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SIMILARITY MEASURE ON CUBIC INTUITIONISTIC FUZZY SETS AND ITS RELATIONSHIP WITH ENTROPY MEASURE

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Abstract. The objective of the study is to introduce the axioms defining similarity measures on cubic intuitionistic fuzzy sets. The proposed similarity measure integrates the principles of similarity measures on interval-valued intuitionistic fuzzy sets and intuitionistic fuzzy sets. Additionally, a novel approach is presented to construct similarity measures using entropy measures specific to cubic intuitionistic fuzzy sets. The transformation of entropy measures into similarity measures is formalized through several key theorems that adhere to the established axioms. Illustrative example is provided to demonstrate and validate the proposed definitions.

Keywords: Intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, cubic intuitionistic fuzzy set, entropy measure, similarity measure.

AMS Subject Classification: 03E72, 28E10, 94D05.

1. Introduction

For the first time, Zadeh introduced the concept of fuzzy sets (FSs) in 1965 [37]. Situations that are vague or imprecise have been described using FS theory, where fuzziness can be measured by entropy. The term "entropy" is chosen due to its intrinsic similarity to equations in Shannon entropy [27]. In 1968, Zadeh [38] introduced fuzzy entropy for the first time, and in 1972, De Luca and Termini [5] developed the axiomatic framework for fuzzy entropy and referred to Shannon's probability entropy and interpreting it as a measure of the amount of information. A similarity measure is defined to compare the information carried by various things. In 1994, Hyung et al. [13] presented two similarity measures for both FSs and elements.

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The generalization of FSs known as intuitionistic fuzzy sets (IFSs), which includes degrees of membership and non-membership, is first described by Atanassov [2] in 1986. Interval-valued intuitionistic fuzzy sets (IVIFSs) are first described by Atanassov and Gargov [3] in 1989. A non-probabilistic type of entropy measure with a geometric interpretation for IFSs are proposed by Szmidt and Kacprzyk [29] in 2001. In 2007, Park et al. [24] introduced the concept of entropy and similarity measures for IVIFSs and discussed the relationship between similarity and entropy measures.

In 2009, Zhang et al. [41] axiomatically defined entropy for interval-valued fuzzy sets (IVFSs) and explored the relationship between entropy and similarity measures. In the same year, Xu and Yager [36] studied preference relations and defined similarity measures in IVF and IVIF environment. In 2011, Wei et al. [35] derived a generalized measure of entropy for IVIFSs. Sun and Liu [30], along with Hu and Li [11] proposed entropy and similarity measures, explored their interrelationship for IVIFSs.

In 2012, Jun, Kim, and Yang [16] introduced cubic sets (CSs), which combine IVFSs and FSs. CSs do not include non-membership degrees. In the same year, Zhang et al. [43] discussed the close relationship among entropy, similarity measures, and inclusion measures for IFSs. In 2013, Jing [14] developed a new class of similarity measure for IVIFSs based on the proposed entropy measure. In 2015, Meng and Chen [20] introduced the entropy measure of Atannasov's IVIFSs. Tiwari and Gupta [32] extended the entropy for IVIFSs based on distance, by considering the hesitancy degree in 2018.

The cubic intuitionistic fuzzy set (CIFS) is a combination of IVIFS and IFS, is developed in 2018 by Kaur and Garg [18]. They discussed the concepts of P-order and R-order in CIFSs, as well as the operations P(R)-union, P(R)-intersection, P(R)-addition, and P(R)-product. Additionally, they classified CIFSs as internal and external CIFSs. CIFSs capture both membership and non-membership degrees, provide a more comprehensive representation of uncertainty, motivating further research in the CIF framework.

In the same year, Garg and Kaur [7] proposed a series of distance measures based on Hamming, Euclidean, and Hausdorff measures for CIFSs and derived various relationships among them. Subsequently, in 2019 Garg and Kaur [8] presented a novel multi-criteria group decision-making method under the CIF environment, integrating it with the extended technique for order preference by similarity to an ideal solution (TOPSIS) method. They developed various concepts for CIFSs, including a series of aggregation operators, score functions, accuracy functions, correlation coefficients, and different types of distance measures. These concepts have been applied to various decision-making methods. However, a significant research gap remains: while CIFSs have been primarily applied to decision-making problems, their potential applications in image processing remain underexplored. The problem statement is to identify various measures for CIFSs to assess the quality of original and enhanced images, thereby determining their significance in the context of image processing applications.

In 2019, Song et al. [28] proposed similarity measure based on the direct operation on the membership function, non-membership function, hesitation function and the upper bound of membership function of two IFSs. They demonstrated its application in pattern recognition, medical diagnosis, and cluster analysis through numerical examples. In 2020, Verma and Merigo [34] proposed cosine similarity measure for IVIFSs and applied it to a contractor selection problem, demonstrating its effectiveness in real-life scenarios. In the same year, Jeevaraj [15] introduced a similarity measure for IVIF numbers based on the non-hesitance score function and applied it to pattern recognition problems.

In 2022, Chen and Liu [4] introduced a new class of similarity measures, called IF value similarity measures, with two components: a similarity measure and a non-similarity measure for two IFSs. These measures are demonstrated through applications in pattern recognition. In the same year, Gohain et al. [9] introduced similarity measures for IFSs that incorporate cross-evaluation factors and hesitancy differences, applying them to pattern recognition, face-mask selection, and clustering problems. In 2023, Talukdar and Dutta [31] proposed similarity measures for IFSs and defined entropy measures based on these similarities. They applied the proposed entropy measure in multi-criteria decision-making problems.

In 2024, Alolyian et al. [1] presented a new similarity measure for IVIFSs, demonstrating its superiority over existing measures and developing an algorithm for its application in multi-criteria decision-making problems. In the same year, Huang et al. [12] proposed a knowledge-based similarity measure for IFSs with applications in pattern recognition.

In 2024, Palanisamy and Periyasamy [23] explored cosine similarity measures for IVIFSs as a technique for assessing associations between objects in real-world scenarios. In 2024, Patel et al. [25] proposed a novel similarity measure for intuitionistic fuzzy sets and applied it in face recognition and software quality evaluation. Also in 2024, Vishnukumar et al. [33] introduced distance and similarity measures based on accuracy functions and applied them to solve multi-criteria decision-making problems using the TOPSIS technique.

Various authors [10, 17 & 21] have used different fuzzy extensions such as bipolar fuzzy, IVIFSs in various fields in recent years. Many studies focus on the similarity and the entropy measure on IFSs [19 & 39] and IVIFSs [40 & 42]. Most of the literatures indicate that the entropy measure has a strong relationship with the similarity measures for both IFSs and IVIFSs. Moreover, previous studies proved that entropy and similarity measures on IFSs and IVIFSs can be transformed by each other. However, none of the work is done using CIFS. The core objective of this research endeavour is to extend the similarity of IFSs and IVIFSs to CIFSs.

The rest of the paper is organized as follows: In Section 2, some necessary definitions related to CIFSs are reviewed. In Section 3, a definition of the similarity measure on CIFS is introduced and in Section 4, the relationship between the similarity measure and the entropy measure on CIFSs is investigated. In Section 5, novel CIF decision-making problems based on the TOPSIS method are proposed.

2. Preliminaries

In this section, some fundamental concepts and notations are provided.

Definition 2.1. [6] *A CIFS $\mathcal{A}$ defined over the universal set $\mathcal{X}$ is an ordered pair which is defined as follows:*

$$\mathcal{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in \mathcal{X}\}$$

where $A = \{\langle x, [\mu^-(x), \mu^+(x)], [\nu^-(x), \nu^+(x)]\rangle \mid x \in \mathcal{X}\}$ represents the IVIFS defined on $\mathcal{X}$ while $\lambda = \{x, \langle \mu(x), \nu(x)\rangle \mid x \in \mathcal{X}\}$ represents an IFS such that $0 \leq \mu^-(x) \leq \mu^+(x) \leq 1,\quad 0 \leq \nu^-(x) \leq \nu^+(x) \leq 1$ and $0 \leq \mu^+(x) + \nu^+(x) \leq 1$. Also, $0 \leq \mu(x), \nu(x) \leq 1$ and $\mu(x)+\nu(x) \leq 1$. For the sake of simplicity, we denote these pairs as $\mathcal{A} = \langle A, \lambda\rangle,$ where $\mathcal{A} = \langle [\mu^-, \mu^+], [\nu^-, \nu^+]\rangle$ and $\lambda = \langle \mu, \nu\rangle$ and call them the cubic intuitionistic fuzzy number(CIFN).

Definition 2.2. [26] *A real-valued function $\varepsilon : CIFS(\mathcal{X}) \to [0, 1]$ is called an entropy measure on a CIFS $\mathcal{A}$ if it satisfies the following axioms:*

(E1) $\varepsilon(\mathcal{A}) = 0, if \mathcal{A}$ *is a crisp set;*
(E2) $\varepsilon(\mathcal{A}) = 1$ *if and only if* $[\mu^-_{\mathcal{A}}(x_i), \mu^+_{\mathcal{A}}(x_i)] = [\nu^-_{\mathcal{A}}(x_i), \nu^+_{\mathcal{A}}(x_i)]$ *and* $\mu_{\mathcal{A}}(x_i) = \nu_{\mathcal{A}}(x_i)$ *for all* $x_i \in \mathcal{X}$*;*

(E3) $\varepsilon(\mathcal{A}) = \varepsilon(\mathcal{A}^c)$, *where* $\mathcal{A}^c$ *is the complement of* $\mathcal{A}$;
(E4) $\varepsilon(\mathcal{A}) \le \varepsilon(\mathcal{B})$,
if $\mathcal{A} \subseteq_P \mathcal{B}$ *with* $[\mu^-_{\mathcal{B}}(x_i), \mu^+_{\mathcal{B}}(x_i)] \le [\nu^-_{\mathcal{B}}(x_i), \nu^+_{\mathcal{B}}(x_i)], \mu_{\mathcal{B}}(x_i) \le \nu_{\mathcal{B}}(x_i)\ \forall x_i \in \mathcal{X}$;

or

if $\mathcal{B} \subseteq_P \mathcal{A}$ *with* $[\mu^-_{\mathcal{B}}(x_i), \mu^+_{\mathcal{B}}(x_i)] \ge [\nu^-_{\mathcal{B}}(x_i), \nu^+_{\mathcal{B}}(x_i)], \mu_{\mathcal{B}}(x_i) \ge \nu_{\mathcal{B}}(x_i)\ \forall x_i \in \mathcal{X}$.

Definition 2.3. [26] *A real-valued function* $\check{\epsilon} : CIFS(\mathcal{X}) \to [0, 1]$ *is said to be an entropy measure using distance on CIFS* $\mathcal{A}$ *on the universe* $\mathcal{X}$, *if* $\check{\epsilon}$ *satisfies the following properties:*

(E1) $\check{\varepsilon}(\mathcal{A}) = 0$, *if* $\mathcal{A}$ *is a crisp set*
(E2) $\check{\varepsilon}(\mathcal{A}) = 1$ *if and only if* $[\mu^-_{\mathcal{A}}(x_i), \mu^+_{\mathcal{A}}(x_i)] = [\nu^-_{\mathcal{A}}(x_i), \nu^+_{\mathcal{A}}(x_i)] = \left[\frac{1}{2}, \frac{1}{2}\right]$ *and* $\mu_{\mathcal{A}}(x_i) = \nu_{\mathcal{A}}(x_i) = \frac{1}{2}$ *for all* $x_i \in \mathcal{X}$; *i.e.,* $\Leftrightarrow \mathcal{A} = (\langle[\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}]\rangle, \langle\frac{1}{2}, \frac{1}{2}\rangle)$
(E3) *If* $d(\mathcal{A}, (\langle[\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}]\rangle\langle\frac{1}{2}, \frac{1}{2}\rangle)) \ge d(\mathcal{B}, (\langle[\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}]\rangle\langle\frac{1}{2}, \frac{1}{2}\rangle))$, *then* $\check{\varepsilon}(\mathcal{A}) \le \check{\varepsilon}(\mathcal{B})$ $\forall \mathcal{A}, \mathcal{B} \in CIFS(\mathcal{X})$ *where* d *is a distance measure on the CIFSs.*
(E4) $\check{\varepsilon}(\mathcal{A}) = \check{\varepsilon}(\mathcal{A}^c)$ *where* $\mathcal{A}^c$ *is the complement of* $\mathcal{A}$.

3. Similarity measure on cubic intuitionistic fuzzy sets

In this part of the article, an effective similarity measure on CIFSs is introduced.

Let $\mathcal{X} = \{x_i | i = 1, 2, \ldots, n\}$ be a universe of discourse. The family of all CIFSs in $\mathcal{X}$ is denoted by $CIFS(\mathcal{X})$.

Definition 3.1. *A mapping* $\mathcal{S} : CIFS(\mathcal{X}) \times CIFS(\mathcal{X}) \to [0, 1]$ *is called a similarity measure on the CIFSs* $\mathcal{A}, \mathcal{B}$, *and* $\mathcal{C}$ *if it satisfies the following properties:*

(S1) $\mathcal{S}(\mathcal{A}, \mathcal{A}^c) = 0$, *if* $\mathcal{A}$ *is a crisp set;*
(S2) $\mathcal{S}(\mathcal{A}, \mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$;
(S3) $\mathcal{S}(\mathcal{A}, \mathcal{B}) = \mathcal{S}(\mathcal{B}, \mathcal{A})$;
(S4) *If* $\mathcal{A} \subseteq_P \mathcal{B} \subseteq_P \mathcal{C}$ *then* $\mathcal{S}(\mathcal{A}, \mathcal{C}) \le \mathcal{S}(\mathcal{A}, \mathcal{B})$ *and* $\mathcal{S}(\mathcal{A}, \mathcal{C}) \le \mathcal{S}(\mathcal{B}, \mathcal{C})$.

Hereafter $\mu^-_A(x_i), \mu^+_A(x_i), \nu^-_A(x_i), \nu^+_A(x_i), \mu_A(x_i)$ and $\nu_A(x_i)$ will be denoted as $\mu^-_A, \mu^+_A, \nu^-_A$, ν^+_A, μ_A and ν_A.

Proposition 3.1. *Assume* $\mathcal{S}$ *is a similarity measure of CIFS, let* $\mathcal{A} \in CIFS(\mathcal{X})$, *then* $\mathcal{S}(\mathcal{A}, \mathcal{A}^c)$ *is an entropy of CIFS* $\mathcal{A}$.

Proof. Let $\mathcal{S}$ be a similarity measure of a CIFS $\mathcal{A}$, where $\mathcal{A} = \{\langle[\mu^-_{\mathcal{A}}, \mu^+_{\mathcal{A}}], [\nu^-_{\mathcal{A}}, \nu^+_{\mathcal{A}}]\rangle, \langle\mu_{\mathcal{A}}, \nu_{\mathcal{A}}\rangle\}$ and $\mathcal{A}^c$ is the complement of $\mathcal{A}$.
To prove $\mathcal{S}(\mathcal{A}, \mathcal{A}^c)$ is an entropy measure, it should satisfy the conditions of Definition 2.2.

(E1) If $\mathcal{A}$ is a crisp set, then $\mathcal{S}(\mathcal{A}, \mathcal{A}^c) = 0$, by Definition 3.1 (S1).
(E2) Let $[\mu^-_{\mathcal{A}}, \mu^+_{\mathcal{A}}] = [\nu^-_{\mathcal{A}}, \nu^+_{\mathcal{A}}]$ and $\mu_{\mathcal{A}} = \nu_{\mathcal{A}}$.Then this is possible if and only if $\mathcal{A} = \mathcal{A}^c$.
$\therefore$ From the Definition 3.1 (S2), we have, $\mathcal{S}(\mathcal{A}, \mathcal{A}^c) = 1$.
(E3) From the Definition 3.1 (S3), we have, $\mathcal{S}(\mathcal{A}, \mathcal{A}^c) = \mathcal{S}(\mathcal{A}^c, \mathcal{A})$.
(E4) Let $\mathcal{A} \subseteq_P \mathcal{B}$ with $\mu^-_{\mathcal{A}} \le \mu^-_{\mathcal{B}} \le \nu^-_{\mathcal{B}} \le \nu^-_{\mathcal{A}}, \mu^+_{\mathcal{A}} \le \mu^+_{\mathcal{B}} \le \nu^+_{\mathcal{B}} \le \nu^+_{\mathcal{A}}$ and $\mu_{\mathcal{A}} \le \mu_{\mathcal{B}} \le \nu_{\mathcal{B}} \le \nu_{\mathcal{A}}$, then

$$\mathcal{A} \subseteq_P \mathcal{B} \subseteq_P \mathcal{B}^c \subseteq_P \mathcal{A}^c.$$

$\therefore$ By the Definition 3.1 (S4), we have, $\mathcal{S}(\mathcal{A}, \mathcal{A}^c) \le \mathcal{S}(\mathcal{B}, \mathcal{A}^c) \le \mathcal{S}(\mathcal{B}, \mathcal{B}^c)$. This implies,

$$\mathcal{S}(\mathcal{A}, \mathcal{A}^c) \le \mathcal{S}(\mathcal{B}, \mathcal{B}^c).$$

Let $\mathcal{B} \subseteq_P \mathcal{A}$ with $\nu^-_{\mathcal{A}} \le \nu^-_{\mathcal{B}} \le \mu^-_{\mathcal{B}} \le \mu^-_{\mathcal{A}}, \nu^+_{\mathcal{A}} \le \nu^+_{\mathcal{B}} \le \mu^+_{\mathcal{B}} \le \mu^+_{\mathcal{A}}$ and $\nu_{\mathcal{A}} \le \nu_{\mathcal{B}} \le \mu_{\mathcal{B}} \le \mu_{\mathcal{A}}$, then

$$\mathcal{A}^c \subseteq_P \mathcal{B}^c \subseteq_P \mathcal{B} \subseteq_P \mathcal{A}.$$

$\therefore$ By the Definition 3.1 (S4), we have, $S(\mathcal{A},\mathcal{A}^c) \leq S(\mathcal{B},\mathcal{A}^c) \leq S(\mathcal{B},\mathcal{B}^c)$. This implies,

$$S(\mathcal{A},\mathcal{A}^c) \leq S(\mathcal{B},\mathcal{B}^c).$$

Since $\mathcal{S}(\mathcal{A},\mathcal{A}^c)$ satisfies all the conditions of Definition 2.2, it is an entropy measure for $\mathcal{A}$. $\square$

4. Relationship between entropy measure and similarity measure on cubic intuitionistic fuzzy sets

In this section, we have discussed the relationship between entropy and the similarity measures of CIFSs based on the proposed definition.

Definition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be any two distinct CIFSs. We define a new CIFS denoted by $\mathcal{M}(\mathcal{A},\mathcal{B})$ as follows:*

$$\mathcal{M}(\mathcal{A},\mathcal{B}) = \{\langle[\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}], [\nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}]\rangle, \langle\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}\rangle\}$$

where,

$$\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \min\{\mathcal{M}_{\mathcal{AB}1}, \mathcal{M}_{\mathcal{AB}2}\}, \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \max\{\mathcal{M}_{\mathcal{AB}1}, \mathcal{M}_{\mathcal{AB}2}\}$$

$$\nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \min\{\mathcal{M}_{\mathcal{AB}3}, \mathcal{M}_{\mathcal{AB}4}\}, \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \max\{\mathcal{M}_{\mathcal{AB}3}, \mathcal{M}_{\mathcal{AB}4}\}$$

$$\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \mathcal{M}_{\mathcal{AB}5} \quad \textit{and} \quad \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \mathcal{M}_{\mathcal{AB}6}, \ \textit{and}$$

$$\mathcal{M}_{\mathcal{AB}1} = \frac{1+2min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}})|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}})|\}}{3}, \quad \mathcal{M}_{\mathcal{AB}2} = \frac{1+2min\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{3}$$

$$\mathcal{M}_{\mathcal{AB}3} = \frac{1-max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}}{3}, \quad \mathcal{M}_{\mathcal{AB}4} = \frac{1-max\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{3}$$

$$\mathcal{M}_{\mathcal{AB}5} = \frac{1+2min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{3}, \quad \mathcal{M}_{\mathcal{AB}6} = \frac{1-max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{3}$$

Theorem 4.1. *Let ε be an entropy measure of the CIFS $\mathcal{M}(\mathcal{A},\mathcal{B})$, then it is a similarity measure.*

Proof. To prove $\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B}))$ satisfy the conditions of Definition 3.1.

(S1) Let $\mathcal{A}$ be a crisp set, i.e., $\mathcal{A} = (\langle[1,1],[0,0]\rangle, \langle 1,0\rangle)$ or $\mathcal{A} = (\langle[0,0],[1,1]\rangle, \langle 0,1\rangle)$. Then $\mathcal{A}^c = (\langle[0,0],[1,1]\rangle, \langle 0,1\rangle)$ or $\mathcal{A}^c = (\langle[1,1],[0,0]\rangle, \langle 1,0\rangle)$
Now, as $\mathcal{M}_{\mathcal{A}\mathcal{A}^c1} = 1, \mathcal{M}_{\mathcal{A}\mathcal{A}^c2} = 1, \mathcal{M}_{\mathcal{A}\mathcal{A}^c3} = 0, \mathcal{M}_{\mathcal{A}\mathcal{A}^c4} = 0, \mathcal{M}_{\mathcal{A}\mathcal{A}^c5} = 1$ and $\mathcal{M}_{\mathcal{A}\mathcal{A}^c6} = 0$, we have,
$\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{A}^c)} = 1, \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{A}^c)} = 1, \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{A}^c)} = 0, \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{A}^c)} = 0, \mu_{\mathcal{M}(\mathcal{A},\mathcal{A}^c)} = 1$, and $\nu_{\mathcal{M}(\mathcal{A},\mathcal{A}^c)} = 0$.
Thus, $\mathcal{M}(\mathcal{A},\mathcal{A}^c) = (\langle[1,1],[0,0]\rangle, \langle 1,0\rangle)$ is a crisp set in $\mathcal{X}$, and from the Definition 2.2 (E1) we have, $\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{A}^c)) = 0$

(S2) From the Definition 2.2 (E2) we have, $\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B})) = 1$
$\Leftrightarrow \mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}$ and $\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} = \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}$
$\Leftrightarrow |\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}| = 0, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}| = 0, |\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}| = 0, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}| = 0, |\mu_{\mathcal{A}} - \mu_{\mathcal{B}}| = 0$ and $|\nu_{\mathcal{A}} - \nu_{\mathcal{B}}| = 0$.

$$\Leftrightarrow \mathcal{A} = \mathcal{B}.$$

(S3) By the definition of $\mathcal{M}(\mathcal{A},\mathcal{B})$, it is obvious that $\mathcal{M}(\mathcal{A},\mathcal{B}) = \mathcal{M}(\mathcal{A},\mathcal{B})^c$. This implies,

$$\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B})) = \varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B})^c)$$

$$\text{and hence, } \varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B})) = \varepsilon(\mathcal{M}(\mathcal{B},\mathcal{A})).$$

(S4) If $\mathcal{A} \subseteq_P \mathcal{B} \subseteq_P \mathcal{C}$, then we have,
$\mu^-_{\mathcal{A}} \leq \mu^-_{\mathcal{B}} \leq \mu^-_{\mathcal{C}}, \mu^+_{\mathcal{A}} \leq \mu^+_{\mathcal{B}} \leq \mu^+_{\mathcal{C}}, \nu^-_{\mathcal{A}} \geq \nu^-_{\mathcal{B}} \geq \nu^-_{\mathcal{C}}, \nu^+_{\mathcal{A}} \geq \nu^+_{\mathcal{B}} \geq \nu^+_{\mathcal{C}}, \mu_{\mathcal{A}} \leq \mu_{\mathcal{B}} \leq \mu_{\mathcal{C}}$ and $\nu_{\mathcal{A}} \geq \nu_{\mathcal{B}} \geq \nu_{\mathcal{C}}$.
Hence,

$$|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{C}}| \geq |\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{C}}| \geq |\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{C}}| \geq |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|,$$
$$|\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{C}}| \geq |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|, |\mu_{\mathcal{A}} - \mu_{\mathcal{C}}| \geq |\mu_{\mathcal{A}} - \mu_{\mathcal{B}}| \text{ and } |\nu_{\mathcal{A}} - \nu_{\mathcal{C}}| \geq |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|.$$

It follows that,

$$\begin{aligned}
min\big(|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{C}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{C}}|\big) &\geq min\big(|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\big)\\
min\big(|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{C}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{C}}|\big) &\geq min\big(|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\big)\\
max\big(|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{C}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{C}}|\big) &\geq max\big(|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\big)\\
max\big(|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{C}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{C}}|\big) &\geq max\big(|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\big)\\
min\big(|\mu_{\mathcal{A}} - \mu_{\mathcal{C}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{C}}|\big) &\geq min\big(|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\big)\\
max\big(|\mu_{\mathcal{A}} - \mu_{\mathcal{C}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{C}}|\big) &\geq max\big(|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\big)
\end{aligned}$$

and we have,
$\mathcal{M}_{\mathcal{AC}1} \geq \mathcal{M}_{\mathcal{AB}1}, \mathcal{M}_{\mathcal{AC}2} \geq \mathcal{M}_{\mathcal{AB}2}, \mathcal{M}_{\mathcal{AC}3} \leq \mathcal{M}_{\mathcal{AB}3}, \mathcal{M}_{\mathcal{AC}4} \leq \mathcal{M}_{\mathcal{AB}4},$
$\mathcal{M}_{\mathcal{AC}5} \geq \mathcal{M}_{\mathcal{AB}5}, \mathcal{M}_{\mathcal{AC}6} \leq \mathcal{M}_{\mathcal{AB}6}$.

$$\text{Thus, } \mathcal{M}(\mathcal{A}, \mathcal{C}) \subseteq_P \mathcal{M}(\mathcal{A}, \mathcal{B}).$$

From the Definition 2.2 (E4), we have

$$\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{C})) \leq \varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{B})).$$

Similarly, we can prove, $\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{C})) \leq \varepsilon(\mathcal{M}(\mathcal{B}, \mathcal{C}))$.

$\therefore \varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{C})) \leq \varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{B}))$ and $\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{C})) \leq \varepsilon(\mathcal{M}(\mathcal{B}, \mathcal{C}))$ if $\mathcal{A} \subseteq_P \mathcal{B} \subseteq_P \mathcal{C}$.

Since all the conditions for a similarity measure on CIFS are satisfied by $\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{B}))$, it is a similarity measure. □

Corollary 4.1. *Let ε be an entropy measure for $\mathcal{M}(\mathcal{A}, \mathcal{B}) \in CIFS(\mathcal{X})$, then $\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{B})^c)$ is also a similarity measure.*

Proof. Straightforward. □

Theorem 4.2. *For each $\mathcal{A}, \mathcal{B} \in CIFS(\mathcal{X})$, the entropy measure $\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{B}))$ define by*

$$\varepsilon(\mathcal{M}(\mathcal{A}, \mathcal{B})) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\begin{array}{c}\min(\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}) + \min(\mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}) + \\ \min(\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}) + \pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}\end{array}}{\begin{array}{c}\max(\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}) + \max(\mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}) + \\ \max(\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}) + \pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}\end{array}}\right] \tag{1}$$

where, $\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} = 1 - \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} - \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} = 1 - \mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} - \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}$ and $\pi_{\mathcal{M}(\mathcal{A},\mathcal{B})} = 1 - \mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} - \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}$ is equal to the similarity measure

$$\mathcal{S}(\mathcal{A}, \mathcal{B}) = \frac{1}{n}\sum_{i=1}^{n}\frac{6 - 2\min(\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}) - 2\min(\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}) - 2\min(\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}})}{6 + \max(\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}) + \max(\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}) + \max(\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}})}$$

where, $\mu^-_{\mathcal{AB}} = |\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, \mu^+_{\mathcal{AB}} = |\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, \nu^-_{\mathcal{AB}} = |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|, \nu^+_{\mathcal{AB}} = |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|, \mu_{\mathcal{AB}} = |\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|$ *and* $\nu_{\mathcal{AB}} = |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|$.

Proof. By the definition of $\mathcal{M}(\mathcal{A},\mathcal{B})$, we have $\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} \geq \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} \geq \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}$ and $\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} \geq \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}$. Then (1) becomes,

$$\begin{aligned}
&\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B}))\\
&\quad= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}}{\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}}\right]\\
&\quad= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\mathcal{M}_{\mathcal{AB}3} + \mathcal{M}_{\mathcal{AB}4} + \mathcal{M}_{\mathcal{AB}6} + \pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}}{\mathcal{M}_{\mathcal{AB}1} + \mathcal{M}_{\mathcal{AB}2} + \mathcal{M}_{\mathcal{AB}5} + \pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}}\right]
\end{aligned} \tag{2}$$

In (2), consider $\mathcal{M}_{\mathcal{AB}3} + \mathcal{M}_{\mathcal{AB}4} + \mathcal{M}_{\mathcal{AB}6}$

$$\begin{aligned}
&= \frac{1 - \max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}}{3} + \frac{1 - \max\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{3}\\
&\quad + \frac{1 - \max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{3}\\
&= \left[\frac{\begin{array}{r}3 - \max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\} - \max\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}\\ - \max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}\end{array}}{3}\right]
\end{aligned} \tag{3}$$

and $\mathcal{M}_{\mathcal{AB}1} + \mathcal{M}_{\mathcal{AB}2} + \mathcal{M}_{\mathcal{AB}5}$

$$\begin{aligned}
&= \frac{1 + 2\min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}}{3} + \frac{1 + 2\min\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{3}\\
&\quad + \frac{1 + 2\min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{3}\\
&= \left[\frac{\begin{array}{r}3 + 2\min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\} + 2\min\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}\\ + 2\min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}\end{array}}{3}\right]
\end{aligned} \tag{4}$$

Sub (3) & (4) in (2), we get $\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B}))$

$$\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\begin{array}{r}3[\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}] + 3 - \max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}\\ - \max\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\} - \max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}\end{array}}{\begin{array}{r}3[\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}] + 3 + 2\min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}\\ + 2\min\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\} + 2\min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}\end{array}}\right]\\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\begin{array}{r}3[\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}] + 3 - \max(\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}})\\ - \max(\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}) - \max(\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}})\end{array}}{\begin{array}{r}3[\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}] + 3 + 2\min(\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}})\\ + 2\min(\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}) + 2\min(\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}})\end{array}}\right]
\end{aligned} \tag{5}$$

Since $\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} = 1 - \mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} - \nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})}, \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} = 1 - \mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} - \nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})}$ and $\pi_{\mathcal{M}(\mathcal{A},\mathcal{B})} = 1 - \mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} - \nu_{\mathcal{M}(\mathcal{A},\mathcal{B})}$, we have,

$$
\begin{aligned}
&3[\pi^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} + \pi_{\mathcal{M}(\mathcal{A},\mathcal{B})}] \\
&=9 - 3\mu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} - 3\nu^+_{\mathcal{M}(\mathcal{A},\mathcal{B})} - 3\mu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} - 3\nu^-_{\mathcal{M}(\mathcal{A},\mathcal{B})} - 3\mu_{\mathcal{M}(\mathcal{A},\mathcal{B})} - 3\nu_{\mathcal{M}(\mathcal{A},\mathcal{B})} \\
&=9 - 3(\max(\mathcal{M}_{\mathcal{AB}1}, \mathcal{M}_{\mathcal{AB}2})) - 3(\max(\mathcal{M}_{\mathcal{AB}3}, \mathcal{M}_{\mathcal{AB}4})) - 3(\min(\mathcal{M}_{\mathcal{AB}1}, \mathcal{M}_{\mathcal{AB}2})) \\
&\quad - 3(\min(\mathcal{M}_{\mathcal{AB}3}, \mathcal{M}_{\mathcal{AB}4})) - 3\mathcal{M}_{\mathcal{AB}5} - 3\mathcal{M}_{\mathcal{AB}6} \\
&=9 - 3\mathcal{M}_{\mathcal{AB}2} - 3\mathcal{M}_{\mathcal{AB}3} - 3\mathcal{M}_{\mathcal{AB}1} - 3\mathcal{M}_{\mathcal{AB}4} - 3\mathcal{M}_{\mathcal{AB}5} - 3\mathcal{M}_{\mathcal{AB}6} \\
&=9 - 3\left(\frac{1 + 2\min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}}{3}\right) - 3\left(\frac{1 - \max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\}}{3}\right) \\
&\quad - 3\left(\frac{1 + 2\min\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{3}\right) - 3\left(\frac{1 - \max\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{3}\right) \\
&\quad - 3\left(\frac{1 + 2\min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{3}\right) - 3\left(\frac{1 - \max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{3}\right) \\
&=9 - 6 + \max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\} + \max\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\} + \max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, \\
&\quad |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\} - 2\min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}|\} - 2\min\{|\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\} \\
&\quad - 2\min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\} \\
&=3 + \max\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} + \max\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} + \max\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} - 2\min\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} \\
&\quad - 2\min\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} - 2\min\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\}
\end{aligned}
\tag{6}
$$

Sub (6) in (5), we get

$$
\begin{aligned}
\varepsilon(\mathcal{M}(\mathcal{A},\mathcal{B})) &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\begin{array}{r} 6 - \max\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} - \max\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} - \max\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} \\ + \max\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} + \max\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} + \max\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} \\ -2\min\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} - 2\min\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} - 2\min\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} \end{array}}{\begin{array}{r} 6 + 2\min\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} + 2\min\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} + 2\min\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} \\ + \max\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} + \max\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} + \max\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} \\ -2\min\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} - 2\min\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} - 2\min\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\} \end{array}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{6 - 2\min\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} - 2\min\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} - 2\min\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\}}{6 + \max\{\mu^-_{\mathcal{AB}}, \nu^-_{\mathcal{AB}}\} + \max\{\mu^+_{\mathcal{AB}}, \nu^+_{\mathcal{AB}}\} + \max\{\mu_{\mathcal{AB}}, \nu_{\mathcal{AB}}\}}\right) \\
&= \mathcal{S}(\mathcal{A},\mathcal{B})
\end{aligned}
$$

□

Definition 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two CIFSs in universe $\mathcal{X}$, then a CIFS denoted by $\mathcal{N}(\mathcal{A},\mathcal{B})$ is defined as follows:*

$$\mathcal{N}(\mathcal{A},\mathcal{B}) = \{\langle[\mathcal{N}_{\mathcal{AB}1}, \mathcal{N}_{\mathcal{AB}2}], [\mathcal{N}_{\mathcal{AB}3}, \mathcal{N}_{\mathcal{AB}4}]\rangle, \langle\mathcal{N}_{\mathcal{AB}5}, \mathcal{N}_{\mathcal{AB}6}\rangle\}$$

where, $\mathcal{N}_{\mathcal{AB}1} = \mathcal{N}_{\mathcal{AB}2} = \dfrac{1 + \min\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}| \vee |\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}| \vee |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{2}$

$$\mathcal{N}_{\mathcal{AB}3} = \mathcal{N}_{\mathcal{AB}4} = \frac{1 - \max\{|\mu^-_{\mathcal{A}} - \mu^-_{\mathcal{B}}| \vee |\mu^+_{\mathcal{A}} - \mu^+_{\mathcal{B}}|, |\nu^-_{\mathcal{A}} - \nu^-_{\mathcal{B}}| \vee |\nu^+_{\mathcal{A}} - \nu^+_{\mathcal{B}}|\}}{2}$$

$$\mathcal{N}_{\mathcal{AB}5} = \frac{1 + \min\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{2} \text{ and } \mathcal{N}_{\mathcal{AB}6} = \frac{1 - \max\{|\mu_{\mathcal{A}} - \mu_{\mathcal{B}}|, |\nu_{\mathcal{A}} - \nu_{\mathcal{B}}|\}}{2}$$

Theorem 4.3. *Let ε be an entropy measure for the CIFS $\mathcal{N}(\mathcal{A},\mathcal{B})$, then it is a similarity measure for $\mathcal{N}(\mathcal{A},\mathcal{B})$.*

Proof. Straightforward. □

Corollary 4.2. *An entropy measure ε for the CIFS $\mathcal{N}(\mathcal{A},\mathcal{B})^c$, is a similarity measure for $\mathcal{N}(\mathcal{A},\mathcal{B})^c$.*

Proof. Straightforward. □

Definition 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two CIFSs in universe $\mathcal{X}$. For any positive integer n, we define another new CIFS denoted by $\mathcal{O}(\mathcal{A},\mathcal{B})$ as follows:*

$$\mathcal{O}(\mathcal{A},\mathcal{B}) =\{\langle[\mathcal{O}_{\mathcal{AB}1},\mathcal{O}_{\mathcal{AB}2}],[\mathcal{O}_{\mathcal{AB}3},\mathcal{O}_{\mathcal{AB}4}]\rangle,\langle\mathcal{O}_{\mathcal{AB}5},\mathcal{O}_{\mathcal{AB}6}\rangle\}$$

where,

$$\mathcal{O}_{\mathcal{AB}1} =\mathcal{O}_{\mathcal{AB}2} = \frac{1+\min\{(|\mu^-_{\mathcal{A}}-\mu^-_{\mathcal{B}}|\vee|\mu^+_{\mathcal{A}}-\mu^+_{\mathcal{B}}|)^n,(|\nu^-_{\mathcal{A}}-\nu^-_{\mathcal{B}}|\vee|\nu^+_{\mathcal{A}}-\nu^+_{\mathcal{B}}|)^n\}}{2}$$

$$\mathcal{O}_{\mathcal{AB}3} =\mathcal{O}_{\mathcal{AB}4} = \frac{1-\max\{(|\mu^-_{\mathcal{A}}-\mu^-_{\mathcal{B}})|\vee|\mu^+_{\mathcal{A}}-\mu^+_{\mathcal{B}}|)^n,(|\nu^-_{\mathcal{A}}-\nu^-_{\mathcal{B}}|\vee|\nu^+_{\mathcal{A}}-\nu^+_{\mathcal{B}}|)^n\}}{2}$$

$$\mathcal{O}_{\mathcal{AB}5} =\frac{1+\min\{|\mu_{\mathcal{A}}-\mu_{\mathcal{B}}|^n,|\nu_{\mathcal{A}}-\nu_{\mathcal{B}}|^n\}}{2} \text{ and } \mathcal{O}_{\mathcal{AB}6} = \frac{1-\max\{|\mu_{\mathcal{A}}-\mu_{\mathcal{B}}|^n,|\nu_{\mathcal{A}}-\nu_{\mathcal{B}}|^n\}}{2}$$

Theorem 4.4. *Let ε be an entropy measure for the CIFS $\mathcal{O}(\mathcal{A},\mathcal{B})$, then it is a similarity measure for $\mathcal{O}(\mathcal{A},\mathcal{B})$.*

Proof. Straightforward. □

Corollary 4.3. *An entropy measure ε for the CIFS $\mathcal{O}(\mathcal{A},\mathcal{B})^c$, is a similarity measure for $\mathcal{O}(\mathcal{A},\mathcal{B})^c$.*

Proof. Straightforward. □

Definition 4.4. *Let $\mathcal{A}$ be a CIFS in universe $\mathcal{X}$. For any positive integer n, we define new CIFSs $\mathscr{f}(\mathcal{A})$ and $\mathscr{g}(\mathcal{A})$ as follows:*

$$\mathscr{f}(\mathcal{A}) =\{\langle[\mathscr{f}_{\mathcal{A}1},\mathscr{f}_{\mathcal{A}2}],[\mathscr{f}_{\mathcal{A}3},\mathscr{f}_{\mathcal{A}4}]\rangle,\langle\mathscr{f}_{\mathcal{A}5},\mathscr{f}_{\mathcal{A}6}\rangle\} \text{ and}$$
$$\mathscr{g}(\mathcal{A}) =\{\langle[\mathscr{g}_{\mathcal{A}1},\mathscr{g}_{\mathcal{A}2}],[\mathscr{g}_{\mathcal{A}3},\mathscr{g}_{\mathcal{A}4}]\rangle,\langle\mathscr{g}_{\mathcal{A}5},\mathscr{g}_{\mathcal{A}6}\rangle\}$$

where,

$$\mathscr{f}_{\mathcal{A}1} =\mathscr{f}_{\mathcal{A}2} = \frac{1+(|\mu^-_{\mathcal{A}}-\nu^-_{\mathcal{A}}|\vee|\mu^+_{\mathcal{A}}-\nu^+_{\mathcal{A}}|)^n}{2}, \mathscr{f}_{\mathcal{A}3} = \mathscr{f}_{\mathcal{A}4} = \frac{1-(|\mu^-_{\mathcal{A}}-\nu^-_{\mathcal{A}}|\vee|\mu^+_{\mathcal{A}}-\nu^+_{\mathcal{A}}|)}{2},$$
$$\mathscr{f}_{\mathcal{A}5} =\frac{1+(|\mu_{\mathcal{A}}-\nu_{\mathcal{A}}|)^n}{2} \quad \text{and} \quad \mathscr{f}_{\mathcal{A}6} = \frac{1-(|\mu_{\mathcal{A}}-\nu_{\mathcal{A}}|)}{2},$$

$$\mathscr{g}_{\mathcal{A}1} =\mathscr{g}_{\mathcal{A}2} = \frac{1-(|\mu^-_{\mathcal{A}}-\nu^-_{\mathcal{A}}|\vee|\mu^+_{\mathcal{A}}-\nu^+_{\mathcal{A}}|)}{2}, \mathscr{g}_{\mathcal{A}3} = \mathscr{g}_{\mathcal{A}4} = \frac{1+(|\mu^-_{\mathcal{A}}-\nu^-_{\mathcal{A}}|\vee|\mu^+_{\mathcal{A}}-\nu^+_{\mathcal{A}}|)^n}{2},$$
$$\mathscr{g}_{\mathcal{A}5} =\frac{1-(|\mu_{\mathcal{A}}-\nu_{\mathcal{A}}|)}{2} \quad \text{and} \quad \mathscr{g}_{\mathcal{A}6} = \frac{1+(|\mu_{\mathcal{A}}-\nu_{\mathcal{A}}|)^n}{2}.$$

Theorem 4.5. *A similarity measure $\mathcal{S}$ of the CIFSs $\mathscr{f}(\mathcal{A})$ and $\mathscr{g}(\mathcal{A})$, is an entropy measure of $\mathscr{f}(\mathcal{A})$ and $\mathscr{g}(\mathcal{A})$.*

Proof. Straightforward. □

Corollary 4.4. *A similarity measure $\mathcal{S}$ of the CIFSs $\mathscr{f}(\mathcal{A})^c$ and $\mathscr{g}(\mathcal{A})^c$, is an entropy measure of $\mathscr{f}(\mathcal{A})^c$ and $\mathscr{g}(\mathcal{A})^c$.*

Proof. Straightforward. □

Theorem 4.6. *Assume $\mathcal{S}_i, i = 1, 2, \ldots, n$ is the similarity measure of a CIFS generated by the distance d_i, then for any $\mathcal{A} \in CIFS(\mathcal{X})$,*

$$\varepsilon_i(\mathcal{A}) = 2\{\mathcal{S}_i(\mathcal{A}, (\langle[\tfrac{1}{2}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{1}{2}]\rangle, \langle\tfrac{1}{2}, \tfrac{1}{2}\rangle))\} - 1,\ i = 1, 2, \ldots, n$$

is an entropy measure for CIFSs based on the corresponding similarity measures $\mathcal{S}_i,\ i = 1, 2, \ldots, n$.

Proof. (E1) Let $\mathcal{A}$ be a crisp set,

i.e., $\mathcal{A} = (\langle[1, 1], [0, 0]\rangle, \langle 1, 0\rangle)$ or $\mathcal{A} = (\langle[0, 0], [1, 1]\rangle, \langle 0, 1\rangle)$ and

$\mathcal{S}_i$ and d_i, the similarity measures and distance measures of $\mathcal{A}$. Then we have,

$$\begin{aligned}\mathcal{S}_i\left(\mathcal{A}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right) =& 1 - \left\{d_i\left(\mathcal{A}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\} \\ =& 1 - \frac{1}{2} = \frac{1}{2}\end{aligned}$$

$$\text{Hence, } \varepsilon_i(\mathcal{A}) = 2\left(\frac{1}{2}\right) - 1 = 0$$

(E2) $\varepsilon_i(\mathcal{A}) = 2\{\mathcal{S}_i(\mathcal{A}, (\langle[\tfrac{1}{2}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{1}{2}]\rangle, \langle\tfrac{1}{2}, \tfrac{1}{2}\rangle))\} - 1 = 1$

$$\begin{aligned}&\Leftrightarrow\left\{\mathcal{S}_i\left(\mathcal{A}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\} = 1 \\ &\Leftrightarrow\mathcal{A} = \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\end{aligned}$$

(E3) If

$$\left\{d_i\left(\mathcal{A}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\} \geq \left\{d_i\left(\mathcal{B}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\},$$

then,

$$\begin{aligned}\varepsilon_i(\mathcal{A}) =& 2\left\{\mathcal{S}_i\left(\mathcal{A}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\} - 1 \\ \leq& 2\left\{\mathcal{S}_i\left(\mathcal{B}, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\} - 1 \\ =& \varepsilon_i(\mathcal{B})\end{aligned}$$

(E4) $\varepsilon_i(\mathcal{A}) = 2\{\mathcal{S}_i(\mathcal{A}, (\langle[\tfrac{1}{2}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{1}{2}]\rangle, \langle\tfrac{1}{2}, \tfrac{1}{2}\rangle))\} - 1$

$$\begin{aligned}&= 2\left\{\mathcal{S}_i\left(\mathcal{A}^c, \left(\left\langle\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2}\right]\right\rangle, \left\langle\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right)\right\} - 1 \\ &= \varepsilon_i(\mathcal{A}^c)\end{aligned}$$

Since all the conditions for entropy measures in Definition 2.3 are satisfied $\varepsilon_i(\mathcal{A})$ is an entropy measure. □

5. Cubic intuitionistic fuzzy multi-criteria decision-making problem based on similarity measure

In this section, TOPSIS approach in the CIF environment is introduced to solve multi-criteria decision-making problem, using the proposed similarity measure.

Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ be the set of m alternatives which are evaluated under a set of n criteria, denoted by $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\}$ and their preferences are given in the form of the CIFNs,

$\mathcal{A}_i = \{x_j, \langle[\mu^-_{i,j}(x_j), \mu^+_{i,j}(x_j)], [\nu^-_{i,j}(x_j), \nu^+_{i,j}(x_j)]\rangle, \langle \mu_{i,j}(x_j), \nu_{i,j}(x_j)\rangle | x_j \in X\}$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The TOPSIS method can be depicted as follows:

Step 1: Compute the CIF positive ideal sets and negative ideal sets corresponding to the alternatives $\mathcal{A}_i, i = 1, 2, \ldots, m$ by applying the following conditions:

$$\begin{aligned} A^+ =& \{\langle[\mu^-_{\mathcal{A}^+}(x_i), \mu^+_{\mathcal{A}^+}(x_i)], [\nu^-_{\mathcal{A}^+}(x_i), \nu^+_{\mathcal{A}^+}(x_i)]\rangle, \langle \mu_{\mathcal{A}^+}(x_i), \nu_{\mathcal{A}^+}(x_i)\rangle | x_i \in X\} \\ A^- =& \{\langle[\mu^-_{\mathcal{A}^-}(x_i), \mu^+_{\mathcal{A}^-}(x_i)], [\nu^-_{\mathcal{A}^-}(x_i), \nu^+_{\mathcal{A}^-}(x_i)]\rangle, \langle \mu_{\mathcal{A}^-}(x_i), \nu_{\mathcal{A}^-}(x_i)\rangle | x_i \in X\} \end{aligned}$$

where, $[\mu^-_{\mathcal{A}^+}(x_i), \mu^+_{\mathcal{A}^+}(x_i)] = [\max_i \mu^-_{\mathcal{A}}(x_i), \max_i \mu^+_{\mathcal{A}}(x_i)], [\nu^-_{\mathcal{A}^+}(x_i), \nu^+_{\mathcal{A}^+}(x_i)]$ $= [\min_i \nu^-_{\mathcal{A}}(x_i), \min_i \nu^+_{\mathcal{A}}(x_i)], \mu_{\mathcal{A}^+}(x_i) = \max_i \mu_{\mathcal{A}}(x_i), \nu_{\mathcal{A}^+}(x_i) = \min_i \nu_{\mathcal{A}}(x_i)$, $[\mu^-_{\mathcal{A}^-}(x_i), \mu^+_{\mathcal{A}^-}(x_i)] = [\min_i \mu^-_{\mathcal{A}}(x_i), \min_i \mu^+_{\mathcal{A}}(x_i)], [\nu^-_{\mathcal{A}^-}(x_i), \nu^+_{\mathcal{A}^-}(x_i)] = [\max_i \nu^-_{\mathcal{A}}(x_i)$, $\max_i \nu^+_{\mathcal{A}}(x_i)], \mu_{\mathcal{A}^-}(x_i) = \min_i \mu_{\mathcal{A}}(x_i)$ and $\nu_{\mathcal{A}^-}(x_i) = \max_i \nu_{\mathcal{A}}(x_i)$.

Step 2: Calculate the similarity measure between the positive ideal CIFS $\mathcal{A}^+$ and the alternatives $\mathcal{A}_i$, as well as the similarity measure between the negative ideal CIFS $\mathcal{A}^-$ and the alternatives $\mathcal{A}_i$, respectively as follows:

$$\mathcal{S}(\mathcal{A}^+, \mathcal{A}_i) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{6-2\min\left\{\mu^-_{\mathcal{A}^+\mathcal{A}_i}, \nu^-_{\mathcal{A}^+\mathcal{A}_i}\right\}-2\min\left\{\mu^+_{\mathcal{A}^+\mathcal{A}_i}, \nu^+_{\mathcal{A}^+\mathcal{A}_i}\right\}-2\min\left\{\mu_{\mathcal{A}^+\mathcal{A}_i}, \nu_{\mathcal{A}^+\mathcal{A}_i}\right\}}{6+\max\left\{\mu^-_{\mathcal{A}^+\mathcal{A}_i}, \nu^-_{\mathcal{A}^+\mathcal{A}_i}\right\}+\max\left\{\mu^+_{\mathcal{A}^+\mathcal{A}_i}, \nu^+_{\mathcal{A}^+\mathcal{A}_i}\right\}+\max\left\{\mu_{\mathcal{A}^+\mathcal{A}_i}, \nu_{\mathcal{A}^+\mathcal{A}_i}\right\}}\right]$$

where, $\mu^-_{\mathcal{A}^+\mathcal{A}_i} = |\mu^-_{\mathcal{A}^+} - \mu^-_{\mathcal{A}_i}|, \mu^+_{\mathcal{A}^+\mathcal{A}_i} = |\mu^+_{\mathcal{A}^+} - \mu^+_{\mathcal{A}_i}|, \nu^-_{\mathcal{A}^+\mathcal{A}_i} = |\nu^-_{\mathcal{A}^+} - \nu^-_{\mathcal{A}_i}|$, $\nu^+_{\mathcal{A}^+\mathcal{A}_i} = |\nu^+_{\mathcal{A}^+} - \nu^+_{\mathcal{A}_i}|, \mu_{\mathcal{A}^+\mathcal{A}_i} = |\mu_{\mathcal{A}^+} - \mu_{\mathcal{A}_i}|$ and $\nu_{\mathcal{A}^+\mathcal{A}_i} = |\nu_{\mathcal{A}^+} - \nu_{\mathcal{A}_i}|$, and

$$\mathcal{S}(\mathcal{A}^-, \mathcal{A}_i) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{6-2\min\left\{\mu^-_{\mathcal{A}^-\mathcal{A}_i}, \nu^-_{\mathcal{A}^-\mathcal{A}_i}\right\}-2\min\left\{\mu^+_{\mathcal{A}^-\mathcal{A}_i}, \nu^+_{\mathcal{A}^-\mathcal{A}_i}\right\}-2\min\left\{\mu_{\mathcal{A}^-\mathcal{A}_i}, \nu_{\mathcal{A}^-\mathcal{A}_i}\right\}}{6+\max\left\{\mu^-_{\mathcal{A}^-\mathcal{A}_i}, \nu^-_{\mathcal{A}^-\mathcal{A}_i}\right\}+\max\left\{\mu^+_{\mathcal{A}^-\mathcal{A}_i}, \nu^+_{\mathcal{A}^-\mathcal{A}_i}\right\}+\max\left\{\mu_{\mathcal{A}^-\mathcal{A}_i}, \nu_{\mathcal{A}^-\mathcal{A}_i}\right\}}\right]$$

where $\mu^-_{\mathcal{A}^-\mathcal{A}_i} = |\mu^-_{\mathcal{A}^-} - \mu^-_{\mathcal{A}_i}|, \mu^+_{\mathcal{A}^-\mathcal{A}_i} = |\mu^+_{\mathcal{A}^-} - \mu^+_{\mathcal{A}_i}|, \nu^-_{\mathcal{A}^-\mathcal{A}_i} = |\nu^-_{\mathcal{A}^-} - \nu^-_{\mathcal{A}_i}|, \nu^+_{\mathcal{A}^-\mathcal{A}_i} = |\nu^+_{\mathcal{A}^-} - \nu^+_{\mathcal{A}_i}|, \mu_{\mathcal{A}^-\mathcal{A}_i} = |\mu_{\mathcal{A}^-} - \mu_{\mathcal{A}_i}|$ and $\nu_{\mathcal{A}^-\mathcal{A}_i} = |\nu_{\mathcal{A}^-} - \nu_{\mathcal{A}_i}|$.

Step 3: Determine the relative similarity measure corresponding to the alternatives $\mathcal{A}_i$:

$$\mathscr{S}_i = \frac{\mathcal{S}(\mathcal{A}^+, \mathcal{A}_i)}{\mathcal{S}(\mathcal{A}^+, \mathcal{A}_i) + \mathcal{S}(\mathcal{A}^-, \mathcal{A}_i)}, i = 1, 2, \ldots, n$$

Step 4: Ranking the alternatives by selecting the one with the largest value, say $\mathscr{S}_k, k = 1, 2, 3, 4$, among the alternatives $\mathscr{S}_i, i = 1, 2, \ldots, m$. Consequently, the alternative $\mathscr{S}_k, k = 1, 2, 3, 4$ is determined to be the optimal choice.

Now consider the following example in which the newly introduced similarity measure is applied.

Example 5.1. *This example is adopted from Nayagam et al., 2011* [22] *and Talukdar and Dutta, 2023* [31].

Assume that there exists a panel with four possible alternatives for investment purpose. $\mathcal{A}_1$ is a car company, $\mathcal{A}_2$ is a food company, $\mathcal{A}_3$ is a computer company and $\mathcal{A}_4$ is an arms *company. This investment entity must make a decision according to the following criteria:* $\mathcal{C}_1$ *(risk),* $\mathcal{C}_2$ *(growth) and* $\mathcal{C}_3$ *(environment impact).*

The evaluation of these alternatives is made using the CIFN by the decision-maker under the above three general characteristics. The CIF decision matrix is given in Table 1.

Table 1. The alternatives in terms of CIFNs

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{A}_1$	$\langle[0.4,0.5],[0.3,0.4]\rangle$, $\langle 0.5,0.4\rangle$	$\langle[0.4,0.6],[0.2,0.4]\rangle$, $\langle 0.6,0.4\rangle$	$\langle[0.1,0.3],[0.5,0.6]\rangle$, $\langle 0.3,0.6\rangle$
$\mathcal{A}_2$	$\langle[0.6,0.7],[0.2,0.3]\rangle$, $\langle 0.7,0.3\rangle$	$\langle[0.6,0.7],[0.2,0.3]\rangle$, $\langle 0.7,0.3\rangle$	$\langle[0.4,0.8],[0.1,0.2]\rangle$, $\langle 0.8,0.2\rangle$
$\mathcal{A}_3$	$\langle[0.3,0.6],[0.3,0.4]\rangle$, $\langle 0.6,0.4\rangle$	$\langle[0.5,0.6],[0.3,0.4]\rangle$, $\langle 0.6,0.4\rangle$	$\langle[0.4,0.5],[0.1,0.3]\rangle$, $\langle 0.5,0.3\rangle$
$\mathcal{A}_4$	$\langle[0.7,0.8],[0.1,0.2]\rangle$, $\langle 0.8,0.2\rangle$	$\langle[0.6,0.7],[0.1,0.3]\rangle$, $\langle 0.7,0.3\rangle$	$\langle[0.3,0.4],[0.1,0.2]\rangle$, $\langle 0.4,0.2\rangle$

Note: *Table values taken from Nayagam et al., 2011* [22] *and Talukdar and Dutta, 2023* [31].

The applicability of the existing similarity measures on IVIFSs is presented in Table 2.

Table 2. Similarity values for IVIFSs

Existing Approaches		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	***Ranking order***
IVIFSs	*Wei et al., 2011* [35]	*0.3886*	*0.5868*	*0.4748*	*0.5909*	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
	Sun and Liu, 2012 [30]	*0.4616*	*0.5283*	*0.4893*	*0.5296*	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
	Meng and Chen, 2015 [20]	*0.4422*	*0.5453*	*0.4884*	*0.5465*	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$

To apply the proposed similarity measures on CIFSs using TOPSIS method, the steps to be followed are given below:

Step 1: *Compute the positive ideal CIFS and the negative ideal CIFS for the alternatives* $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$*, as outlined in the table below:*

Table 3. Positive and negative ideal CIFSs

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{A}^+$	$\langle[0.7,0.8],[0.1,0.2]\rangle$, $\langle 0.8,0.2\rangle$	$\langle[0.6,0.7],[0.1,0.3]\rangle$, $\langle 0.7,0.3\rangle$	$\langle[0.4,0.8],[0.1,0.2]\rangle$, $\langle 0.8,0.2\rangle$
$\mathcal{A}^-$	$\langle[0.3,0.5],[0.3,0.4]\rangle$, $\langle 0.5,0.4\rangle$	$\langle[0.4,0.6],[0.3,0.4]\rangle$, $\langle 0.6,0.4\rangle$	$\langle[0.1,0.3],[0.5,0.6]\rangle$, $\langle 0.3,0.6\rangle$

Step 2: *The degree of similarity between the positive ideal CIFS* $\mathcal{A}^+$ *and alternatives* $\mathcal{A}_i$*, as well as between the negative ideal CIFS* $\mathcal{A}^-$ *and alternatives* $\mathcal{A}_i$*, are calculated and they are presented in Table 4.*

Step 3: *The relative similarity measures* $\mathcal{S}_k,\ k = 1, 2, 3, 4$ *with respect to* $\mathcal{A}^+$ *and* $\mathcal{A}^-$ *are calculated and are given in Table 5.*

TABLE 4. Degree of similarity measures

	$\mathcal{A}_1$		$\mathcal{A}_2$		$\mathcal{A}_3$		$\mathcal{A}_4$	
	${\mathcal{A}_1}^+$	$\mathcal{A}_1^-$	$\mathcal{A}_2^+$	$\mathcal{A}_2^-$	$\mathcal{A}_3^+$	$\mathcal{A}_3^-$	$\mathcal{A}_4^+$	$\mathcal{A}_4^-$
$\mathcal{S}$	*0.5132*	*0.7418*	*0.7102*	*0.5408*	*0.5995*	*0.6521*	*0.7174*	*0.5551*

TABLE 5. Relative similarity measures

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$	***Ranking order***
$\mathcal{S}$	*0.4089*	*0.5677*	*0.4790*	*0.5638*	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$

5.1. **Results and Discussion.** These results indicate that, IVIFSs provide an incorrect ranking order. In contrast, the similarity measures on CIFSs identifies $\mathcal{A}_2$ (food company) as the better alternative with a stable ranking order, which coincides with Nayagam et al., 2011 [22] and Talukdar and Dutta, 2023 [31].

6. CONCLUSION

Entropy and similarity measures have received significant attention over the last few decades and are crucial tools in image processing, decision-making, pattern recognition, medical diagnosis, neural networks, clustering problems, and data mining applications. In this article, we introduced the concept of similarity measures on CIFSs. In certain situations entropy measures cannot be directly applied, they have to be converted into similarity measures and then applied. To address this, we have developed the relationship between similarity and entropy measures in CIF environment. To validate the effectiveness of the proposed similarity measure, it is applied to a multi-criteria decision-making problem. Also, a comparative study is presented between IVIFSs, and CIFSs. This study addressed the decision-making problem using only secondary data, which is a limitation of the work.

In future studies, we aim to define additional similarity measures based on set-theoretic approach, geometric distance models, and matching functions. Furthermore, we plan to explore applications of similarity measures on CIFSs in image recognition.

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