TWMS J. App. and Eng. Math. V.15, N.2, 2025, pp. 353-362

STRUCTURAL PROPERTIES OF SIGNED GRAPHS ADMITTING ROMAN DOMINATING FUNCTION

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Abstract. A Roman dominating function(RDF) on a signed graph $S = (G, \sigma)$ is a function $f : V(S) \to \{0, 1, 2\}$ such that

(i) $f(N[v]) = f(v) + \sum_{u \in N(v)} \sigma(uv)f(u) \geq 1$ for every vertex $v \in V(S)$ and

(ii) for any vertex v with $f(v) = 0$ there exists a vertex $u \in N^+(v)$ having $f(u) = 2$.

In this article we explore structural properties of signed graphs admitting an RDF. Further, signed graphs with 3-regular graph as their underlying graph are examined and characterisation of one of its subclasses, net-regular signed graphs admitting an RDF is obtained.

Keywords: signed graphs, roman dominating function, roman domination number.

AMS Subject Classification: 05C22, 05C69

1. Introduction

The concept of signed graphs, introduced by Harary [1, 7], has been studied extensively by researchers [14] because of its wide use in modelling socio-psychological processes. A graph in which the edges are assigned positive or negative sign is called a signed graph. Formally, a signed graph is an ordered pair $S = (G, \sigma)$ where G is called the underlying graph and σ is a function from the edge set $E(G)$ to the set $\{-1, 1\}$ known as the *signing* of G or the *signature* of S. The set of all vertices adjacent to a vertex v, denoted by $N(v)$, is known as the neighbourhood of v and $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. The positive and negative neighbourhoods of v are $N^+(v) = \{u \in N(v) | \sigma(uv) = 1\}$ and $N^-(v) = \{u \in N(v) | \sigma(uv) = -1\}$, respectively. Given $X \subseteq V(S)$, we denote the number of vertices in X with empty positive neighbourhood by $n^+_\emptyset(X)$. The degree of a vertex v is given as $d(v) = |N(v)|$. Further, $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$

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§ Manuscript received: : August 24, 2023; accepted: December 15, 2023.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.2;

are referred to as positive and negative degrees of v. The *net-degree* [2] of a vertex v is defined as $d^+(v) - d^-(v)$, denoted as $d^{\pm}(v)$. A signed graph S is *net-regular* [10, 11] if the net-degrees of all its vertices are equal and the common net-degree is denoted using $d^{\pm}(S)$. For a signed graph $S = (G, \sigma)$ and $X \subseteq V(S)$, the *subsigned graph* induced by X is the signed graph $S' = (G', \sigma')$, where G' is the subgraph of G induced by X and σ' is the restriction of σ to the edge set of G'. For definitions and notations used in this article we refer to [13].

Roman dominating functions form an interesting class of dominating functions that have been discussed in the recent literature [3, 4, 5]. The idea of Roman domination emerged from the defence scheme of Roman Emporer Constantine [12]. Cockayne et al. [6] were the first to mathematically formulate the concept of Roman dominating functions. A Roman dominating function on a graph G is a function f that maps the vertices to the set $\{0, 1, 2\}$ such that at least one neighbour u of each vertex v with $f(v) = 0$, have $f(u) = 2$.

The positive and negative signs on the edges of signed graphs can represent not only cooperative relationships but also adversarial or potentially harmful interactions. Thus, incorporating signed graph theory into network security enhances our ability to model, analyze, and respond to complex interactions and relationships within a networked environment. It offers a more nuanced understanding of the dynamics at play and contributes to developing robust and effective security strategies.

In this context Joseph and Joseph [8, 9] initiated a study on Roman dominating functions in the realm of signed graphs. A function $f : V \to \{0, 1, 2\}$ on a signed graph $S = (G, \sigma)$ is a *Roman dominating function(RDF)*, if

(i) $f(N[v]) = f(v) + \sum_{u \in N(v)} \sigma(uv) f(u) \geq 1$, for each $v \in V$ and

(ii) there is a vertex u in $N^+(v)$ with $f(u) = 2$ corresponding to each vertex v having $f(v) = 0$.

The weight of f is given by $\omega(f) = \sum_{v \in V} f(v)$ and the Roman domination number $\gamma_R(S)$ is the minimum weight among all the RDFs on S. An RDF on S with the minimum weight is known as a $\gamma_R(S)$-function. The functions $f : V \to \{0, 1, 2\}$ on a signed graph induce an ordered partition $(\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$ of the vertex set, where $\mathcal{V}_i = \{v \in V | f(v) = i\}$ for $i = 0, 1, 2$. There is always a one-one correspondence between these functions and the ordered partitions induced by them and we may write $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$.

In this article some structural properties of signed graphs admitting an RDF are examined. Further, some classes of signed graphs not admitting an RDF are identified. Several results on signed graphs whose underlying graph is 3-regular is obtained. This includes a characterisation of net-regular signed graphs having a 3-regular graph as its underlying graph.

2. Results

In any signed graph $S = (G, \sigma)$ that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$, a vertex v of $\mathcal{V}_0$ must have a neighbour u in $\mathcal{V}_2$ such that $\sigma(uv) = 1$. Therefore, when $N^+(v) = \emptyset$, $v \notin \mathcal{V}_0$.

Proposition 2.1. *Let $S = (G, \sigma)$ be a signed graph admitting an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. For every vertex v of S having $N^+(v) = \emptyset$, $N^-(v) \subseteq \mathcal{V}_0$, with the exception that one vertex may belong to $\mathcal{V}_1$ when $v \in \mathcal{V}_2$.*

Proof. Suppose that $S = (G, \sigma)$ is a signed graph that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. Let v be a vertex with $N^-(v) = \emptyset$. By the definition of RDF on signed graphs, $v \in \mathcal{V}_1 \cup \mathcal{V}_2$.

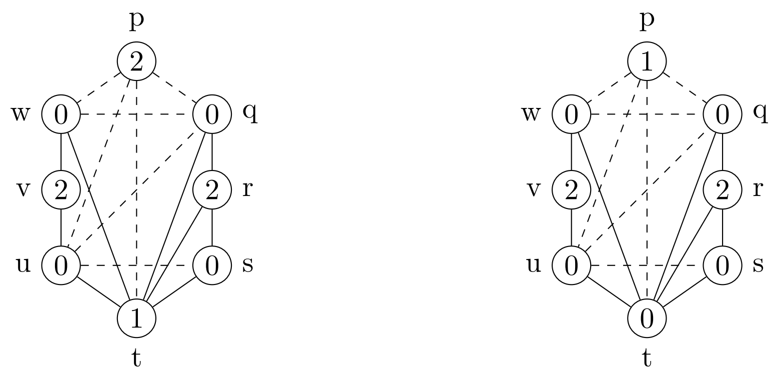
Now, observe that for any vertex $u \in N^-(v)$, $f(u) \neq 2$. For, if there exists a vertex $u \in N^-(v)$ such that $f(u) = 2$, then $f(N[v]) < 1$, which is not possible as f is an RDF. Therefore, for every vertex $u \in N^-(v)$, $u \in \mathcal{V}_0 \cup \mathcal{V}_1$.

Assume that $v \in \mathcal{V}_1$. If there exists a vertex $u \in N^-(v)$ such that $u \in \mathcal{V}_1$, then $f(N[v]) < 1$, which is a contradiction to the fact that f is an RDF. Therefore, $N^-(v) \subseteq \mathcal{V}_0$.

Now, assume that $v \in \mathcal{V}_2$. By similar arguments we get a contradiction when there is more than one vertex in $N^-(v)$ that belong to $\mathcal{V}_1$. Therefore, at most one vertex in $N^-(v)$ belong to $\mathcal{V}_1$, when $v \in \mathcal{V}_2$. □

Now, we present an example that illustrates the result given in Proposition 2.1.

Example 2.1. *Consider the signed graph S shown in Figure 1. The function f_1 defined by the ordered partition $(\{q, s, u, w\}, \{t\}, \{p, r, v\})$ is an RDF of S. Observe that $N^+(p) = \emptyset$ with $f_1(p) = 2$ and only the vertex $v \in N^-(p)$ has the value 1 under f_1. Further, f_1 is the only RDF under which p takes the value 2. Now, define the function f_2 by the ordered partition $(\{q, s, t, u, w\}, \{p\}, \{v, r\})$. Note that f_2 is an RDF with $f_2(p) = 1$ and all the vertices in $N^-(p)$ take the value 0 under f_2. Also, the only RDF with p having the value 1 is f_2.*



(A) $f_1 = (\{q, s, u, w\}, \{t\}, \{p, r, v\})$ is an RDF of the signed graph S.

(B) $f_2 = (\{q, s, t, u, w\}, \{p\}, \{v, r\})$ is an RDF of the signed graph S.

FIGURE 1. Illustration of Example 2.1

In view of Proposition 2.1 we have the following observation.

Observation 2.2. *Let $S = (G, \sigma)$ be a signed graph admitting an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. Then for any vertex v with empty positive neighbourhood none of the vertices of $N^-(v)$ have empty positive neighbourhoods. Moreover, for a vertex $w \in N^-(v)$ if $w \in \mathcal{V}_1$, then $v \in \mathcal{V}_2$. If $v \in \mathcal{V}_2$, then at least $d^-(v) - 1$ vertices in $N^-(v)$ are incident with at least two positive edges each.*

We require the following known results for further discussions.

Lemma 2.3. [8] *Signed graphs on n vertices and containing a vertex u with $d^-(u) = n-1$ do not admit an RDF.*

Lemma 2.4. [8] *Signed graphs with a pair of adjacent vertices both having empty positive neighbourhoods do not admit an RDF.*

A maximal path of a graph G such that its internal vertices have degree 2 in G is an *ear* [13]. The next two theorems give families of signed graphs not admitting an RDF.

Theorem 2.5. *Signed graphs that are not paths and whose underlying graph contains an ear $uvwx$ with vertices u, v, w, x such that $d^+(u) = d^+(x) = 0$, do not admit an RDF.*

Proof. Suppose that $S = (G, \sigma)$ is a signed graph that is not a path such that G contains an ear $uvwx$ and $d^+(u) = d^+(x) = 0$. Then there are two cases to be considered, when $\sigma(vw) = -1$ and $\sigma(vw) = 1$. In each case we show that S do not admit an RDF.

Case 1: $\sigma(vw) = -1$. In this case by Lemma 2.4 the result is immediate as u and v become adjacent vertices with empty positive neighbourhoods.

Case 2: $\sigma(vw) = 1$. We claim that S does not admit an RDF. On the contrary assume that S admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. Since $N^+(u) = N^+(x) = \emptyset$ from Proposition 2.1 it follows that $v, w \notin \mathcal{V}_2$, which implies that $v, w \notin \mathcal{V}_0$. Therefore $v, w \in \mathcal{V}_1$ and hence $u, x \in \mathcal{V}_2$, by Observation 2.2. But, then $f(N[v]) < 1$ and $f(N[w]) < 1$, which is a contradiction. Therefore, S does not admit an RDF. □

Theorem 2.6. *If any vertex v of a signed graph have $N^+(v) \neq \emptyset$ and $2d^+(v) \leq n^+_\emptyset(N^-(v))$, then it does not admit an RDF.*

Proof. Consider a signed graph $S = (G, \sigma)$ satisfying the hypothesis. We have to prove that S does not admit an RDF. If possible assume that S admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. Then $v \in \mathcal{V}_0 \cup \mathcal{V}_1$, by Proposition 2.1. Suppose that $v \in \mathcal{V}_1$. Then every vertex $w \in N^-(v)$ with $N^+(w) = \emptyset$ belong to $\mathcal{V}_2$, by Observation 2.2. Therefore, $f(N[v]) \leq 1 + 2d^+(v) - 2n^+_\emptyset(N^-(v)) < 1$, which is a contradiction. Now, suppose that $v \in \mathcal{V}_0$. Then $f(N[v]) \leq 0 + 2d^+(v) - n^+_\emptyset(N^-(v)) \leq 0$, which is not possible. Therefore, S does not admit an RDF. □

Now we explore the signed graphs containing vertices v with $d^+(v) < d^-(v)$. First we examine the family of signed trees.

Theorem 2.7. *Signed trees T with $d^+(v) < d^-(v)$ for each vertex $v \in V(T)$ do not admit an RDF.*

Proof. Suppose that T is a signed tree such that $d^+(v) < d^-(v)$ for each vertex v of T. Note that all the pendant edges of T are negative. If there exists a pair of adjacent vertices with empty positive neighbourhoods, then by Lemma 2.4, T does not admit an RDF. Now, assume that for no two adjacent vertices u and v, $N^+(u) = N^+(v) = \emptyset$. In this case, if v is any vertex with $N^+(v) = \emptyset$, then $N^+(w) \neq \emptyset$ whenever $w \in N^-(v)$. Therefore, every support vertex of T will be incident with at least one positive edge. Further, there exists at least one support vertex that is incident with exaclty one positive edge and two or more negative pendant edges. Hence by Theorem 2.6, T does not admit an RDF. □

The above result for signed trees appears to be true for any signed graph. However, the general case of signed graphs with $d^+(v) < d^-(v)$ for every vertex v requires a lengthy and detailed analysis of various possible functional values of the vertices. We present it as a conjecture.

Conjecture 2.1. *Any signed graph $S = (G, \sigma)$ with $d^+(v) < d^-(v)$ for every vertex v of S, does not admit an RDF.*

Note that any signed graph with $d^+(v) \geq d^-(v)$ for every vertex v admit an RDF as proved in the following theorem.

Theorem 2.8. *Any signed graph $S = (G, \sigma)$ such that $d^+(v) \geq d^-(v)$ for each vertex v in $V(S)$, admits an RDF.*

Proof. Consider a signed graph $S = (G, \sigma)$ with $d^+(v) \geq d^-(v)$ for each vertex v of S. Define the function $f = (\emptyset, V(S), \emptyset)$ on S. Since $\mathcal{V}_0 = \emptyset$, to show that f is an RDF on S we prove that $f(N[v]) \geq 1$ for all vertices v. Now for any vertex v of S, $f(N[v]) = 1 + d^+(v) - d^-(v)$ and hence by our assumption $f(N[v]) \geq 1$. □

The RDFs on signed graphs containing a vertex v with $d^+(v) < n^+_\emptyset(N^-(v)) < 2d^+(v)$ follows certain properties, which are discussed in the next result.

Theorem 2.9. *Let $S = (G, \sigma)$ be a signed graph that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. If S has a vertex v with $d^+(v) < n^+_\emptyset(N^-(v)) < 2d^+(v)$, then $v \in \mathcal{V}_0$. Further,*

(i) *among the vertices in $N^-(v)$ with empty positive neighbourhoods, at least three vertices belong to $\mathcal{V}_1$ and*

(ii) *in $N^+(v)$ at least one vertex belongs to $\mathcal{V}_2$ and at most $\left\lfloor \frac{d^+(v)-2}{2} \right\rfloor$ vertices belong to $\mathcal{V}_0$.*

Proof. Consider a signed graph $S = (G, \sigma)$ that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, V_2)$. Given that v is a vertex of S with $d^+(v) < n^+_\emptyset(N^-(v)) < 2d^+(v)$. Then $d^+(v) \geq 2$ and $n^+_\emptyset(N^-(v)) \geq 3$. Now, it follows that $v \in \mathcal{V}_0 \cup \mathcal{V}_1$, by Proposition 2.1. We prove that $v \in \mathcal{V}_0$. Assume that $v \in \mathcal{V}_1$. Then every neighbour w of v with $N^+(w) = \emptyset$ belongs to $\mathcal{V}_2$, by Observation 2.2. Hence, $f(N[v]) \leq 1 + 2d^+(v) - 2n^+_\emptyset(N^-(v)) < 1$, which is not possible as f is an RDF. Therefore, $v \in \mathcal{V}_0$. Now we examine the vertices in the neighbourhood of v.

Note that any vertex in $N^-(v)$ with an empty positive neighbourhood belongs to $\mathcal{V}_1 \cup \mathcal{V}_2$ by Proposition 2.1. We show that at least three vertices in $N^-(v)$ with empty positive neighbourhoods belong to $\mathcal{V}_1$. On the contrary assume that among the $n^+_\emptyset(N^-(v))$ vertices in $N^-(v)$, at most two vertices belong to $\mathcal{V}_1$. Then the remaining vertices belong to $\mathcal{V}_2$. This shows that $f(N[v]) \leq 2d^+(v) - 2n^+_\emptyset(N^-(v)) + 2 < 1$, which contradicts the fact that f is an RDF.

Now it remains to prove that at least one vertex in $N^+(v)$ belongs to $\mathcal{V}_2$ and at most $\left\lfloor \frac{d^+(v)-2}{2} \right\rfloor$ vertices in $N^+(v)$ belong to $\mathcal{V}_0$. Since $v \in \mathcal{V}_0$, $N^+(v) \cap \mathcal{V}_2$ is non-empty. Now, if possible assume that $\left\lfloor \frac{d^+(v)-2}{2} \right\rfloor + 1$ neighbours of v in $N^+(v)$ belongs to $\mathcal{V}_0$. Then $f(N[v]) \leq d^+(v) - n^+_\emptyset(N^-(v)) + 1 \leq 0$, which is not possible as f is an RDF. Therefore, it follows that at most $\left\lfloor \frac{d^+(v)-2}{2} \right\rfloor$ vertices in $N^+(v)$ belong to $\mathcal{V}_0$. □

Next we examine the signed graphs that admits an RDF when the underlying graph belong to a specific class of graphs. We begin by exploring the case when underlying graph is regular. Signed cycles admitting an RDF has been studied in [8]. Therefore, for further discussion we consider those signed graphs with a 3-regular underlying graph.

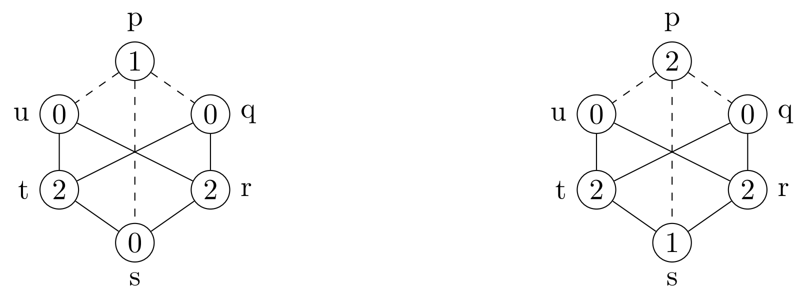
All the signed graphs considered for further discussion in this article have a 3-regular graph as underlying graph.

Proposition 2.10. *Let $S = (G, \sigma)$ be a signed graph that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$ and v be a vertex of S with an empty positive neighbourhood. If v belongs to $\mathcal{V}_2$, then at least two neighbours of v have positive degree equal to two such that their positive neighbourhoods are contained in $\mathcal{V}_1 \cup \mathcal{V}_2$.*

Proof. Suppose that $S = (G, \sigma)$ be a signed graph that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$, that has vertex v with empty positive neighbourhood. By Proposition 2.1, when $v \in \mathcal{V}_2$, at least two vertices in $N^-(v)$ belong to $\mathcal{V}_0$. Let u be a vertex in $N^-(v)$ such that $u \in \mathcal{V}_0$. By the definition of RDF on signed graphs, u has a neighbour w that belong to $\mathcal{V}_2$ such that $\sigma(uw) = 1$. Let $N(u) \setminus \{v, w\} = \{x\}$. Then $f(N[u]) = \pm f(x)$, since the underlying graph G is 3-regular. Now, observe that $f(N[u]) \geq 1$ if and only if $\sigma(ux) = 1$ and $x \in \mathcal{V}_1 \cup \mathcal{V}_2$. Therefore, if $v \in \mathcal{V}_2$ then there exists at least two vertices in $N^-(v)$ having positive degree 2 such that their neighbourhoods are in $\mathcal{V}_1 \cup \mathcal{V}_2$. $\square$

Now, we present an example to demonstrate the converse of Proposition 2.10 is not true.

Example 2.2. *In the signed graph S_1 shown in Figure 2, $N^+(p) = \emptyset$ and all the vertices in $N^-(p)$ have positive degree equal to two. The function $(\{q, s, u\}, \{p\}, \{r, t\})$ is an RDF such that $\{r, t\} \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$ whereas $p \in \mathcal{V}_1$, proving that the converse of Proposition 2.10 is not true. However, $(\{q, u\}, \{s\}, \{p, r, t\})$ is an RDF of S_1, illustrating the result of Proposition 2.10.*
On the other hand, for the signed graph S_2 depicted in Figure 3, $N^+(d) = \emptyset$ and $N^+(j) = \emptyset$ and all the neighbours of d and j have positive degree two. However, the ordered partition $(\{a, c, e, g, i, k\}, \{d, j\}, \{b, h\})$ is the only RDF on the signed graph S_2.



(A) $(\{q, s, u\}, \{p\}, \{r, t\})$ is an RDF

(B) $(\{q, u\}, \{s\}, \{p, r, t\})$ is an RDF

FIGURE 2. Signed graph S_1

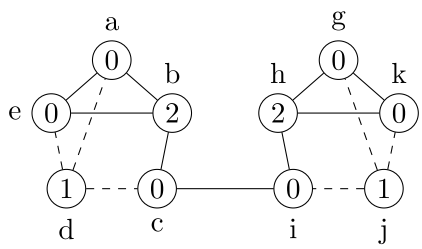


FIGURE 3. Signed graph S_2 and RDF $(\{a, c, e, g, i, k\}, \{d, j\}, \{b, h\})$

The following result is a characterisation of net-regular signed graphs admitting an RDF.

Theorem 2.11. *A net-regular signed graph $S = (G, \sigma)$ admits an RDF if and only if $d^{\pm}(S) > 0$.*

Proof. Let $S = (G, \sigma)$ be a net-regular signed graph. First we prove the sufficiency part. Assume that $d^{\pm}(S) > 0$. Then $d^+(v) > d^-(v)$ for each vertex v of S. Therefore, by Theorem 2.8, S admits an RDF.

To prove the necessary part, it suffices to show that if $d^{\pm}(S) \leq 0$, then S does not admit an RDF. Assume that $d^{\pm}(S) \leq 0$. Then there are two cases, $d^{\pm}(S) = -3$ and $d^{\pm}(S) = -1$.

Case 1: $d^{\pm}(S) = -3$. Then S is all-negative and hence does not admit an RDF.

Case 2: $d^{\pm}(S) = -1$. In this case $d^+(v) = 1$ and $d^-(v) = 2$ for every vertex v. If possible assume that S admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$. We prove that $V(S) = \mathcal{V}_1 \cup \mathcal{V}_2$. Suppose on the contrary that there is a vertex v that belongs to $\mathcal{V}_0$. Let $N^+(v) = \{u\}$. Since f is an RDF, $u \in \mathcal{V}_2$. Now, let $N^-(u) = \{x, y\}$, $N^+(x) = \{r\}$, and $N^-(x) = \{u, t\}$. Note that if both x and y belong to $\mathcal{V}_1 \cup \mathcal{V}_2$, then $f(N[u]) < 1$. Therefore, either x or y belong to $\mathcal{V}_0$. Consider the case when $x \in \mathcal{V}_0$. Then $f(N[x]) = -f(t) \leq 0$ as $r \in \mathcal{V}_2$, which is a contradiction. Similarly, whenever y is in $\mathcal{V}_0$ we arrive a contradiction. Therefore $V(S) = \mathcal{V}_1 \cup \mathcal{V}_2$.
Next we show that $\mathcal{V}_2 = \emptyset$. If possible suppose that $\mathcal{V}_2 \neq \emptyset$ and let $v \in \mathcal{V}_2$. Then $N^-(v) \cap \mathcal{V}_1$ is non-empty. For, if $N^-(v) \subseteq \mathcal{V}_2$, then $f(N[v]) < 1$. Let $u \in N^-(v) \cap \mathcal{V}_1$ and $N^-(u) = \{v, w\}$. Now $f(N[u]) \leq 1 - f(w) < 1$, as w belongs to $\mathcal{V}_1 \cup \mathcal{V}_2$. This is a contradiction as f is an RDF and hence $\mathcal{V}_2 = \emptyset$.
Thus we conclude that all the vertices of S belong to $\mathcal{V}_1$. But, then for each vertex v of S, $f(N[v]) = 1 + d^+(v) - d^-(v) = 0$, which is again a contradiction. Thus S does not admit an RDF. □

The next two results give properties of the neighbourhoods of vertices with empty positive neighbourhoods.

Lemma 2.12. *If $S = (G, \sigma)$ is a signed graph that admits an RDF, then for any pair of vertices u and v having $N^+(u) = N^+(v) = \emptyset$, $N^-(u) \cap N^-(v) = \emptyset$.*

Proof. Suppose that $S = (G, \sigma)$ admits an RDF. Let u and v be any two vertices of S such that $N^+(u) = N^+(v) = \emptyset$. We have to prove that $N^-(u) \cap N^-(v) = \emptyset$. If possible assume that $N^-(u) \cap N^-(v) \neq \emptyset$ and let $w \in N^-(u) \cap N^-(v)$. Then $N^+(w) \neq \emptyset$, since S admits an RDF. Therefore $d^+(w) = 1$. Also, observe that $2d^+(w) = n^+_{\emptyset}(N^-(w))$. Hence by Theorem 2.6, S does not admit an RDF. Thus we get a contradiction and hence proving that $N^-(u) \cap N^-(v) = \emptyset$. □

Theorem 2.13. *Let $S = (G, \sigma)$ be a signed graph containing a vertex v with empty positive neighbourhood. If S admits an RDF, then the subsigned graph induced by $N^-(v)$ is disconnected. Moreover, if two vertices x and y belonging to $N^-(v)$ are adjacent, then $d^+(x) = d^+(y) = 1$ or 2 whenever $\sigma(xy) = -1$ or $\sigma(xy) = 1$, respectively.*

Proof. Suppose that $S = (G, \sigma)$ admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$ and it contains a vertex v with $N^+(v) = \emptyset$. We claim that the subsigned graph induced by $N^-(v)$ is disconnected. Let $N^-(v) = \{w, x, y\}$.
We assume the contrary that the subsigned graph induced by $N^-(v)$ is connected . Then the subsigned graph induced by $N^-(v)$ is either a signed cycle or a signed path on 3 vertices. Suppose that the subsigned graph induced by $N^-(v)$ is a signed cycle. Then since G is 3-regular, $G = K_4$ and $d^-(v) = 3$. Hence, by Lemma 2.3, S does not admit an RDF.

Now, suppose that the subsigned graph induced by $N^-(v)$ is a signed path wxy. Then at least one edge of the path wxy is positive. For, if both edges are negative, then there will be a pair of adjacent vertices with empty positive neighbourhoods, which is not possible as S admits an RDF. Without loss generality assume that $\sigma(xy) = 1$. Then $\sigma(wx) = -1$ or 1. By Proposition 2.1, for a vertex $u \in N^-(v)$, $f(u) \neq 2$. Therefore, $f(x) = 1$ and hence, by Observation 2.2, $f(v) = 2$. Further, $x, w \in \mathcal{V}_0$. This shows that$f(N[x]) < 1$, which is not possible as f is an RDF. Clearly, this contradicts our assumption and therefore, the subsigned graph induced by $N^-(v)$ is not connected.

Next, suppose that x and y are two adjacent vertices in $N^-(v)$. Since S admits an RDF, by Lemma 2.4, S cannot have a pair of adjacent vertices with empty positive neighbourhood. Therefore, if $\sigma(xy) = -1$, then their positive degrees are equal to 1. Now, consider the case when $\sigma(xy) = 1$. We wish to prove that $d^+(x) = d^+(y) = 2$. On the contrary assume that $d^+(x) = 1$. By Proposition 2.1, $y \notin \mathcal{V}_2$. Therefore, $x \in \mathcal{V}_1$ and hence, $v \in \mathcal{V}_2$, $\{y, w\} \subseteq \mathcal{V}_0$, by Observation 2.2. Now observe that $f(N[x]) \leq -1 - f(z) < 1$, where $z \in N^-(x) \setminus \{v\}$. This contradicts the fact that f is an RDF and hence, our assumption is wrong. Therefore, $d^+(x) = 2$. In a similar manner we can prove that $d^+(y) = 2$.

□

Now, we examine the number of vertices with an empty positive neighbourhood in a signed graph of diameter 2.

Theorem 2.14. *Let $S = (G, \sigma)$ be a signed graph admittig an RDF. If the diameter of S is 2, then the number of vertices in S with empty positive neighbourhood is at most one.*

Proof. Consider a signed graph $S = (G, \sigma)$ that admits an RDF such that its diameter is 2. We show that S has at most one vertex with an empty positive neighbourhood. On the contrary assume that there exists two vertices u and v having empty positive neighbourhoods. By Lemma 2.4, u and v are non-adjacent. Therefore, u and v have a common neighbour as S is of diameter 2, which is not possible by Lemma 2.12. Hence, our assumption is wrong and therefore, S has at most one vertex with an empty positive neighbourhood. □

If we consider a positive edge of a signed graph with a 3-regular underlying graph, then its end vertices are adjacent to at most 2 negative edges each. Suppose that uv is a positive edge of a signed graph that admits an RDF. If both $d^-(u)$ and $d^-(v)$ are less than or equal to 1, then for every vertex x belonging to $N^+(u) \cup N^+(v)$, $n^+_\emptyset(N^-(x)) \leq 1$. Further, in the case when $d^-(u) = 0$ and $d^-(v) = 2$, for any vertex x in $N^+(u)$, $n^+_\emptyset(N^-(x)) \leq 1$. Now we examine the cases when $d^-(u) = d^-(v) = 2$ and $d^-(u) = 1$, $d^-(v) = 2$.

Theorem 2.15. *If $S = (G, \sigma)$ admits an RDF, then for any positive edge $uv \in E(S)$ such that u and v are incident with two negative edges each,*

(i) $N^-(u) \neq N^-(v)$.
(ii) $n^+_\emptyset(N^-(u) \cup N^-(v)) \leq 1$.

Proof. Suppose that $S = (G, \sigma)$ is a signed graph that admits an RDF $f = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$ and uv be an edge of S with $\sigma(uv) = 1$ and $d^-(u) = d^-(v) = 2$.

(i) We prove that $N^-(u) \neq N^-(v)$. Assume on the contrary that $N^-(u) = N^-(v)$. Let $N^-(u) = N^-(v) = \{x, y\}$.
First, suppose that $N^+(x) = \emptyset$ or $N^+(y) = \emptyset$. Then the vertices u and v do not belong to $\mathcal{V}_2$, by Proposition 2.1. Since f is an RDF, u and v do not belong to $\mathcal{V}_0$ as well. This proves that u and v are in $\mathcal{V}_1$ and $f(N[x]) < 1$ or $f(N[y]) < 1$, which contradicts the fact that f is an RDF.
Now, let $N^+(x) \neq \emptyset$ and $N^+(y) \neq \emptyset$. Since $d^+(u) = d^+(v) = 1$ and f being an RDF,

$f(u) = f(v) = 0$ and $\{f(u), f(v)\} = \{0, 1\}$ are not possible. The remaining possibilities are either $\{f(u), f(v)\} = \{0, 2\}$ or $\{f(u), f(v)\} \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$. In either of the cases we find that both $f(x)$ and $f(y)$ belong to $\mathcal{V}_1 \cup \mathcal{V}_2$. This proves that $f(N[u]) < 1$ and $f(N[v]) < 1$, which is a contradiction. Hence our assumption is wrong and therefore, $N^-(u) \neq N^-(v)$.

(ii) To prove $n^+_\emptyset(N^-(u) \cup N^-(v)) \leq 1$, suppose on the contrary that $n^+_\emptyset(N^-(u) \cup N^-(v))$ is at least 2. Let x and y be any two vertices in $N^-(u) \cup N^-(v)$ with empty positive neighbourhoods. Suppose that $N^-(u) \cap N^-(v) = \emptyset$. By using Lemma 2.12, the neighbourhoods of x and y are disjoint. Without loss of generality assume that $x \in N^-(u)$ and $y \in N^-(v)$. Then $u, v \in \mathcal{V}_1$ and $x, y \in \mathcal{V}_2$, by Proposition 2.1 and Observation 2.2. Now it follows that $f(N[u]) < 1$ and $f(N[v]) < 1$, contradicting the fact that f is an RDF.
Now, let $N^-(u) \cap N^-(v) \neq \emptyset$. If $x \in N^-(u) \cap N^-(v)$, then by similar argument as in *(i)*, it follows that u and v belong to $\mathcal{V}_1$ and hence, $f(N[x]) < 1$, which is not possible. Now assume that $x \notin N^-(u) \cap N^-(v)$. Let $x \in N^-(u)$ and $N^-(u) \cap N^-(v) = \{y\}$. Then $x \in \mathcal{V}_1 \cup \mathcal{V}_2$ and $u \in \mathcal{V}_0 \cup \mathcal{V}_1$, by Proposition 2.1. Further, observe that $x \in \mathcal{V}_2$ and $u \in \mathcal{V}_1$. This shows that v belongs to $\mathcal{V}_2$ and hence, $y \in \mathcal{V}_1 \cup \mathcal{V}_2$. Then $f(N[u]) \leq 1 - f(y) < 1$, which is a contradiction as f is an RDF. Hence our assumption is wrong.
Therefore, from both the cases we get, $n^+_\emptyset(N^-(u) \cup N^-(v)) \leq 1$. □

Similarly, on examining the signed graphs that admits an RDF and has a positive edge uv with $d^-(u) = 1$ and $d^-(v) = 2$ we obtain the following result. The proof is omitted as it is similar to that of Theorem 2.15.

Theorem 2.16. *For any signed graph $S = (G, \sigma)$ that admits an RDF and containing a positive edge uv with $d^-(u) = 1$ and $d^-(v) = 2$, $n^+_\emptyset(N^-(u) \cup N^-(v)) \leq 1$.*

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