\right)}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha}}{(1-\alpha)!}-\frac{b}{m}y_1x. \tag{6}$$

To solve the equation (4) using the method of successive approximations [33, 34] and its solution has the form of the Neumann series

$$y\left(x\right)=y_1x+\int_{x_0}^{x}\left(x-t\right)F\left(t\right)dt+\sum_{j=0}^{\infty}\int_{x_0}^{x}\left(x-t\right)F\left(t\right)dt\int_{t}^{x}K_j\left(\xi-t\right)d\xi. \tag{7}$$

$$K_n\left(x-t\right)=-\sum_{k=0}^{n+1}C_{n+1}^{k}\left(\frac{a}{m}\right)^{(n+1)-k}\left(\frac{b}{m}\right)^{k}\frac{\left(x-t\right)^{2(n+1)-1-(n+1-k)\alpha}}{(2(n+1)-1-\left(n+1-k\right)\alpha)!}, \tag{8}$$

Let us consider (7) in functional (3):

$$J(\alpha)=\min_{\alpha}\left(y_1l+\int_{x_0}^{l}\left(l-t\right)F\left(t\right)dt+\sum_{j=0}^{\infty}\int_{x_0}^{l}\left(l-t\right)F\left(t\right)dt\int_{t}^{l}K_j\left(\xi-t\right)d\xi-\sum_{p=1}^{s}\frac{y_p}{s}\right)^2. \tag{9}$$

Using the least squares method, the following condition is checked to determine the parameter α:

$$\frac{\partial J(\alpha)}{\partial \alpha} \approx \frac{J(\alpha+h)-J(\alpha)}{h} \approx 0. \tag{10}$$

So, let us present the following algorithm for solving problems (1)-(3).

Algorithm.

(1) Enter the values of the parameters $m,\ a,\ b,\ y_1,\ f,\ n,\ k,\ l,\ x_0$included in problem (1)-(2).
(2) Substitute the expression for $F(t)$ according (6).
(3) Substitute the expression for $K_n\,(x-t)$ according (8).
(4) Enter the statistical data $y_p,\ \ p=\overline{1,s}$.
(5) Construct the functional (9).
(6) Using the least squares method, we check condition (10) to determine the parameter α.

3. Algorithm realization

Let the parameters in problem (1)-(3) as follows [14] :

$$m=10^5,\ a=3,\ b=1,\ y_1=0,\ f=8,\ n=1,\ k=1,\ l=1, s=11,$$

$y_1(x)=0,\ y_2(x)=-0.67,\ y_3(x)=-0.34,\ y_4(x)=0.81,\ y_5(x)=1.22,\ y_6(x)=1.44,$
$y_7(x)=1.57, y_8(x)=1.66, y_9(x)=1.72, y_{10}(x)=1.77, y_{11}(x)=1.81.$

Note that some steps of the algorithm above need explanations. The first step does not require much additional work. The following explanations are necessary for the subsequent steps:

3.1. **Construction of the function $F(t)$.** We can write for each α_i $(1<\alpha_i<2,\ i=\overline{1,10})$ in equation (9):

$$F_i(t)=\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i}}{(1-\alpha_i)!}-\frac{b}{m}y_1 t. \tag{11}$$

Lets write the expression of $(1-\alpha_i)!$ through the Green's function:

$$\Gamma(\alpha)=(\alpha-1)!=\int_0^\infty e^{-t}t^{-\alpha}dt, \tag{12}$$

$$(1-\alpha_i)!=\int_0^\infty e^{-t}t^{2-\alpha_i}dt. \tag{13}$$

Let us consider the expression (13) in (11):

$$F_i(t)=\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t}t^{2-\alpha_i}dt}-\frac{b}{m}y_1 t. \tag{14}$$

Then we substitute the expression (14) in the second term of expression (7):

$$\int_{x_0}^{l}(l-t)\,F_i\,(t)\,dt=\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t}t^{2-\alpha_i}dt}-\frac{b}{m}y_1 t\right)dt. \tag{15}$$

Using Matlab Software thedefinite integral (15) is calculated and entered into a table as follows:

	$\int_{x_0}^{l}(l-t)\,F_i(t)\,dt,\ \ i=\overline{1,9}$
$\alpha = 1.1$	2399/600000-1/51300/gamma(9/10)
$\alpha = 1.2$	2399/600000-1/38400/gamma(4/5)
$\alpha = 1.3$	2399/600000-3/83300/gamma(7/10)
$\alpha = 1.4$	2399/600000-1/19200/gamma(3/5)
$\alpha = 1.5$	2399/600000-1/12500/pi∧(1/2)
$\alpha = 1.6$	2399/600000-3/22400/pi*sin(2/5*pi)*gamma(3/5)
$\alpha = 1.7$	2399/600000-1/3900/pi*sin(3/10*pi)*gamma(7/10)
$\alpha = 1.8$	2399/600000-1/1600*sin(1/5*pi)*gamma(4/5)/pi
$\alpha = 1.9$	2399/600000-3/1100/pi*sin(1/10*pi)*gamma(9/10)

Table 1. Calculating $\int_{x_0}^{l}(l-t)\,F_i(t)\,dt$**.**

3.2. **Constructing the Kernel.** $K_n(x-t)$ We write for each α_i $(1<\alpha_i<2,\ i=\overline{1,9})$ in equation (8):

$$\sum_{j=0}^{\infty}K_{ij}(\xi-t)=-\sum_{j=0}^{\infty}\sum_{k=0}^{n}C_n^k\left(\frac{a}{m}\right)^{n-k}\left(\frac{b}{m}\right)^{k}\frac{(\xi-t)^{2n-1-(n-k)\alpha_i}}{(2n-1-(n-k)\,\alpha_i)!}. \tag{16}$$

Then we write through the Greens' function (12) as

$$(2n-1-(n-k)\,\alpha_i)!=\int_0^{\infty}e^{-t}t^{2n-(n-k)\alpha_i}dt. \tag{17}$$

Further we consider the expression (17) in (16):

$$\sum_{j=0}^{\infty}K_{ij}(\xi-t)=-\sum_{j=0}^{\infty}\sum_{k=0}^{n}C_n^k\left(\frac{a}{m}\right)^{n-k}\left(\frac{b}{m}\right)^{k}\frac{(\xi-t)^{2n-1-(n-k)\alpha_i}}{\int_0^{\infty}e^{-t}t^{2n-(n-k)\alpha_i}dt}. \tag{18}$$

Let us substitute the expressions (15) and (18) in the third term of expression (7):

$$\sum_{j=0}^{\infty}\int_0^{l}(l-t)\,F_i(t)\,dt\int_t^{l}K_{ij}(\xi-t)\,d\xi=-\sum_{j=0}^{\infty}\int_0^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i}}{\int_0^{\infty}e^{-t}t^{2-\alpha_i}dt}-\frac{b}{m}y_1t\right)dt\times$$

$$\times\int_t^{l}\sum_{j=0}^{\infty}\sum_{k=0}^{n}C_n^k\left(\frac{a}{m}\right)^{n-k}\left(\frac{b}{m}\right)^{k}\frac{(\xi-t)^{2n-1-(n-k)\alpha_i}}{\int_0^{\infty}e^{-t}t^{2n-(n-k)\alpha_i}dt}d\xi. \tag{19}$$

Using Matlab Software the definite integral (19) is calculated and entered into a table as follows:

	$\sum_{j=0}^{\infty}\int_{x_0}^{l}(l-t)\,F_i(t)\,dt\int_t^{l}K_{ij}(\xi-t)\,d\xi,\ \ i=\overline{1,9}$
$\alpha = 1.1$	1.0010e-005
$\alpha = 1.2$	1.1928e-005
$\alpha = 1.3$	1.4361e-005
$\alpha = 1.4$	1.7541e-005
$\alpha = 1.5$	2.1879e-005
$\alpha = 1.6$	2.8185e-005
$\alpha = 1.7$	3.8315e-005
$\alpha = 1.8$	5.7681e-005
$\alpha = 1.9$	1.1153e-004

Table 2. Calculating $\sum_{j=0}^{\infty}\int_{x_0}^{l}(l-t)\,F_i(t)\,dt\int_t^{l}K_{ij}(\xi-t)\,d\xi$**.**

3.3. **Calculating the expression (7).** Let us consider the expressions (15) and (19) in expression (7):

$$y_i(l) = \int_{x_0}^{l} (l-t)\left(\frac{f}{m} - \frac{a}{m} y_1 \frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t} t^{2-\alpha_i} dt} - \frac{b}{m} y_1 t\right) dt -$$
$$-\sum_{j=0}^{\infty} \int_{x_0}^{l} (l-t)\left(\frac{f}{m} - \frac{a}{m} y_1 \frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t} t^{2-\alpha_i} dt} - \frac{b}{m} y_1 t\right) dt \times$$
$$\times \int_t^l \sum_{j=0}^{\infty} \sum_{k=0}^{n} C_n^k \left(\frac{a}{m}\right)^{n-k} \left(\frac{b}{m}\right)^k \frac{(\xi - t)^{2n-1-(n-k)\alpha_i}}{\int_0^\infty e^{-t} t^{2n-(n-k)\alpha_i} dt} d\xi. \tag{20}$$

Using Matlab Software the definite integral (20) is calculated and entered into a table as follows:

	$y_i(l)$
$\alpha = 1.1$	6044781134965157995499 63/15111572745182864683827200000
$\alpha = 1.2$	6044810491661967716249 63/15111572745182864683827200000
$\alpha = 1.3$	1511211960297191023038 97/3777893186295716170956800000
$\alpha = 1.4$	6044896846070795654874 63/15111572745182864683827200000
$\alpha = 1.5$	3022482022134889274296 69/7555786372591432341913600000
$\alpha = 1.6$	3778164028298191820019 3/944473296573929042739200000
$\alpha = 1.7$	3778263951199670510019 3/944473296573929042739200000
$\alpha = 1.8$	7556917979796037505351 1/1888946593147858085478400000
$\alpha = 1.9$	2361895314937669572473/590295810358705651712 00000

Table 3. Calculating $y_i(l)$.

3.4. **Constructing the functional $J(\alpha_i)$.** Let us consider the expression (20) and the statistical data in (9):

$$J(\alpha_i) = \min_{\alpha_i} \left(\int_{x_0}^{l} (l-t)\left(\frac{f}{m} - \frac{a}{m} y_1 \frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t} t^{2-\alpha_i} dt} - \frac{b}{m} y_1 t\right) dt - \right.$$
$$-\sum_{j=0}^{\infty} \int_{x_0}^{l} (l-t)\left(\frac{f}{m} - \frac{a}{m} y_1 \frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t} t^{2-\alpha_i} dt} - \frac{b}{m} y_1 t\right) dt \times$$
$$\left. \times \int_t^l \sum_{j=0}^{\infty} \sum_{k=0}^{n} C_n^k \left(\frac{a}{m}\right)^{n-k} \left(\frac{b}{m}\right)^k \frac{(\xi - t)^{2n-1-(n-k)\alpha_i}}{\int_0^\infty e^{-t} t^{2n-(n-k)\alpha_i} dt} d\xi - \sum_{p=1}^{s} \frac{y_p(x)}{s} \right)^2. \tag{21}$$

Using Matlab Software the definite integral (21) is calculated and entered into a table as follows:

	$J(\alpha_i)$
$\alpha = 1.1$	1.4399039991854506749399714749094
$\alpha = 1.2$	1.4399039987192270400422118620174
$\alpha = 1.3$	1.4399039981260665678548636218649
$\alpha = 1.4$	1.4399039973478034557978132288056
$\alpha = 1.5$	1.4399039962806061219767630649706
$\alpha = 1.6$	1.4399039947178664324068548419640
$\alpha = 1.7$	1.4399039921788113854528329993152
$\alpha = 1.8$	1.4399039872228505054173814614358
$\alpha = 1.9$	1.4399039726497787317126594810796

Table 4. Calculating $J(\alpha_i)$.

3.5. **Constructing $J(\alpha_i+h)$.** Let us write the expression (21) at the point (α_i+h_v), $v=\overline{1,5}$:

$$J(\alpha_i+h_v)=\min_{\alpha_i}\left(\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i-h_v}}{\int_0^\infty e^{-t}t^{2-\alpha_i-h_v}dt}-\frac{b}{m}y_1t\right)dt-\right.$$

$$-\sum_{j=0}^{\infty}\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i-h_v}}{\int_0^\infty e^{-t}t^{2-\alpha_i-h_v}dt}-\frac{b}{m}y_1t\right)dt\times$$

$$\left.\times\int_t^l\sum_{j=0}^{\infty}\sum_{k=0}^{n}C_n^k\left(\frac{a}{m}\right)^{n-k}\left(\frac{b}{m}\right)^k\frac{(\xi-t)^{2n-1-(n-k)(\alpha_i+h_v)}}{\int_0^\infty e^{-t}t^{2n-(n-k)(\alpha_i+h_v)}dt}d\xi-\sum_{p=1}^{s}\frac{y_p(x)}{s}\right)^2. \tag{22}$$

Using Matlab Software thedefinite integral (22) is calculated and entered into a table as follows:

	$J(\alpha_i+h_v)$ $\alpha=1.1$	$J(\alpha_i+h_v)$ $\alpha=1.2$	$J(\alpha_i+h_v)$ $\alpha=1.4$	$J(\alpha_i+h_v)$ $\alpha=1.85$
$h_1=10^{-1}$	1.439903998719	1.439903998126	1.43990399628	1.4399039437
$h_2=10^{-2}$	1.439903999143	1.439903998666	1.43990399725	1.4399039809
$h_3=10^{-3}$	1.439903999181	1.439903998713	1.43990399733	1.4399039822
$h_4=10^{-4}$	1.439903999185	1.439903998718	1.43990399734	1.4399039823
$h_5=10^{-5}$	1.439903999185	1.439903998719	1.43990399734	1.4399039823

Table 5. Calculating $J(\alpha_i+h_v)$.

3.6. **Constructing the variation.** $\frac{\partial J(\alpha)}{\partial\alpha}$ Let us consider the expressions (21) and (22) in (10):

$$\frac{\partial J(\alpha_i)}{\partial\alpha_i}=\left\{\left(\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i-h_v}}{\int_0^\infty e^{-t}t^{2-\alpha_i-h_v}dt}-\frac{b}{m}y_1t\right)dt-\right.\right.$$

$$-\sum_{j=0}^{\infty}\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i-h_v}}{\int_0^\infty e^{-t}t^{2-\alpha_i-h_v}dt}-\frac{b}{m}y_1t\right)dt\times$$

$$\left.\times\int_t^l\sum_{j=0}^{\infty}\sum_{k=0}^{n}C_n^k\left(\frac{a}{m}\right)^{n-k}\left(\frac{b}{m}\right)^k\frac{(\xi-t)^{2n-1-(n-k)(\alpha_i+h_v)}}{\int_0^\infty e^{-t}t^{2n-(n-k)(\alpha_i+h_v)}dt}d\xi-\sum_{p=1}^{s}\frac{y_p(x)}{s}\right)^2-$$

$$-\left(\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t}t^{2-\alpha_i}dt}-\frac{b}{m}y_1t\right)dt-\right.$$

$$-\sum_{j=0}^{\infty}\int_{x_0}^{l}(l-t)\left(\frac{f}{m}-\frac{a}{m}y_1\frac{x^{1-\alpha_i}}{\int_0^\infty e^{-t}t^{2-\alpha_i}dt}-\frac{b}{m}y_1t\right)dt\times$$

$$\left.\left.\times\int_t^l\sum_{j=0}^{\infty}\sum_{k=0}^{n}C_n^k\left(\frac{a}{m}\right)^{n-k}\left(\frac{b}{m}\right)^k\frac{(\xi-t)^{2n-1-(n-k)\alpha_i}}{\int_0^\infty e^{-t}t^{2n-(n-k)\alpha_i}dt}d\xi-\sum_{p=1}^{s}\frac{y_p(x)}{s}\right)^2\right\}\Big/h_v. \tag{23}$$

Using Matlab Software the definite integral (23) is calculated and entered into a table as follows:

	$\frac{\partial J(\alpha_i)}{\partial \alpha_i}$, $\alpha = 1.1$	$\frac{\partial J(\alpha_i)}{\partial \alpha_i}$, $\alpha = 1.2$	$\frac{\partial J(\alpha_i)}{\partial \alpha_i}$, $\alpha = 1.4$	$\frac{\partial J(\alpha_i)}{\partial \alpha_i}$, $\alpha = 1.85$
h = 10^{-1}	$-0.46622 \cdot 10^{-8}$	$-0.5931 \cdot 10^{-8}$	$-0.1067 \cdot 10^{-7}$	$-0.3858 \cdot 10^{-6}$
h = 10^{-2}	$-0.42037 \cdot 10^{-8}$	$-0\ .5287 \cdot 10^{-8}$	$-0.9140 \cdot 10^{-8}$	$-0.1388 \cdot 10^{-6}$
h = 10^{-3}	$-0.41625 \cdot 10^{-8}$	$-0\ .5230 \cdot 10^{-8}$	$-0.9011 \cdot 10^{-8}$	$-0.1305 \cdot 10^{-6}$
h = 10^{-4}	$-0.41584 \cdot 10^{-8}$	$-0\ .5224 \cdot 10^{-8}$	$-0.8998 \cdot 10^{-8}$	$-0.1298 \cdot 10^{-6}$
h = 10^{-5}	$-0.41580 \cdot 10^{-8}$	$-0.5223 \cdot 10^{-8}$	$-0.8997 \cdot 10^{-8}$	$-0.1297 \cdot 10^{-6}$

Table 6. Constructing the variation $\frac{\partial J(\alpha_i)}{\partial \alpha_i}$.

As can be seen from the Table 6,when $\alpha = 1.1$,the expression (23), that is, the first variation of the functional (9) approaches zero with an accuracy of 10^{-8}. But for $\alpha = 1.85$ obtained in the previous problem [3], the first variation of the functional approaches zero with an accuracy of 10^{-4}. Even when we check $\alpha = 1.85$ with our method, the first variation of the functional approaches zero with an accuracy of 10^{-6}. From this it is clear once again that the most effective fractional order is

4. Conclusions

In the paper the calculation algorithm for defining the fractional order in oscillating systems with liquid damper. Using Matlab Software some steps of this algorithm has been explained.

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Hajiyeva Nazile She has graduated from the faculty of Mechanics and Mathematics of Baku State University in 2009 with honors degree. She received her M.Sc. degree with honors degree in 2011 at the faculty of Mechanics and Mathematics of Baku State University. She received the Ph.D. degree in 2016 from the Institute of Applied Mathematics of Baku State University. She is leading researcher of the Department of “Inverse Problems and Image Recognition” at the Institute of Applied Mathematics of Baku State University. Professional Interests are optimization, numerical analysis, fractional order differential equations. She has 32 scientific papers, 18 of them are in the list of Web of Science.

Aliyeva Irade She has graduated from the faculty of Mechanics and Mathematics of Baku State University in 2023. She received her M.Sc. degree in 225 at the faculty of Mechanics and Mathematics of Baku State University. Professional Interests are mechanics, optimization, numerical analysis, fractional order differential equations. She has 10 scientific papers..

F.A. Aliev for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.7, No.1.