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SOLVING HIGHER-ORDER FUZZY DIFFERENTIAL EQUATIONS VIA LAPLACE RESIDUAL POWER SERIES APPROACH

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Abstract. In this paper, we have applied a new method called Laplace residual power series method (LRPSM) introduced by (T. Eriqate et. al.) for solving n-order fuzzy linear differential equations. The algorithm gains powerful results for this kind of problem. Initially, we analyze in general the Laplace residual power series technique, and then we expand it and use it to solve 2-order and 4-order fuzzy linear differential equations with extended Hukuhara differentiability as a two examples to demonstrate efficiency and accuracy of the approch.

Keywords: Laplace residual power series method, fuzzy Laplace transform, Generalized Hukuhara differentiability, fuzzy differential equations.

AMS Subject Classification: 34A07, 39A26.

1. Introduction

The concept of FDEs has received extensive focus in current history and has been quickly evolving. Multiple scientists thoroughly researched it (see [13, 18, 22]). Other academics have created certain numerical approaches and algorithms for solving these types of problems (see [1, 5, 12, 20]).

The residual power series (RPS) approach is a numerical analytical tool used to handle partial, ordinary and fuzzy differential equations, as well as fractional-order integro-differential equations. It is a useful optimization approach because it provides solutions in a mathematical expression of known functions. The RPS approach has effectively solved a variety of Integrodifferential equations, fuzzy FDEs, and FDEs. Furthermore, the RPS technique enables us to design the precise solution of initial value issues with polynomials results. The RPS method analyzes several mathematical frameworks, covering differential equations that include the framework of vibration model of big membranes, fractional Black-Scholes option pricing equations, and temporal fractional fuzzy vibration equation of large membranes. For instance, you can see [2–4].

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The fuzzy Laplace transform technique is an effective solving mechanism many significant models that are emerging in numerous sectors of sciences. Combining analytical methodologies with the fuzzy Laplace transform operator increases accuracy while taking less time to solve non-linear problems. For instance [9].

Eriqat et al. [17] developed the LRPSM to deal with certain types of fractional differential equations through the integration of the LT and the RPSM. This novel approach promises to be more simple than RPSM for generating precise and estimated solutions to linear and nonlinear FDEs, such as neutral FDEs [17] and high-order FPDEs (see [11, 15–17, 19, 21, 23–26]).

The major purpose of this work is to investigate analytic and approximation solutions for nonlinear and linear FDE phenomena utilising the Laplace residual power series (LRPSM) technique. The LRPSM process combines the Fuzzy Laplace transform strategy and the RPS mechanism to provide precise and approximated results as faster power series (PS) solutions while turning the core dilemma to Fuzzy Laplace area and coming up with solutions for the novel equations, and eventually the solution to the key concern could be acquired by fuzzy Laplace inverse of the study findings. In contrast to the RPS approach, which relies on the derivative and may take time to compute the numerous derivatives in phases of discovering solutions, the unknown factors in a new fuzzy Laplace expression may be calculated by using limit principle. The LRPSM approach has modest computing needs that need less time and more precision.

This research is divided into different sections: Section 2 contains some essential fundamental conclusions for fuzzy numbers, Hukuhara difference, extended Hukuhara differentiability, and Fuzzy Laplace transform. Section 3 describes the proposed approach for getting solutions for FDE systems. Section 4 investigates the simplicity and application of the LRPSM technique by solving two systems of FDEs. Finally, in Section 5, we make conclusions about our findings.

2. Preliminaries

This part covers the fundamentals of fuzzy theory. In addition, we discuss the Fuzzy Laplace transform and its relationship to generalized Hukuhara differentiability.

Assume that E represents the set of all fuzzy numbers on $\mathbb{R}$.

Definition 2.1. *[28] $\pi : \mathbb{R} \to [0,1]$ A fuzzy membership function is said to as a fuzzy number if and only if the very next cases are met:*

(1) *π, is normal. This indicates that there's a x_0 such as $u(x_0) = 1$.*
(2) *π, is fuzzy convex.*
(3) *π, is upper semi-continuous.*
(4) $\text{Supp}(\pi) = \{x \in R \mid \pi(x) > 0\}$ *is a compact set as a support set.*

If π is a fuzzy number on $\mathbb{R}$, therefore, the r-cut of π is $[\pi]^r = \{s \in \mathbb{R} \mid \pi(s) \geq r\}$, for $r \in (0,1]$.

Since $[\pi]^r$ is a compact set of all $r \in [0,1]$, then we can represent $[\pi]^r$ by $[\underline{\pi}(r), \overline{\pi}(r)]$.

Definition 2.2. *[28] Assume that Φ and Ψ are two level-wise fuzzy numbers. The Hukuhara difference $\Phi \ominus \Psi$ is defined as follows:*

$$\exists \Omega; \quad \Phi \ominus \Psi = \Omega \Leftrightarrow \Phi = \Psi \oplus w \tag{1}$$

The presence of the difference is clearly reliant on the existence of the fuzzy value Ω.

Definition 2.3. *A function $f : (a,b) \to E$ is what we call it strongly generalized differentiable of the n-order at $x_0 \in (a,b)$; if there's a component $f^n(x_0) \in E$ such as*

(i) $\forall h > 0$ *that are adequately little, there's a* $f^{n-1}(x_0+h) \ominus f^{n-1}(x_0), f^{n-1}(x_0) \ominus f^{n-1}(x_0-h)$, *as well as the limitations*

$$\lim_{h\to 0^+} \frac{f^{n-1}(x_0+h) \ominus f^{n-1}(x_0)}{h} = \lim_{h\to 0^+} \frac{f^{n-1}(x_0) \ominus f^{n-1}(x_0-h)}{h} = f^n(x_0)$$

or

(ii) $\forall h > 0$ *that are adequately little, there's a* $f^{n-1}(x_0) \ominus f^{n-1}(x_0+h), f^{n-1}(x_0-h) \ominus f^{n-1}(x_0)$, *as well as the limitations*

$$\lim_{h\to 0^+} \frac{f^{n-1}(x_0) \ominus f^{n-1}(x_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{f^{n-1}(x_0-h) \ominus f^{n-1}(x_0)}{(-h)} = f^n(x_0)$$

or

(iii) $\forall h > 0$ *that are adequately little, there's a* $f^{n-1}(x_0+h)\ominus f^{n-1}(x_0), f^{n-1}(x_0-h)\ominus f^{n-1}(x_0)$, *as well as the limitations*

$$\lim_{h\to 0^+} \frac{f^{n-1}(x_0+h) \ominus f^{n-1}(x_0)}{h} = \lim_{h\to 0^+} \frac{f^{n-1}(x_0-h) \ominus f^{n-1}(x_0)}{(-h)} = f^n(x_0)$$

or

(iv) $\forall h > 0$ *that are adequately little, there's a* $f^{n-1}(x_0) \ominus f^{n-1}(x_0+h), f^{n-1}(x_0) \ominus f^{n-1}(x_0-h)$, *as well as the limitations*

$$\lim_{h\to 0^+} \frac{f^{n-1}(x_0) \ominus f^{n-1}(x_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{f^{n-1}(x_0) \ominus f^{n-1}(x_0-h)}{h} = f^n(x_0).$$

Definition 2.4. *[27] Any fuzzy value function f have exponential scale p if there are values $M > 0$ as well as p so for $t_0 \geq 0, |f(t)| \leq Me^{pt} \cdot \tilde{1}, t \geq t_0$.*

Definition 2.5. *(see [6]) Allow χ to be a continuous fuzzy value function. Assume that $e^{-px}\chi(x)$ is fuzzy Riemann improper integrable on $[0,\infty[$, so $\int_0^\infty e^{-px} \odot \chi(x)dx$ is known as the χ fuzzy Laplace transform and is identified as*

$$\mathbf{L}[\chi(x)] = \int_0^\infty e^{-px} \odot \chi(x)dx, \quad p > 0$$

Indicate by $\mathcal{L}(\chi)$ the classic Laplace transform of a crisp function χ. As such

$$\int_0^\infty e^{-px}\chi(x)dx = \left(\int_0^\infty e^{-px}\chi(x,r)dx, \int_0^\infty e^{-px}\bar{\chi}(x,r)dx\right),$$

therefore

$$\mathbf{L}[\chi(x)] = (\mathcal{L}(\underline{\chi}(x,r)), \mathcal{L}(\underline{\chi}(x,r))).$$

Theorem 2.1. *[14] Assume that $u(t), u'(t), \ldots, u^{(n-1)}(t)$ are continuous fuzzy value functions on $[0,\infty)$ and of exponential order and that $u^{(n)}(t)$ is piece-wise continuous fuzzy value function on $[0,\infty)$. Allow $u^{(i_1)}(t), u^{(i_2)}(t), \ldots, u^{(i_m)}(t)$ to be (ii)-differentiable functions for $0 \leq i_1 < i_2 < \ldots < i_m \leq n-1$ and $u^{(p)}$ be (i)-differentiable function for $p \neq i_j, j = 1, 2, \ldots, m$ and $u(t) = (\underline{u}(t,\alpha), \bar{u}(t,\alpha))$; then*

(1) *If m is an even number, we obtain*

$$\mathbf{L}\left(u^{(n)}(t)\right) = s^n\mathbf{L}(u(t)) \ominus s^{n-1}u(0) \otimes \sum_{k=1}^{n-1} s^{n-(k+1)}u^{(k)}(0), \tag{2}$$

such as

$$\otimes = \begin{cases} \ominus, & \text{if the number of } (ii) - \text{ differentiable functions } g^{(i)}, \text{ supplied that } i < k \text{ is an even number} \\ -, & \text{if the number of } (ii) - \text{ differentiable functions } g^{(i)}, \text{ supplied that } i < k \text{ is an odd number} \end{cases}$$

(2) *If m is an odd number, we get*

$$\mathbf{L}\left(u^{(n)}(t)\right) = -s^{n-1}u(0) \ominus (-s^n)\,\mathbf{L}(u(t)) \otimes \sum_{k=1}^{n-1} s^{n-(k+1)}u^{(k)}(0), \tag{3}$$

such as

$$\otimes = \begin{cases} \ominus, & \text{if the number of (ii) } - \text{ differentiable functions } g^{(i)}, \text{ supplied that i} < k \text{ is an odd number} \\ -, & \text{if the number of (ii) } - \text{ differentiable functions } g^{(i)}, \text{ supplied that i} < k \text{ is an even number} \end{cases}$$

3. Analysis of Laplace residual power series method

In this part, we provide the algorithm for solving FDEs using the Fuzzy Laplace residual power series approach. Consider the FDE of the next structure:

$$u^{(\alpha)}(t) = f(t, u(t), {}^{C}_{gH}D_t^{\alpha-1}u(t)), \quad t > 0, \tag{4}$$

subordinate to the preceding condition:

$$u(0) = u_0 = (\underline{u}_0, \bar{u}_0) \in E, \tag{5}$$

where $u(t) = (\underline{u}(t,r), \bar{u}(t,r))$ is a fuzzy function and $f\left(t, u(t), {}^{C}_{gH}D_t^{\alpha-1}u(t)\right)$ is a fuzzy value function, which is linear in regard to $\left(u(t), {}^{C}_{gH}D_t^{\alpha-1}u(t)\right)$.

To initiate adopting FLRPSM, we must first execute the Fuzzy Laplace transform to Eq. (4),

$$\mathbf{L}\left[{}^{C}_{gH}D_t^{\alpha}u(t)\right] = \mathbf{L}\left(f(t, u(t), {}^{C}_{gH}D_t^{\alpha-1}u(t))\right). \tag{6}$$

Assume that m is an even number, then according to 2 we have

$$\mathbf{L}\left({}^{C}_{gH}D_t^{\alpha}u(t)\right) = s^{\alpha}U(s) \ominus s^{\alpha-1}u(0) \otimes \sum_{n=1}^{\alpha-1} s^{\alpha-(n+1)}u^{(n)}(0),$$

due to the starting condition (5), Equation (6) may be described as

$$U(s) = \frac{1}{s} \odot u_0 \oplus \frac{1}{s^{\alpha}} \odot \sum_{n=1}^{\alpha-1} s^{\alpha-(n+1)}u^{(n)}(0) \oplus \frac{1}{s^{\alpha}} \odot F(s), \tag{7}$$

where $U(s) = \mathbf{L}[u(t)], \quad F(s) = \mathbf{L}\left(f(t, u(t), {}^{C}_{gH}D_t^{\alpha-1}u(t))\right)$.

As a consequence, we presume that $U(s)$, the fuzzy converted function, has the series representation shown below.

$$U(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n\alpha+1}}, \tag{8}$$

and the form of the k-th Fuzzy truncated series of the Eq. (8) has the form

$$U_k(s) = \sum_{n=0}^{k} \frac{a_n}{s^{n\alpha+1}} = \frac{a_0}{s} \oplus \sum_{n=1}^{k} \frac{a_n}{s^{n\alpha+1}}. \tag{9}$$

The Fuzzy Laplace residual function of Eq. (7) is now computed as

$$\mathcal{L}\,\mathrm{Res}(s) = U(s) \ominus \frac{1}{s} \odot u_0 \ominus \frac{1}{s^\alpha} \odot \sum_{n=1}^{\alpha-1} s^{\alpha-(n+1)} u^{(n)}(0) \ominus \frac{1}{s^\alpha} \odot F(s), \tag{10}$$

and the k-th Fuzzy Laplace residual function of Eq. (10) is provided by

$$\mathcal{L}\,\mathrm{Res}_k(s) = U_k(s) \ominus \frac{1}{s} \odot u_0 \ominus \frac{1}{s^\alpha} \odot \sum_{n=1}^{\alpha-1} s^{\alpha-(n+1)} u^{(n)}(0) \ominus \frac{1}{s^\alpha} \odot F(s), k = 1, 2, \cdots \tag{11}$$

Interestingly, as in [7, 8, 10], the following outcomes are critical in determining FPS solutions.

$$\mathcal{L}\,\mathrm{Res}(s) = 0 \text{ and } \lim_{n\to\infty} \mathcal{L}\,\mathrm{Res}_n(s) = \mathcal{L}\,\mathrm{Res}(s) \text{ for all } s > 0$$

$$\lim_{s\to\infty} s \odot \mathcal{L}\,\mathrm{Res}(s) = 0 \text{ where it reveals } \lim_{s\to\infty} s \odot \mathcal{L}\,\mathrm{Res}_n(s) = 0.$$

$$\lim_{s\to\infty} s^{n\alpha+1} \mathcal{L}\,\mathrm{Res}(s) = \lim_{s\to\infty} s^{n\alpha+1} \mathcal{L}\,\mathrm{Res}_n(s) = 0, n = 1, 2, \cdots .$$

Furthermore, in order to derive the factors a_n, we should first resolve the preceding system's recurrence relations.

$$\lim_{s\to\infty} s^{k\alpha+1} \odot \mathcal{L}\,\mathrm{Res}_k(s) = 0, k = 1, 2, \cdots .$$

Therefore, we take the a_n findings and plug them into the series expression (9) to establish the form of the k-th Laplace series solution, $U_k(s)$. Ultimately, we use the fuzzy inverse Laplace transform on the form of $U_k(s)$ to obtain the k-th LFPS approximation solution to the first issue (4).

4. Applications of LRPSM

The proposed approach is used to solve two interesting problems in this part. The study demonstrates that FLRPSM is scalable and has the potential to be an effective approach for solving FDEs.

Example 4.1.

$$\begin{cases} x^{(2)}(t) = \sigma_0, \quad \sigma_0 \in E, \\ x(0) = x_0, \quad x_0 \in E, \\ x'(0) = \tilde{0} \in E. \end{cases} \tag{12}$$

In light of the previous description of the Fuzzy Laplace RPS strategy, we begin through using the Fuzzy Laplace transform to Eq. (12), which generates the following system utilizing the starting data:

$$X(s) = \frac{1}{s} \odot x_0 \oplus \frac{1}{s^3} \odot \sigma_0, \tag{13}$$

Presume that the system's Fuzzy Laplace series solutions (13) adopt the pursuing type:

$$X(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{2n+1}}, \quad s > 0, \tag{14}$$

It is certain that $\lim_{s\to\infty} sX(s) = a_0 = x(0) = x_0$. *As a corollary, the k-th Fuzzy Laplace series solutions of (13) will be stated as follows:*

$$X_k(s) = \frac{x_0}{s} \oplus \sum_{n=1}^{k} \frac{a_n}{s^{2n+1}}. \tag{15}$$

where the undetermined factors a_n may be identified for $n = 1, 2, 3, \cdots$ by generating the k-th Fuzzy Laplace residual functions $\mathbf{L}\,\mathrm{Res}^k\,(X(s))$ of system (13) in such a way that

$$\mathbf{L}\,\mathrm{Res}^k\,(X(s)) = X_k(s) \ominus \frac{1}{s} \odot x_0 \ominus \frac{1}{s^3} \odot \sigma_0, \tag{16}$$

For the first-Laplace series solutions, take $k = 1$ in the Fuzzy Laplace residual Eq. (16).

$$\mathbf{L}\,\mathrm{Res}^1\,(X(s)) = \frac{x_0}{s} \oplus \frac{a_1}{s^3} \ominus \frac{1}{s} \odot x_0 \ominus \frac{1}{s^3} \odot \sigma_0, \tag{17}$$

Thus, by multiplying either parts of the resultant equation (17) by s^3, one may obtain:

$$s^3\mathbf{L}\,\mathrm{Res}^1\,(X(s)) = a_1 \ominus \sigma_0. \tag{18}$$

Following that, by resolving, $\lim_{s\to\infty} s^3\mathbf{L}\,\mathrm{Res}^1\,(X(s)) = 0$, *one could earn $a_1 = \sigma_0$. As an end, the system's 1st-Laplace series solutions (13) will be stated as:*

$$X_1(s) = \frac{x_0}{s} \oplus \frac{\sigma_0}{s^3}.$$

Similarly, to determine the shape of the Eq's 2nd-Fuzzy Laplace series solutions (13), we need

$$\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = \frac{x_0}{s} \oplus \frac{\sigma_0}{s^3} \oplus \frac{a_2}{s^5} \ominus \frac{1}{s} \odot x_0 \ominus \frac{1}{s^3} \odot \sigma_0, \tag{19}$$

After that, multiply either sides of $\mathcal{L}\,\mathrm{Res}^2\,(Y(s))$ by s^5 to get

$$s^5\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = a_2, \tag{20}$$

Finally, in order to acquire the constant, a_2, one able to solve, $\lim_{s\to\infty} s^5\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = 0$, *in order to confirm that $a_2 = 0$ Furthermore, the system's 2nd-Fuzzy Laplace series solutions (13) may be expressed as*

$$X_2(s) = \frac{x_0}{s} \oplus \frac{\sigma_0}{s^3}. \tag{21}$$

As a result, by adopting the fuzzy inverse Laplace over sides of the resultant expression of (21), the Fuzzy Laplace PS solutions for the system FDE's (12) may be written as the very next expansion:

$$x_2(t) = x_0 \oplus \frac{t^2}{2}\sigma_0. \tag{22}$$

Example 4.2.

$$\begin{cases} x^{(4)}(t) = x(t), \\ x(0) = x_0, \quad x_0 \in E, \\ x'(0) = x''(0) = x'''(0) = \tilde{0} \in E. \end{cases} \tag{23}$$

We urge m to be an odd integer, and applying 3 on the system (23) and the beginning information, we get the following system:

$$X(s) = \frac{-s^3}{(1 - s^4)}x_0, \tag{24}$$

Suppose that the system's Fuzzy Laplace series solutions (13) adopt the following structure:

$$X(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{2n+1}}, \quad s > 0, \tag{25}$$

It is obvious that $\lim s \to \infty sX(s) = a0 = x(0) = x0$. *As an outcome, the k-th Fuzzy Laplace series solutions of the Eq. (24) will be displayed as follows:*

$$X_k(s) = \frac{x_0}{s} \oplus \sum_{n=1}^{k} \frac{a_n}{s^{2n+1}}. \tag{26}$$

where the missing factors an may be recovered for $n = 1, 2, 3, \cdots$ *by creating the k-th Fuzzy Laplace residual functions* $\mathbf{L}\,\mathrm{Res}^k\,(X(s))$ *of system (24) in such a way that*

$$\mathbf{L}\,\mathrm{Res}^k\,(X(s)) = X_k(s) \ominus \frac{-s^3}{(1-s^4)}x_0, \tag{27}$$

Considering $k = 1$ *in the Fuzzy Laplace residual Eq. (27) to derive the first-Laplace series solutions.*

$$\mathbf{L}\,\mathrm{Res}^1\,(X(s)) = \frac{x_0}{s} \oplus \frac{a_1}{s^3} \ominus \frac{-s^3}{(1-s^4)}x_0, \tag{28}$$

However, multiplying either parts of the resultant Eq. (28) by $s3$ *yields:*

$$s^3\mathbf{L}\,\mathrm{Res}^1\,(X(s)) = a_1 \oplus \frac{s^2}{1-s^4}x_0. \tag{29}$$

Following that, by finding, $\lim_{s\to\infty} s^3\mathbf{L}\,\mathrm{Res}^1\,(X(s)) = 0$, *one could attain* $a_1 = 0$. *As an end, the system's first-Laplace series solutions (24) will be given as:*

$$X_1(s) = \frac{x_0}{s}.$$

Likewise, to discover the shape of the system's 2nd-Fuzzy Laplace series solutions (24), we require

$$\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = \frac{x_0}{s} \oplus \frac{a_2}{s^5} \ominus \frac{-s^3}{(1-s^4)}x_0, \tag{30}$$

After that, multiply each sides of $\mathcal{L}\,\mathrm{Res}^2\,(Y(s))$ *by* s^5 *to get*

$$s^5\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = a_2 \oplus \frac{s^4}{(1-s^4)}x_0, \tag{31}$$

Lastly, in order to acquire the numbers a_2, *one might solve,*
$\lim_{s\to\infty} s^5\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = 0$, *to state that* $a_2 = x_0$ *As a conclusion, the system's 2nd-Fuzzy Laplace series solutions (24) may be presented as*

$$X_2(s) = \frac{x_0}{s} \oplus \frac{x_0}{s^5}. \tag{32}$$

Similarly, to produce the 3rd-Laplace series solutions of (24), insert $X3(s)$ *of (26) into the 3rd-Laplace residual functions of (27), and afterwards multiply the resulting equation.*

$$s^7\mathbf{L}\,\mathrm{Res}^3\,(X(s)) = a_3 \oplus \frac{s^2}{(1-s^4)}x_0, \tag{33}$$

by $\lim_{s\to\infty} s^7\mathbf{L}\,\mathrm{Res}^2\,(X(s)) = 0$, *we get* $a_3 = 0$ *As a result, the system's 2nd-Fuzzy Laplace series solutions (24) can be phrased as*

$$X_3(s) = \frac{x_0}{s} \oplus \frac{x_0}{s^5}. \tag{34}$$

In the identical statement, for $k = 4$, *and depending on the finding* $\lim\limits_{s\to\infty} s^9\mathbf{L}\,\mathrm{Res}^4\,(X(s)) = 0$, *produces* $a_4 = x_0$. *The answers to the 4th-Lapalce series can be summarized as*

$$X_4(s) = \frac{x_0}{s} \oplus \frac{x_0}{s^5} \oplus \frac{x_0}{s^9}. \tag{35}$$

Moreover, rely on the fact $\lim_{s\to\infty} s^{2k+1}\mathbf{L}\,\mathrm{Res}^k\,(X(s)) = 0$, *for* $k = 5, 6, 7, \cdots$, *The system's Fuzzy Laplace series solutions (23) are found by*

$$X(s) = \frac{x_0}{s} \oplus \frac{x_0}{s^5} \oplus \frac{x_0}{s^9} \oplus \frac{x_0}{s^{13}} \oplus \frac{x_0}{s^{17}} \oplus \cdots \tag{36}$$

As a response, by considering the fuzzy inverse Laplace along both halves of the resultant expansion of (36), the Fuzzy Laplace PS solutions for the system FDE's (23) may be written as the resulting expansion:

$$x(t) = x_0 \oplus \frac{x_0}{\Gamma(5)}t^4 \oplus \frac{x_0}{\Gamma(9)}t^8 \oplus \frac{x_0}{\Gamma(13)}t^{12} \oplus \frac{x_0}{\Gamma(17)}t^{16} \oplus \cdots \tag{37}$$

5. Conclusion

In the present study, we used an innovative approach to solve n-order fuzzy linear differential equations termed the Laplace residual power series method (LRPSM) that was created (T. Eriqate et al.). We first examine the Laplace residual power series methodology in broad terms, next enhance it and apply it to address 2-order and 4-order fuzzy linear differential equations with extending Hukuhara differentiability as instances for showing the approch's accuracy and correctness. In future studies, we plan to investigate this methodology for Solving Nonlinear Caputo Fractional Differential Equations.

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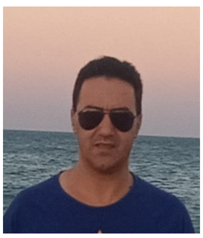

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