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CHARACTERIZATIONS OF FUZZY ⊥-⊤ DISTRIBUTIVE LATTICE IN FUZZY LATTICES

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Abstract. This paper introduces the notion of fuzzy ⊥-distributive lattice in fuzzy lattice. Additionally, we define the concept of fuzzy ⊥-⊤-distributive lattice in fuzzy lattice. We demonstrate that a fuzzy sectionally semi-complemented lattice is a fuzzy distributive lattice if and only if it is a fuzzy ⊥-distributive lattice. Furthermore, we prove that a fuzzy pseudocomplemented lattice is also a fuzzy ⊥-distributive lattice.

Keywords: fuzzy lattice, fuzzy ⊥-distributive, fuzzy ⊤-distributive, fuzzy pseudocomplemented, fuzzy sectionally semi-complemented lattice.

AMS Subject Classification: 03B52, 03E72, 06D72.

1. Introduction

The concept of a 0-distributive lattice was first put forward by Grillet and Varlet [3] as an extension of the distributive lattice. To elaborate on this notion, Varlet [19], Pawar and Thakare [11] have defined 0-distributivity in semilattices, which was further explored by Jayaram [4], Rachůnek and Pawar [13]. Moreover, Pawar and Dhamke [12] have extended this idea to 0-distributivity in posets. In a similar vein, Joshi and Waphare [5] have introduced and investigated 0-distributive posets using a distinct definition. In the broader context of fuzzy algebraic structures, the concept of fuzzy sets was introduced by Zadeh [20], which has inspired various researchers to propose different ideas. For instance, Rosenfeld [15] presented the concept of fuzzy groups, while Ajmal et al. [1] and Chon [2] introduced fuzzy lattices, a topic that was also explored by Mezzomo et. al. [7, 8]. Furthermore, Wasadikar and Khubchandani [16] have introduced the notion of a fuzzy modular pair in fuzzy lattices.

The present research paper draws inspiration from the noteworthy work of Chon [2]. The impetus behind this paper is to further explore the subject matter by building upon Chon's innovative and insightful ideas. The aim is to expand upon the existing understanding of the topic and contribute to the field. The current paper is a testament to the importance of Chon's work and how it has served as a source of inspiration for further research.

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In this paper, we introduce the innovative concepts of fuzzy $\perp$-distributive lattice and its corresponding dual concept of fuzzy $\top$-distributive lattice in the fuzzy lattice. Moreover, we have shown that a fuzzy sectionally semicomplemented lattice is fuzzy distributive if and only if it is fuzzy $\perp$-distributive in the fuzzy lattice. Additionally, we have proved that a fuzzy pseudocomplemented lattice is also a fuzzy $\perp$-distributive lattice in fuzzy lattice.

2. Preliminaries

In this paper, $(X,\ A)$ denotes a fuzzy lattice, where A is a fuzzy partial order relation on a non empty set X.
For the definitions of a fuzzy partial order relation, fuzzy equivalence relation, fuzzy supremum, fuzzy infimum, fuzzy lattice etc. we refer to Chon [2]. We use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$ to distinguish the supremum and infimum of a, b in the lattice sense, if these exist in X.

We recall some known results from Chon [2] which we shall use in this paper.

Definition 2.1. [6, Definition 3.4] *A fuzzy lattice $\mathcal{L} = (X, A)$ is bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for all $a \in X$. In this case, $\perp$ and $\top$ are called bottom and top elements, respectively.*

Proposition 2.1. [2, Proposition 3.3] *and* [7, Proposition 2.4] *Let (X, A) be a fuzzy lattice. For $a, b, c \in X$. The following statements hold:*
(i) $A(a, c) > 0$ and $A(b, c) > 0$ implies $A(a \vee_F b, c) > 0$;
(ii) $A(a, b) > 0$ iff $a \vee_F b = b$;
(iii) $A(a, b) > 0$ iff $a \wedge_F b = a$.

Definition 2.2. *(Chon* [2]*) Let (X, A) be a fuzzy lattice. (X, A) is called a fuzzy distributive lattice, if $a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$ and*
$a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F (a \vee_F c)$ for all $a, b, c \in X$.

We recall definition from Khubchandani and Khubchandani [18]

Definition 2.3. [18, Definition 3.1]*) A fuzzy lattice (X, A) is called fuzzy sectionally semicomplemented lattice (in brief FSSC) if it satisfies the following condition:*
If $a \neq b$ in X, then there exists $c \in X$ such that $c \neq \perp$, $A(c, a) > 0$ and $c \wedge_F b = \perp$.

3. Fuzzy $\perp$-$\top$ Distributive Lattice in Fuzzy Lattices

In their paper, Wasadikar and Khubchandani [17] presented findings related to the fuzzy distributive lattice. The concept of fuzzy $\perp$ and $\top$, as introduced by Mezzomo et al. [7] in the fuzzy lattice, inspired us to investigate the properties of fuzzy $\perp$-$\top$ distributive lattice in the fuzzy lattice.

Definition 3.1. *A fuzzy lattice (X, A) with $\perp$ is called a fuzzy $\perp$-distributive lattice if $a \wedge_F b = \perp$, $a \wedge_F c = \perp$ together imply $a \wedge_F (b \vee_F c) = \perp$, for all $a, b, c \in X$.*

Definition 3.2. *A fuzzy lattice (X, A) with $\top$ is called a fuzzy $\top$-distributive lattice if $a \vee_F b = \top$, $a \vee_F c = \top$ together imply $a \vee_F (b \wedge_F c) = \top$, for all $a, b, c \in X$.*

Definition 3.3. *A fuzzy bounded lattice $\mathcal{L} = (X, A)$ which is both fuzzy $\perp$-distributive lattice and fuzzy $\top$-distributive lattice is called fuzzy $\perp$-$\top$ distributive lattice.*

We illustrate these concepts in the following example. In this example, the fuzzy poset (X, A) is a fuzzy lattice.

Example 3.1. *Let* $X = \{\bot, a, b, c, d, e, \top\}$. *Define a fuzzy relation* $A : X \times X \longrightarrow [0, 1]$ *on* X *as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(\top,\top) = 1,$
$A(\bot,a) = 0.2,\ A(\bot,b) = 0.2,\ A(\bot,c) = 0.2,\ A(\bot,d) = 0.2,\ A(\bot,e) = 0.2,\ A(\bot,\top) = 0.2,$
$A(a,\bot) = 0,\ A(a,b) = 0,\ A(a,c) = 0,\ A(a,d) = 0,\ A(a,e) = 0, A(a,\top) = 0.02,$
$A(b,\bot) = 0,\ A(b,a) = 0,\ A(b,c) = 0.3,\ A(b,d) = 0.3,\ A(b,e) = 0,\ A(b,\top) = 0.02,$
$A(c,\bot) = 0,\ A(c,a) = 0,\ A(c,b) = 0,\ A(c,d) = 0,\ A(c,e) = 0,\ A(c,\top) = 0.02,$
$A(d,\bot) = 0,\ A(d,a) = 0,\ A(d,b) = 0,\ A(d,c) = 0,\ A(d,e) = 0,\ A(d,\top) = 0.02,$
$A(e,\bot) = 0,\ A(e,a) = 0.3,\ A(e,b) = 0,\ A(e,c) = 0.3,\ A(e,d) = 0,\ A(e,\top) = 0.02,$
$A(\top,\bot) = 0,\ A(\top,a) = 0,\ A(\top,b) = 0,\ A(\top,c) = 0,\ A(\top,d) = 0,\ A(\top,e) = 0.$

This fuzzy relation is shown in the following table:

A	⊥	a	b	c	d	e	⊤
⊥	1.0	0.2	0.2	0.2	0.2	0.2	0.2
a	0.0	1.0	0.0	0.0	0.0	0.0	0.02
b	0.0	0.0	1.0	0.3	0.3	0.0	0.02
c	0.0	0.0	0.0	1.0	0.0	0.0	0.02
d	0.0	0.0	0.0	0.0	1.0	0.0	0.02
e	0.0	0.3	0.0	0.3	0.0	1.0	0.02
⊤	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	⊥	a	b	c	d	e	⊤
⊥	⊥	a	b	c	d	e	⊤
a	a	a	⊤	⊤	⊤	a	⊤
b	b	⊤	b	c	d	c	⊤
c	c	⊤	c	c	⊤	c	⊤
d	d	⊤	d	⊤	d	⊤	⊤
e	e	a	c	c	⊤	e	⊤
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤

$\wedge_F$	⊥	a	b	c	d	e	⊤
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
a	⊥	a	⊥	e	⊥	e	a
b	⊥	⊥	b	b	b	⊥	b
c	⊥	e	b	c	b	e	c
d	⊥	⊥	b	b	d	⊥	d
e	⊥	e	⊥	e	⊥	e	e
⊤	⊥	a	b	c	d	e	⊤

We note that (X, A) is a fuzzy lattice.
(X, A) is fuzzy ⊥-distributive lattice as $a \wedge_F d = a \wedge_F b = \bot$ and
$a \wedge_F (d \vee_F b) = a \wedge_F d = \bot$ hold.
But here (X, A) is not fuzzy ⊤-distributive lattice.
Indeed, $c \vee_F a = c \vee_F d = \top$ hold but $c \vee_F (a \wedge_F d) = c \vee_F \bot = c \neq \top$.
Hence (X, A) is fuzzy ⊥-distributive lattice but not fuzzy ⊤-distributive lattice.

Example 3.2. *Let* $X = \{\bot, a, b, c, d, e, f, \top\}$. *Define a fuzzy relation* $A : X \times X \longrightarrow [0, 1]$ *on* X *as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(f,f) = A(\top,\top) = 1,$
$A(\bot,a) = 0.4,\ A(\bot,b) = 0.4,\ A(\bot,c) = 0.4,\ A(\bot,d) = 0.4,\ A(\bot,e) = 0.4,\ A(\bot,f) = 0.4,$
$A(\bot,\top) = 0.4,$
$A(a,\bot) = 0,\ A(a,b) = 0,\ A(a,c) = 0,\ A(a,d) = 0.4,\ A(a,e) = 0.6,\ A(a,f) = 0,$
$A(a,\top) = 0.04,$
$A(b,\bot) = 0,\ A(b,a) = 0,\ A(b,c) = 0,\ A(b,d) = 0,\ A(b,e) = 0,\ A(b,f) = 0,\ A(b,\top) = 0.04,$
$A(c,\bot) = 0,\ A(c,a) = 0,\ A(c,b) = 0.6,\ A(c,d) = 0,\ A(c,e) = 0,\ A(c,f) = 0,$
$A(c,\top) = 0.04,$
$A(d,\bot) = 0,\ A(d,a) = 0,\ A(d,b) = 0,\ A(d,c) = 0,\ A(d,e) = 0.6,\ A(d,f) = 0,$
$A(d,\top) = 0.04,$
$A(e,\bot) = 0,\ A(e,a) = 0,\ A(e,b) = 0,\ A(e,c) = 0,\ A(e,d) = 0,\ A(e,f) = 0,$
$A(e,\top) = 0.04,$

$A(f,\bot)=0,\ A(f,a)=0,\ A(f,b)=0,\ A(f,c)=0.6,\ A(f,d)=0.6,\ A(f,e)=0.6,$
$A(f,\top)=0.04,$
$A(\top,\bot)=0,\ A(\top,a)=0,\ A(\top,b)=0,\ A(\top,c)=0,\ A(\top,d)=0,\ A(\top,e)=0,$
$A(\top,f)=0.$

This fuzzy relation is shown in the following table:

A	⊥	a	b	c	d	e	f	⊤
⊥	1.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4
a	0.0	1.0	0.0	0.0	0.6	0.6	0.0	0.04
b	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.04
c	0.0	0.0	0.6	1.0	0.0	0.0	0.0	0.04
d	0.0	0.0	0.0	0.0	1.0	0.6	0.0	0.04
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.04
f	0.0	0.0	0.0	0.6	0.6	0.6	1.0	0.04
⊤	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	⊥	a	b	c	d	e	f	⊤
⊥	⊥	a	b	c	d	e	f	⊤
a	a	a	⊤	⊤	d	e	d	⊤
b	b	⊤	b	c	⊤	⊤	c	⊤
c	c	⊤	c	c	⊤	⊤	c	⊤
d	d	d	⊤	⊤	d	e	d	⊤
e	e	e	⊤	⊤	e	e	e	⊤
f	f	d	c	c	d	e	f	⊤
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤

$\wedge_F$	⊥	a	b	c	d	e	f	⊤
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
a	⊥	a	⊥	⊥	a	a	⊥	a
b	⊥	⊥	b	b	⊥	⊥	⊥	b
c	⊥	⊥	b	c	f	f	f	c
d	⊥	a	⊥	f	d	d	f	d
e	⊥	a	⊥	f	d	e	f	e
f	⊥	⊥	⊥	f	f	f	f	f
⊤	⊥	a	b	c	d	e	f	⊤

We note that (X,A) is a fuzzy lattice.
(X,A) is fuzzy $\top$-distributive lattice as $a\vee_F b = a\vee_F c=\top$ and
$a\vee_F(b\wedge_F c)=a\vee_F b=\top$ hold.
But (X,A) is not fuzzy $\bot$-distributive lattice.
Indeed, $f\wedge_F a=f\wedge_F b=\bot$ but $f\wedge_F(a\vee_F b)=f\wedge_F\top=f\neq\bot$.
Therefore (X,A) is fuzzy $\top$-distributive lattice but not fuzzy $\bot$-distributive lattice.

Example 3.3. *Let $X=\{\bot,a,b,c,\top\}$. Define a fuzzy relation $A: X\times X\longrightarrow[0,1]$ on X as follows such that*
$A(\bot,\bot)=A(a,a)=A(b,b)=A(c,c)=A(\top,\top)=1,$
$A(\bot,a)=0.7,\ A(\bot,b)=0.7,\ A(\bot,c)=0.7,\ A(\bot,\top)=0.7,$
$A(a,\bot)=0,\ A(a,b)=0,\ A(a,c)=0,\ A(a,\top)=0.03,$
$A(b,\bot)=0,\ A(b,a)=0,\ A(b,c)=0.0,\ A(b,\top)=0.03,$
$A(c,\bot)=0,\ A(c,a)=0,\ A(c,b)=0,\ A(c,\top)=0.03,$
$A(\top,\bot)=0,\ A(\top,a)=0,\ A(\top,b)=0,\ A(\top,c)=0.$

This fuzzy relation is shown in the following table:

A	⊥	a	b	c	⊤
⊥	1.0	0.7	0.7	0.7	0.7
a	0.0	1.0	0.0	0.0	0.03
b	0.0	0.0	1.0	0.0	0.03
c	0.0	0.0	0.0	1.0	0.03
⊤	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	a	b	c	$\top$
a	a	a	$\top$	$\top$	$\top$
b	b	$\top$	b	$\top$	$\top$
c	c	$\top$	$\top$	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	$\bot$	b
c	$\bot$	$\bot$	$\bot$	c	c
$\top$	$\bot$	a	b	c	$\top$

We note that (X, A) is a fuzzy lattice.
As $a \wedge_F b = a \wedge_F c = \bot$ hold. But $a \wedge_F (b \vee_F c) = a \wedge_F \top = a \neq \bot$.
Hence (X, A) is not fuzzy $\bot$-distributive lattice.
Also, $a \vee_F b = a \vee_F c = \top$ hold. But $a \vee_F (b \wedge_F c) = a \vee_F \bot = a \neq \top$.
Hence (X, A) is not fuzzy $\top$-distributive lattice.
Therefore (X, A) is neither fuzzy $\bot$-distributive lattice nor fuzzy $\top$-distributive lattice.

Example 3.4. *Let $X = \{\bot, a, b, c, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot, \bot) = A(a, a) = A(b, b) = A(c, c) = A(\top, \top) = 1,$
$A(\bot, a) = 0.5,\ A(\bot, b) = 0.5,\ A(\bot, c) = 0.5,\ A(\bot, \top) = 0.5,$
$A(a, \bot) = 0,\ A(a, b) = 0,\ A(a, c) = 0,\ A(a, \top) = 0.03,$
$A(b, \bot) = 0,\ A(b, a) = 0,\ A(b, c) = 0.2,\ A(b, \top) = 0.03,$
$A(c, \bot) = 0,\ A(c, a) = 0,\ A(c, b) = 0,\ A(c, \top) = 0.03,$
$A(\top, \bot) = 0,\ A(\top, a) = 0,\ A(\top, b) = 0,\ A(\top, c) = 0.$

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	$\top$
$\bot$	1.0	0.5	0.5	0.5	0.5
a	0.0	1.0	0.0	0.0	0.03
b	0.0	0.0	1.0	0.2	0.03
c	0.0	0.0	0.0	1.0	0.03
$\top$	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	a	b	c	$\top$
a	a	a	$\top$	$\top$	$\top$
b	b	$\top$	b	c	$\top$
c	c	$\top$	c	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	b	b
c	$\bot$	$\bot$	b	c	c
$\top$	$\bot$	a	b	c	$\top$

We note that (X, A) is a fuzzy lattice.
As $a \wedge_F c = a \wedge_F b = \bot$ and $a \wedge_F (b \vee_F c) = a \wedge_F c = \bot$.
Hence (X, A) is fuzzy $\bot$-distributive lattice.
Also, $a \vee_F c = a \vee_F b = \top$ and $a \vee_F (b \wedge_F c) = a \vee_F b = \top$.
Hence (X, A) is fuzzy $\top$-distributive lattice.
Therefore (X, A) is fuzzy $\bot$-$\top$-distributive lattice.
But not fuzzy distributive as $c \wedge_F (a \vee_F b) = c \wedge_F \top = c$
and $(c \wedge_F a) \vee (c \wedge_F b) = \bot \vee_F b = b \neq c$.

Definition 3.4. *Let (X, A) be a fuzzy lattice with $\bot$. An element $x^* \in X$ is said to be fuzzy pseudocomplement of $x \in X$, if $x \wedge_F x^* = \bot$ for any $y \in X$, $x \wedge_F y = \bot$ implies $A(y, x^*) > 0$.*

Definition 3.5. *A fuzzy lattice (X, A) is called fuzzy pseudocomplemented if each element of X has a fuzzy pseudocomplement.*

Example 3.5. *We give an example to illustrate the concept of fuzzy pseudocomplement in a fuzzy pseudocomplement lattice.*

(1) In example 3.4, c is fuzzy pseudocomplement of a in a fuzzy pseudocomplement lattice.
(2) In example 3.2, d and e are fuzzy pseudocomplement of b in a fuzzy pseudocomplement lattice.

we note that if fuzzy pseudocomplement of any element in a fuzzy pseudocomplement lattice if exists need not be unique.

Remark 3.1. *The fuzzy pseudocomplement of $\perp$ is the largest element $\top$. Thus a fuzzy pseudocomplemented lattice contains both the smallest element and the largest element.*

Lemma 3.1. *A fuzzy distributive lattice is fuzzy $\perp$-distributive lattice.*

Proof. Let (X, A) be a fuzzy distributive lattice.
Let $a, b, c \in X$ be such that $a \wedge_F b = a \wedge_F c = \perp$.
By fuzzy distributivity, we have $a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$.
But $(a \wedge_F b) = (a \wedge_F c) = \perp$ hence $a \wedge_F (b \vee_F c) = \perp$.
Thus (X, A) is fuzzy $\perp$-distributive lattice. □

Remark 3.2. *Converse of the above theorem is not true.*
In example 3.4, (X, A) is fuzzy $\perp$-distributive lattice but not fuzzy distributive lattice.

Theorem 3.1. *An $FSSC$ lattice is fuzzy distributive lattice iff it is fuzzy $\perp$-distributive lattice.*

Proof. Let (X, A) be a $FSSC$ lattice. Assume that (X, A) is fuzzy $\perp$-distributive lattice.
Let $x = (a \vee_F b) \wedge_F c$ and $y = (a \wedge_F c) \vee_F (b \wedge_F c)$ for $a, b, c \in X$.
By (ii) of Proposition 2.1 we have

$$A(x, c) > 0. \tag{3.1}$$

To show (X, A) is fuzzy distributive lattice, it is sufficient to show that $A(x, y) > 0$.
Suppose $A(x, y) = 0$. As (X, A) is $FSSC$ lattice there exists $z \in X$ such that $z \neq \perp$,

$$A(z, x) > 0 \tag{3.2}$$

and

$$z \wedge_F y = \perp. \tag{3.3}$$

From (3.1) and (3.2) by fuzzy transitivity of A we have

$$A(z, c) > 0. \tag{3.4}$$

As $y = (a \wedge_F c) \vee_F (b \wedge_F c)$ by (ii) of Proposition 2.1 we get

$$A(a \wedge_F c, y) > 0 \text{ and } A(b \wedge_F c, y) > 0.$$

Taking meet z on both sides we get

$$A(a \wedge_F c \wedge_F z, y \wedge_F z) > 0 \tag{3.5}$$

and

$$A(b \wedge_F c \wedge_F z, y \wedge_F z) > 0. \tag{3.6}$$

Putting (3.3) in (3.5) and (3.6) we get

$$A(a \wedge_F c \wedge_F z, \perp) > 0 \tag{3.7}$$

and

$$A(b \wedge_F c \wedge_F z, \bot) > 0. \tag{3.8}$$

As $A(z,c) > 0$ by (iii) of Proposition 2.1 we get $c \wedge_F z = z$.
Putting $c \wedge_F z = z$ in (3.7) and (3.8) we get

$$A(a \wedge_F z, \bot) > 0 \tag{3.9}$$

and

$$A(b \wedge_F z, \bot) > 0. \tag{3.10}$$

Also,

$$A(\bot, a \wedge_F z) > 0 \tag{3.11}$$

and

$$A(\bot, b \wedge_F z) > 0 \tag{3.12}$$

always holds.
Therefore by (3.9), (3.10), (3.11) and (3.12) by fuzzy antisymmetry we have

$$a \wedge_F z = \bot \text{ and } b \wedge_F z = \bot.$$

Now, by fuzzy $\bot$-distributivity of (X, A) we have

$$z \wedge_F (a \vee_F b) = \bot. \tag{3.13}$$

But since $A(z, x) > 0$ and $A(x, a \vee_F b) > 0$ by fuzzy transitivity of A we have

$$A(z, a \vee_F b) > 0.$$

By (ii) of Proposition 2.1 we have $z \wedge_F (a \vee_F b) = z$.
Therefore equation (3.13) reduces to $z = \bot$, a contradiction to $z \neq \bot$.
Hence (X, A) is fuzzy distributive lattice.
The converse follows from Lemma 3.1. □

Theorem 3.2. *Every fuzzy pseudocomplemented lattice is fuzzy $\bot$-distributive.*

Proof. Let (X, A) be fuzzy pseudocomplemented lattice. Let a^* be fuzzy pseudocomplement of a. Moreover suppose that $a \wedge_F b = a \wedge_F c = \bot$. By the definition of fuzzy pseudocomplement, $A(b, a^*) > 0$ and $A(c, a^*) > 0$. By (i) of Proposition 2.1 we have $A(b \vee_F c, a^*) > 0$. Taking meet a on both sides we get $A(a \wedge_F (b \vee_F c), a \wedge_F a^*) > 0$. As (X, A) is fuzzy pseudocomplemented lattice we get $A(a \wedge_F (b \vee_F c), \bot) > 0$, since $a \wedge_F a^* = \bot$. And $A(\bot, a \wedge_F (b \vee_F c)) > 0$ always holds. Therefore by fuzzy antisymmetry of A we get $a \wedge_F (b \vee_F c) = \bot$. Thus, (X, A) is fuzzy $\bot$-distributive lattice. □

Remark 3.3. *It is well known that fuzzy $\bot$-distributive lattice need not be fuzzy pseudocomplemented lattice.*

Example 3.6. *Let $X = \{\bot, a, a_1, a_2, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot, \bot) = A(a, a) = A(a_1, a_1) = A(a_2, a_2) = A(\top, \top) = 1,$
$A(\bot, a) = 0.6,\ A(\bot, a_1) = 0.6,\ A(\bot, a_2) = 0.6,\ A(\bot, \top) = 0.6,$
$A(a, \bot) = 0,\ A(a, a_1) = 0,\ A(a, a_2) = 0,\ A(a, \top) = 0.06,$
$A(a_1, \bot) = 0,\ A(a_1, a) = 0,\ A(a_1, a_2) = 0.7,\ A(a_1, \top) = 0.06,$
$A(a_2, \bot) = 0,\ A(a_2, a) = 0,\ A(a_2, a_1) = 0,\ A(a_2, \top) = 0.06,$
$A(\top, \bot) = 0,\ A(\top, a) = 0,\ A(\top, a_1) = 0,\ A(\top, a_2) = 0.$

This fuzzy relation is shown in the following table:

A	$\bot$	a	a_1	a_2	$\top$
$\bot$	1.0	0.6	0.6	0.6	0.6
a	0.0	1.0	0.0	0.0	0.06
a_1	0.0	0.0	1.0	0.7	0.06
a_2	0.0	0.0	0.0	1.0	0.06
$\top$	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	a_1	a_2	$\top$
$\bot$	$\bot$	a	a_1	a_2	$\top$
a	a	a	$\top$	$\top$	$\top$
a_1	a_1	a_1	a_1	a_2	$\top$
a_2	a_2	$\top$	a_2	a_2	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	a_1	a_2	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a
a_1	$\bot$	$\bot$	a_1	a_1	a_1
a_2	$\bot$	$\bot$	a_1	a_2	a_2
$\top$	$\bot$	a	a_1	a_2	$\top$

We note that (X, A) is a fuzzy lattice.
(X, A) is fuzzy $\bot$-distributive but not fuzzy pseudocomplement lattice. As a^* does not exists.

Theorem 3.3. *A fuzzy lattice (X, A) with $\bot$ is fuzzy $\bot$-distributive lattice iff it satisfies the following condition (*)*
(*) If $a \wedge_F b = a \wedge_F c = \bot$ and $A(a \vee_F b, b \vee_F c) > 0$ for $a, b, c \in X$, then $a = \bot$.

Proof. Let (X, A) be fuzzy $\bot$-distributive lattice.
To prove the condition (*), we assume $a, b, c \in X$ such that $a \wedge_F b = a \wedge_F c = \bot$ and $A(a \vee_F b, b \vee_F c) > 0$ hold.
By fuzzy $\bot$-distributive lattice, we have

$$a \wedge_F (b \vee_F c) = \bot. \tag{3.14}$$

Since $A(a \vee_F b, b \vee_F c) > 0$. By taking meet a on both sides we get

$$A(a \wedge_F (a \vee_F b), a \wedge_F (b \vee_F c)) > 0.$$

Hence by using (3.14) and absorption property we get

$$A(a, \bot) > 0. \tag{3.15}$$

And

$$A(\bot, a) > 0 \tag{3.16}$$

always holds.
From (3.15) and (3.16) by fuzzy antisymmetry of A we get $a = \bot$.
Conversely, suppose the condition (*) holds.
To prove that (X, A) is fuzzy $\bot$-distributive.
let $a, b, c \in X$ be such that $a \wedge_F b = a \wedge_F c = \bot$. Let $d = a \wedge_F (b \vee_F c)$.
Clearly $d \wedge_F b = d \wedge_F c = \bot$, $A(d \vee_F b, b \vee_F c) > 0$ and $A(d \vee_F c, b \vee_F c) > 0$.
By the condition (*), $d = \bot$ which yields $a \wedge_F (b \vee_F c) = \bot$. □

4. Conclusion

In this paper, we have introduced the notion of fuzzy $\bot$-distributive lattice. Also, we have defined concept of fuzzy $\bot$-$\top$-distributive lattice. We have proved that a fuzzy sectionally semi-complemented lattice is a fuzzy distributive lattice if and only if it is a fuzzy $\bot$-distributive lattice. Moreover, we prove that a fuzzy pseudocomplemented lattice is also a fuzzy $\bot$-distributive lattice.

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