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VERTEX AND EDGE-VERTEX GRACEFUL LABELING ON NEUTROSOPHIC GRAPHS

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Abstract. A graph G with gracefully numbered labels is known as graceful labeling. An intuitionistic fuzzy graph/ neutrosophic graph, which admits graceful labeling, and if all vertex labels are distinct by each membership, then the graph is said to be an intuitionistic fuzzy vertex graceful graph/ neutrosophic vertex graceful graph. Suppose both vertex and edge labels of an intuitionistic fuzzy graceful graph/ neutrosophic graceful graph are distinct by each membership. In that instance, it is referred to as an intuitionistic fuzzy edge-vertex graceful graph/ neutrosophic edge-vertex graceful graph. In this paper, the concept of vertex and edge-vertex graceful labeling has been newly discussed on some intuitionistic and neutrosophic graphs.

Keywords: Intuitionistic fuzzy graph, Neutrosophic graph, Intuitionistic fuzzy vertex graceful labeling, Neutrosophic vertex graceful labeling.

AMS Subject Classification: 05C78

1. Introduction

The crisp set theory had been refined by introducing a fuzzy set, where the continuation of membership grades is possible between $[0, 1]$. L.A. Zadeh[31] initially formulated fuzzy sets and the relations on fuzzy sets in 1965. Applications with fuzzy sets were demonstrated in detail by Zimmermann. H.J. Later, A. Kaufmann's[13] elaborate work on the fuzzy graph and fuzzy graph models based on the fuzzy set definition was modeled by Azriel Rosenfeld[20] in 1975. P. Bhattacharya and K.R. Bhuttani[9, 10] contributed some noteworthy results, exemplifying the fuzzy graph approach and properties. A. Nagoorgani[14, 15, 16] is a notable researcher who yields several results regarding fuzzy graphs and their labeling properties. In 1983, K.T. Atanassov[8] generalized the fuzzy set by adding a degree of non-membership and renamed the set as an intuitionistic fuzzy set(IFS). With the definition of IFS, an intuitionistic fuzzy graph(IFG) was designed by R. Parvathi et al.[17, 18] and M. Akram[1, 2]. A. Nagoorgani discussed its structural properties in detail. Numerous intuitionistic fuzzy works and ideas on graphical environments are done by S. Sahoo and M. Pal[21, 22, 23, 24]. In addressing the inconsistency in

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real-world applications, F. Smarandache[26] introduced a new set named the neutrosophic set, which differs from the prior sets by the space given for the indeterminacy membership function. Wang's[30] single-valued neutrosophic set is the refined form of the neutrosophic set, where the standard unit interval [0,1] has been considered for the truth, indeterminacy, and false membership functions. S. Broumi and F. Smarandache[11] presented the work on single-valued neutrosophic graphs(SVNG), which deals with the degree, order, and size of SVNG.

As for all labeling, graceful labeling was elaborately discussed with varying edge and vertex parameters in graph theory. Graceful labeling was coined by W. Golomb, which was initially named $\beta - labeling$ by A. Rosa[19]. An injection $f : V(G) \rightarrow 0, 1, 2, ..., q$ that results in all of the edge labels being distinct when each edge $xy \in E(G)$ is given by the label $|f(x) - f(y)|$ is a graceful labeling of a graph G with p vertices and q edges. Numerous researchers have produced several works on the graceful labeled graph, which assigns integer values to vertices and edges. However, R. Jahir Hussain et al. have done the first graceful work on fuzzy graphs. Following that, R. Jebesty Shajila and S. Vimala[12, 29] presented fuzzy graceful labeling & fuzzy vertex graceful labeling on some graphs, and this motivated us to introduce graceful labeling to the extension of fuzzy and intuitionistic fuzzy graphs, i.e., neutrosophic graphs (NG). There is a lack of graceful labeling discussion in the dimensions of intuitionistic fuzzy and neutrosophic graphs. In this paper, we explore the study of vertex and edge-vertex graceful labeling on intuitionistic fuzzy graphs and neutrosophic graphs by enhancing the conditions taken to prove vertex graceful labeling for fuzzy graphs[6, 7, 12, 29] and associating it with current trends and technological applications.

2. Preliminaries

Definition 2.1.
A wheel graph W_n is a graph with n vertices $(n \geq 4)$ obtained by the union of a star graph S_n and a cycle with $n-1$ vertices (i.e),S_n+C_{n-1}.

Definition 2.2. [12]
A fan graph $F_{m,n}$ is defined as the graph join $K_m + P_n$, where K_m is the empty graph on m vertices and P_n is the path graph on n vertices. If $m = 1$ in K_m, then it corresponds to the normal fan graph and $m = 2$ corresponds to the double fan graph.

Definition 2.3.
A friendship graph F_n ,$n \geq 1$, is a graph which consists of n copies of cycles with a common vertex.

Definition 2.4.
A star friendship graph $SF(n, m)$ consists of a cycle $C_n(n \geq 3)$ and n sets of m independent vertices where each set joins each of the vertices of C_n.

Definition 2.5.
A generalized butterfly graph $BF_n, n \geq 2$, obtained by inserting vertices to every wing with assumption that sum of inserting vertices to every wing is same and it has $2n + 1$ vertices and $4n - 2$ edges.

Definition 2.6.
An intuitionistic fuzzy graph is of the form $G = (V, E, \sigma, \mu)$, where $\sigma = (T_1, F_1)$ and $\mu = (T_2, F_2)$ with the following conditions,
(i) $V = v_1, v_2, ..., v_n$ such that $T_1 : V \rightarrow [0, 1]$ and $F_1 : V \rightarrow [0, 1]$ denote the degree of membership and nonmembership of the element $v_i \in V$ respectively, and $0 \leq T_1(v_i) +$

$F_1(v_i) \leq 1$ *for every* $v_i \in V$, $(i = 1, 2, ..., n)$,
(ii) $E \subseteq V \times V$, *where* $T_2 : V \times V \to [0, 1]$ *and* $F_2 : V \times V \to [0, 1]$ *are such that*
$T_2(v_i, v_j) \leq min[T_1(v_i), T_1(v_j)]$,
$F_2(v_i, v_j) \leq max[F_1(v_i), F_1(v_j)]$ *and*
$0 \leq T_2(v_i, v_j) + F_2(v_i, v_j) \leq 1$, *for every* $(v_i, v_j) \in E$, $(i, j = 1, 2, \cdots, n)$

Definition 2.7.
A graph $G = (V, E, \sigma, \mu)$, *where* $\sigma = (T_1, F_1)$ *and* $\mu = (T_2, F_2)$ *is said to be intuitionistic fuzzy labeling graph if* $T_1 : V \to [0, 1]$, $F_1 : V \to [0, 1]$, $T_2 : V \times V \to [0, 1]$ *and* $F_2 : V \times V \to [0, 1]$ *are bijective such that* $T_1(v_i)$,$F_1(v_i)$,$T_2(v_i, v_j)$,$F_2(v_i, v_j) \in [0, 1]$ *all are distinct for each node and edge, where* T_1 *is the degree of membership and* F_1 *is the degree of non-membership of nodes. Similarly,* T_2 *and* F_2 *are the degrees of membership and non-membership of edges.*

Definition 2.8.
A neutrosophic graph is of the form $G = (V, \sigma, \mu)$, *where* $\sigma = (T_1, I_1, F_1)$ *and* $\mu = (T_2, I_2, F_2)$ *with the following conditions,*
(i) The functions $T_1 : V \to [0, 1]$, $I_1 : V \to [0, 1]$ *and* $F_1 : V \to [0, 1]$ *denote the degree of truth, indeterminacy and false membership functions of the element* $v_i \in V$, *respectively and* $0 \leq T_1(v_i) + I_1(v_i) + F_1(v_i) \leq 3$, *for all* $v_i \in V$.
(ii) The functions $T_2 : E \subseteq V \times V \to [0, 1]$, $I_2 : E \subseteq V \times V \to [0, 1]$ *and* $F_2 : E \subseteq V \times V \to [0, 1]$ *denote the degree of truth, indeterminacy and false membership functions of the edge* (v_i, v_j) *respectively, such that*
$T_2(v_i, v_j) \leq min[T_1(v_i), T_1(v_j)]$,
$I_2(v_i, v_j) \leq min[I_1(v_i), I_1(v_j)]$,
$F_2(v_i, v_j) \leq max[F_1(v_i), F_1(v_j)]$ *and* $0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$,
for every edge (v_i, v_j).

Definition 2.9.
A neutrosophic graph $G = (V, \sigma, \mu)$, *where* $\sigma = (T_1, I_1, F_1)$ *and* $\mu = (T_2, I_2, F_2)$ *is said to be an neutrosophic labeling graph, if* $T_1 : V \to [0, 1]$, $I_1 : V \to [0, 1]$ $F_1 : V \to [0, 1]$ *and* $T_2 : V \times V \to [0, 1]$, $I_2 : V \times V \to [0, 1]$, $F_2 : V \times V \to [0, 1]$ *are bijective such that the truth, indeterminacy and false membership functions of the vertices and edges are distinct and*
$T_2(v_i, v_j) \leq min[T_1(v_i), T_1(v_j)]$,
$I_2(v_i, v_j) \leq min[I_1(v_i), I_1(v_j)]$,
$F_2(v_i, v_j) \leq max[F_1(v_i), F_1(v_j)]$ *and* $0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$,
for every edge (v_i, v_j).

3. Intuitionistic Vertex/Edge-Vertex Graceful Labeling Graphs

Definition 3.1.
A wheel graph with intuitionistic fuzzy labeling is called an intuitionistic fuzzy wheel graph. An intuitionistic fuzzy wheel graph has a vertex set $V = \{v_c\} \cup \{v_i\}$ *such that* $\mu(v_c v_i) > 0$, *where* i= 1 *to* $n-1$ *and* $\mu(v_i v_{i+1}) > 0$, *where* i= 1 *to* $n-2$. *If all vertices of an intuitionistic fuzzy wheel graph are distinct, then it is called an intuitionistic fuzzy vertex graceful wheel graph.*

Definition 3.2.
A fan graph with intuitionistic fuzzy labeling is called an intuitionistic fuzzy fan graph. An intuitionistic fuzzy fan graph has a vertex set $V = \{v_c\} \cup \{v_i\}$ *such that* $\mu(v_c v_i) > 0$, *where* i=1 *to* n *and* $\mu(v_i v_{i+1}) > 0$, *where* i=1 *to* $n - 1$. *If all vertices of an intuitionistoic fuzzy fan graph are distinct, then it is called an intuitionistic fuzzy vertex graceful fan graph.*

Definition 3.3.
A friendship graph with an intuitionistic fuzzy labeling is called as intuitionistic fuzzy friendship graph which comprises of a vertex set $V = \{v_c\} \cup \{v_i\}$ *such that* $\mu(v_c v_i) > 0$*, where* $i=1$ *to* $n-1$ *and* $\mu(v_i v_{i+1}) > 0$*, where* $i=1$ *to* $n-2$*. If all vertex values of an intuitionistic fuzzy friendship graph are distinct, then it is called an intuitionistic vertex graceful friendship graph.*

Theorem 3.4.
Every intuitionistic fuzzy fan graph $F_{1,n}, n \geq 2$ *admits an intuitionistic fuzzy vertex graceful labeling.*

Proof. Consider an intuitionistic fuzzy fan graph $F_{1,n}, n \geq 2$ consists of two vertex sets V and V_n with $|V|=1$ and $|V_n| > 1$, such that $\mu(V, V_i) > 0$, where $i=1$ to n and $\mu(V_i, V_{i+1}) > 0$, where $i=1$ to $n-1$. Let V be the central vertex and V_i denotes the other vertices of $F_{1,n}$. In an intuitionistic fuzzy fan graph $F_{1,n}, n \geq 2$, the central vertex(V) labeling $\sigma : V \to [0,1]$ satisfies the condition that if each membership value of V starts from $\frac{n-1}{10^k}, \frac{n}{10^k}, \frac{n+1}{10^k}$, etc., where $k \in \mathbb{N}$ then the intuitionistic fuzzy fan graph is termed to have intuitionistic fuzzy vertex graceful labeling. Here, fix $\mu(V, V_1) = (T(V, V_1), F(V, V_1)) = (\frac{0.01}{10^l}, \frac{0.02}{10^l})$, where $l \in \mathbb{W}$ to obtain the value of $\sigma(V_1)$ by the expression, $\sigma(V_1) = \sigma(V) - \mu(V, V_1)$. The labeling for other vertices $\sigma : V_i \to [0,1]$ is defined by $\sigma(V_{i+1}) = \sigma(V) - \mu(V, V_{i+1})$, where $\mu(V, V_{i+1}) = (T(V, V_i) + (\frac{0.01}{10^l})(i+1), F(V, V_i) + (\frac{0.02}{10^l})(i+1))$, where $i=1$ to $n-1$, $l \in \mathbb{W}$ (when the intuitionistic fuzzy labeling condition fails for edges, then minimizing it as per the defined central vertex is necessary). If the intuitionistic fuzzy fan graph is vertex graceful then the outer edges are labeled using $\mu(V_i, V_{i+1}) = \mu(V, V_{i+1}) - \mu(V, V_i)$, where $i=1$ to $n-1$. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every intuitionistic fuzzy fan graph $F_{1,n}, n \geq 2$ is an intuitionistic fuzzy vertex graceful labeling graph.

Example : The fan graphs $F_{1,2}$ & $F_{1,3}$ satisfy the vertex labeling property for k,l=1. For other order of fan graphs, we have to increase the value of k and l to preserve the total sum condition for vertex and edge membership of intuitionistic fuzzy graph. The following examples of intuitionistic fuzzy vertex graceful labeling bears the same condition.

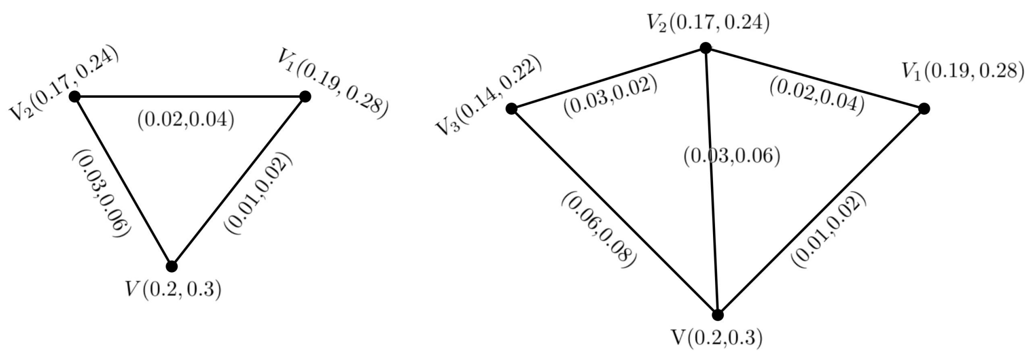

FIGURE 1. Intuitionistic fuzzy vertex graceful $F_{1,2}$ & $F_{1,3}$ graphs

Note 3.5.
The following theorems 3.6 and 3.7 can be proved by proceeding with the same steps of

the above theorem 3.4, from assinging the value for $\mu(V, V_1)$ to generate other edges and vertices of the graph considered.

Theorem 3.6.
Every intuitionistic fuzzy wheel graph $W_n, n \geq 4$ admits an intuitionistic fuzzy vertex graceful labeling.

Proof. Let $W_n, n \geq 4$ be an intuitionistic fuzzy wheel graph consists of two vertex sets U and V with $|U|$=1 and $|V| > 1$, such that $\mu(U, V_i) > 0$, where i=1 to $n-1$ and $\mu(V_i, V_{i+1}) > 0$, where i=1 to $n-2$. Let U be the central vertex and V denotes the other vertices of W_n. In an intuitionistic fuzzy wheel graph $W_n, n \geq 4$ the central vertex(U) labeling $\sigma : U \to [0, 1]$ satisfies the condition that if each membership value of U starts from $\frac{n-3}{10^k}$, $\frac{n-2}{10^k}$ $\frac{n-1}{10^k}$, etc., where $k \in \mathbb{N}$ then the intuitionistic fuzzy wheel graph is termed to have intuitionistic fuzzy vertex graceful labeling. The label values of other vertices and edges are generated as mentioned in Note 3.5. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every intuitionistic fuzzy fan graph $F_{1,n}, n \geq 2$ is an intuitionistic fuzzy vertex graceful labeling graph.

Example :

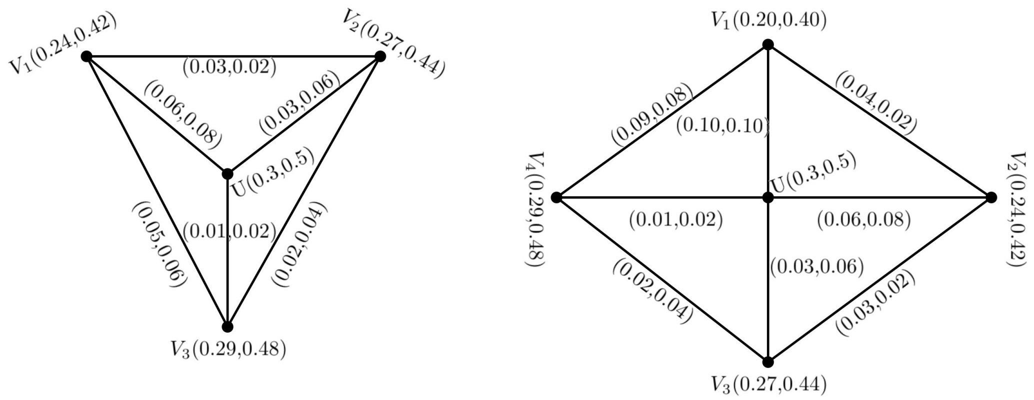

FIGURE 2. Intuitionistic fuzzy vertex graceful W_4 & W_5 graphs

Theorem 3.7.
Every intuitionistic fuzzy friendship graph F_n ,$n \geq 1$ admits an intuitionistic fuzzy vertex graceful labeling.

Proof. Let us consider an intuitionistic fuzzy friendship graph F_n ,$n \geq 1$ consists of two vertex sets U and V with $|U|$=1 and $|V| > 1$, such that $\mu(U, V_i) > 0$, where i=1 to $n-1$ and $\mu(V_i, V_{i+1}) > 0$, where i=1 to $n-2$. Let U be the central vertex and V denotes the other vertices of F_n. In an intuitionistic fuzzy friendship graph $F_n, n \geq 1$ the central vertex(U) labeling $\sigma : U \to [0, 1]$ satisfies the condition that if each membership value of U starts from $\frac{n}{10^k}$, $\frac{n+1}{10^k}$, etc., where $k \in \mathbb{N}$ then the intuitionistic fuzzy friendship graph is termed to have intuitionistic fuzzy vertex graceful labeling. The label values of other vertices and edges are generated as mentioned in Note 3.5. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every intuitionistic fuzzy friendship graph F_n ,$n \geq 1$ is an intuitionistic fuzzy vertex graceful labeling graph.

Example :

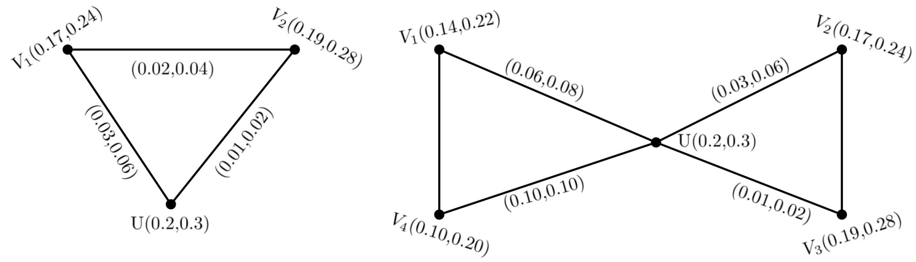

FIGURE 3. Intuitionistic fuzzy vertex graceful F_1 & F_2 graphs

Theorem 3.8.
An intuitionistic fuzzy corona graph $C_4 \odot K_n$ admits an intuitionistic fuzzy edge-vertex graceful labeling.

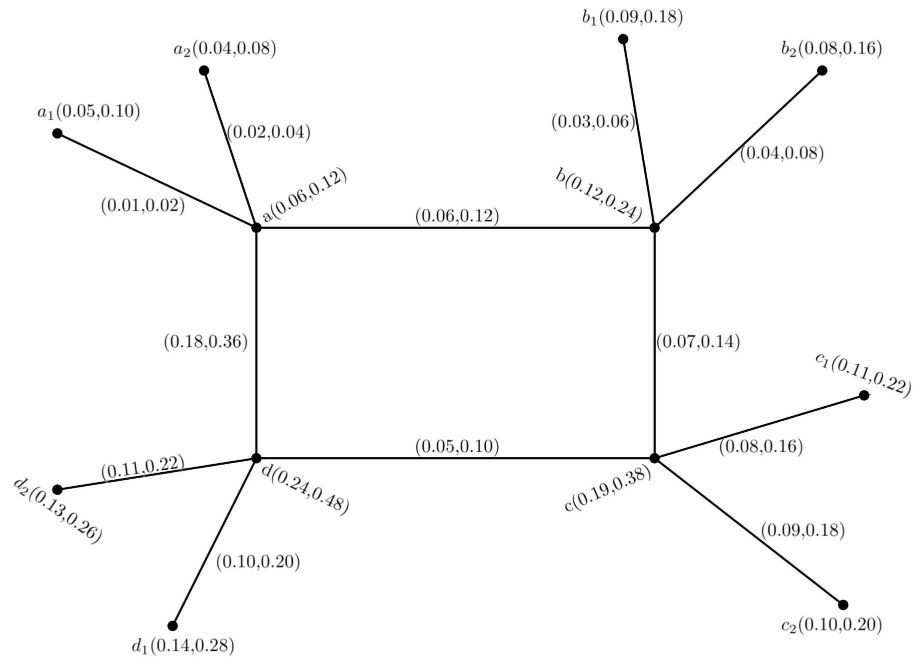

FIGURE 4. Intuitionistic fuzzy edge-vertex graceful $C_4 \odot K_n$

Proof. Let a_i, b_j, c_k, d_l be the pendent vertices of intuitionistic fuzzy corona graph $C_4 \odot K_n$, which emerge from the C_4 vertices a, b, c, d of the graph $C_4 \odot K_n$. Then the vertex set of $C_4 \odot K_n$ is the union of $\{a_i/1 \le i \le n\}, \{b_j/1 \le j \le n\}, \{c_k/1 \le k \le n\}, \{d_l/1 \le l \le n\}$ & $\{a, b, c, d\}$ and the edge set is the union of $\{(a, a_i)/1 \le i \le n\}, \{(b, b_j)/1 \le j \le n\}, \{(c, c_k)/1 \le k \le n\}, \{(d, d_l)/1 \le l \le n\}$ and $\{ab, bc, cd, da\}$. For $1 \le i, j, k, l \le n$, the pendent edges must satisfy the condition: $\mu(a, a_i) > 0$, $\mu(b, b_j) > 0$, $\mu(c, c_k) > 0$, $\mu(d, d_k) > 0$ and there will be no edges between pendent vertices. Each vertex component of $\sigma(V) \in [0, 1]$ and defined by $\sigma(a) = ((n + m)(i - 1) \times 0.01, (n + m)(i - 1) \times 0.02)$ for $i = 2, n = 2$ and $i = 3, n = 3$ and so on, $\sigma(b) = 2(\sigma(a))$, $\sigma(c) = ((3(n + m) + 3i -$

$2) \times 0.01, (3(n+m)+3i-2) \times 0.02)$, $i = 1, 2, \cdots$ and $\sigma(d) = 4(\sigma(a))$. The pendent vertices of vertex a are labeled using $\sigma(a_1) = \sigma(a) - \mu(a, a_1)$, $\sigma(a_2) = \sigma(a) - \mu(a, a_2)$. Similarly, the other pendent vertices of b,c,d are labeled. The intuitionistic fuzzy pendent edge memberships are defined by $\mu(a, a_i) = (i \times 0.01, i \times 0.02)$, $\mu(b, b_j) = ((n+j) \times 0.01, (n+j) \times 0.02)$, $\mu(c, c_k) = ((2n+(m-1)+k) \times 0.01, (2n+(m-1)+k) \times 0.02)$, $\mu(d, d_l) = ((3n+(m-1)+l) \times 0.01, (3n+(m-1)+l) \times 0.02)$ for $1 \leq i, j, k, l \leq n$ and $m = 4$. The edge memberships of cycle C_4 will be attained by the absolute difference between the adjacent vertices of C_4. As a result, every edge membership value satisfy the definition of neutrosophic edge labeling and also each edge and vertex membership values are found to be distinct. Therefore, the corona graph $C_4 \odot K_n$ admits an intuitionistic fuzzy edge-vertex graceful labeling.

4. Neutrosophic Vertex/Edge-Vertex Graceful Labeling Graphs

Definition 4.1.
A generalized butterfly graph with neutrosophic labeling is called as generalized neutrosophic butterfly graph which comprises of a vertex set $V(v) = \{v_c\} \cup \{v_i/1 \leq i \leq 2n\}$ *and the edge set* $E(e) = \{e_j/1 \leq j \leq 4n-2\} = \{v_cv_i/1 \leq i \leq 2n\} \cup \{v_iv_{i+1}/1 \leq i \leq 2n-1\}$ *such that* $\mu(v_cv_i) > 0$ *for* $1 \leq i \leq 2n$ *and* $\mu(v_iv_{i+1}) > 0$, *for* $1 \leq i \leq 2n-1$. *If all the vertex values corresponding to each membership of a neutrosophic generalized butterfly graph are distinct, then it is called a generalized neutrosophic vertex butterfly graph.*

Definition 4.2.
A double wheel graph DW_n *with neutrosophic labeling is called as neutrosophic double wheel graph which is the join of two cycle graphs of same size n and* K_1, *i.e.* $2C_n + K_1$, *where the vertices of the two cycles are connected to a common central vertex. If every vertex value are distinct by each membership and graceful labeling exists, then it is called a neutrosophic vertex graceful double wheel graph.*

Definition 4.3.
The helm graph H_n *is obtained by attaching pendent edges to each vertex of the n-cycle of a wheel graph. If all the vertex(edge) values are distinct by each membership of a neutrosophic helm graph then* H_n *admits neutrosophic vertex(edge) graceful labeling.*

Definition 4.4.
The bistar graph $B_{m,n}$ *is by joining m pendent edges to one end of* K_2 *and n pendent edges to other end of* K_2. *If all the vertex(edge) values are distinct by each membership of neutrosophic bistar graph then* $B_{m,n}$ *admits neutrosophic vertex(edge) graceful labeling.*

Theorem 4.5.
Every generalized neutrosophic butterfly graph $BF_n, n \geq 2$ *admits a neutrosophic vertex graceful labeling.*

Proof. Let us consider a neutrosophic butterfly graph $BF_n, n \geq 2$, which consists of vertex set $V(v) = \{v_c\} \cup \{v_i/1 \leq i \leq 2n\}$ and the edge set $E(e) = \{e_j/1 \leq j \leq 4n-2\} = \{VV_i/1 \leq i \leq 2n\} \cup \{V_iV_{i+1}/1 \leq i \leq 2n-1\}$ such that $\mu(VV_i) > 0$ for $1 \leq i \leq 2n$ and $\mu(V_iV_{i+1}) > 0$, for $1 \leq i \leq 2n-1$. Let V be the central vertex and V_i denotes the other vertices of BF_n. In a generalized neutrosophic butterfly graph $BF_n, n \geq 2$, the central vertex(V) labeling $\sigma : V \to [0,1]$ satisfies the condition that if each membership value of V starts from $\frac{n-1}{10^k}, \frac{n}{10^k}, \frac{n+1}{10^k}$, etc., where $k \in \mathbb{N}$ then the generalized neutrosophic butterfly graph is said to have neutrosophic vertex graceful labeling. Fix $\mu(V, V_1) = (T(V, V_1), I(V, V_1), F(V, V_1)) = (\frac{0.01}{10^l}, \frac{0.02}{10^l}, \frac{0.03}{10^l})$, where $l \in \mathbb{W}$

to obtain the value of $\sigma(V_1)$ by the expression, $\sigma(V_1) = \sigma(V) - \mu(V, V_1)$. The labeling for other vertices $\sigma : V_i \to [0,1]$ is defined by $\sigma(V_{i+1}) = \sigma(V) - \mu(V, V_{i+1})$, where $\mu(V, V_{i+1}) = (T(V, V_i) + (\frac{0.01}{10^l})(i+1), I(V, V_i) + (\frac{0.02}{10^l})(i+1), F(V, V_i) + (\frac{0.03}{10^l})(i+1))$, where i= 1 to consecutive integer sum of n, $l \in \mathbb{W}$. If the generalized neutrosophic butterfly graph is vertex graceful then the outer edges are labeled using $\mu(V_i, V_{i+1}) = \sigma(V_i) - \sigma(V_{i+1})$, where $i \neq n, 2n$ and i= 1 to consecutive integer sum of n. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every generalized neutrosophic butterfly graph $BF_n, n \geq 2$ is a neutrosophic vertex graceful labeling graph.

Example : The butterfly graphs BF_2 & BF_3 satisfy the vertex labeling property for

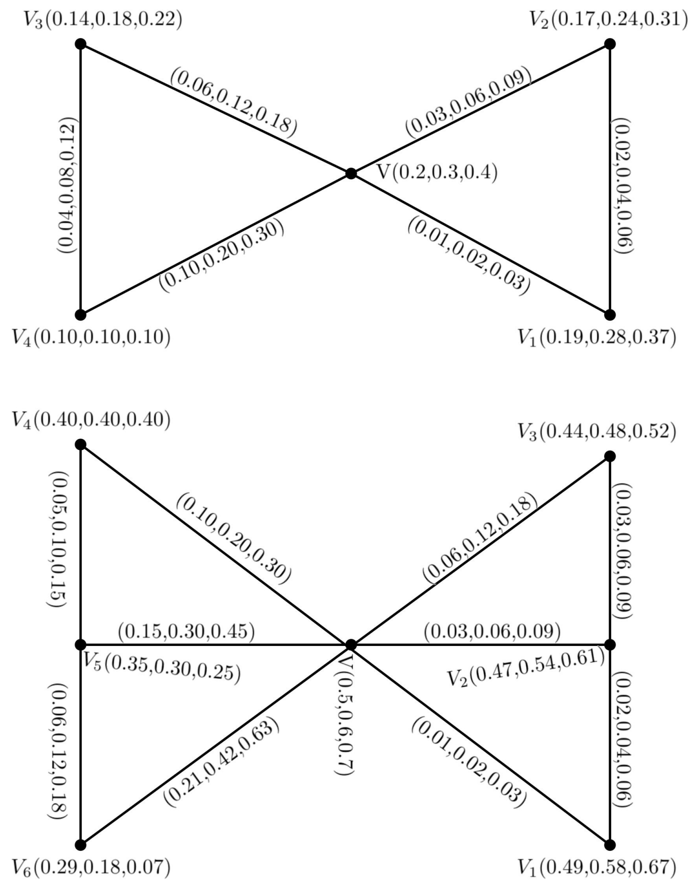

FIGURE 5. Neutrosophic vertex graceful BF_2 & BF_3 graphs

k,l=1. For other order of butterfly graphs, we have to increase the value of k and l to preserve the total sum condition for vertex and edge membership of neutrosophic graph. The following examples of neutrosophic vertex graceful labeling bears the same condition.

Theorem 4.6.
Every neutrosophic double wheel graph $DW_n, n \geq 4$ admits a neutrosophic vertex graceful labeling.

Proof. Consider a neutrosophic double wheel graph $DW_n, n \geq 4$, which comprises of vertex set $V(v) = \{V\} \cup \{V_i/1 \leq i \leq 2n\}$ and the edge set $E(e) = \{E_j/1 \leq j \leq$

$4n\} = \{VV_i/1 \leq i \leq 2n\} \cup \{V_iV_{i+1}/1 \leq i \leq 2n\} \cup \{V_nV_1, V_{2n}V_{n+1}\}$ such that all edges are greater than zero. Let V be the central vertex and V_i denotes the inner and outer cycle vertices of DW_n. In a neutrosophic double wheel graph $DW_n, n \geq 4$, the central vertex(V) labeling $\sigma : V \to [0,1]$ satisfies the condition that if each membership value of V starts from $\frac{n-3}{10^k}, \frac{n-2}{10^k}, \frac{n-1}{10^k}$, etc., where $k \in \mathbb{N}$ then the neutrosophic double wheel graph is said to have neutrosophic vertex graceful labeling. The edges connected to the central vertex of DW_n can be obtained from, $\mu(V, V_i) = (\frac{0.01}{10^l} \times i, \frac{0.011}{10^l} \times i, \frac{0.11}{10^l} \times i)$, where i= 1 to $2n$, $l \in \mathbb{W}$. The labeling value of other vertices are obtained by using the expression, $\sigma(V_{i+1}) = \sigma(V) - \mu(V, V_{i+1})$, where i= 0 to $2n-1$. The outer edge values $\mu(V_i, V_{i+1})$ of both the cycles in DW_n are attained by the absolute difference between its adjacent vertices. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every neutrosophic double wheel graph $DW_n, n \geq 4$ is a neutrosophic vertex graceful labeling graph.

Example :

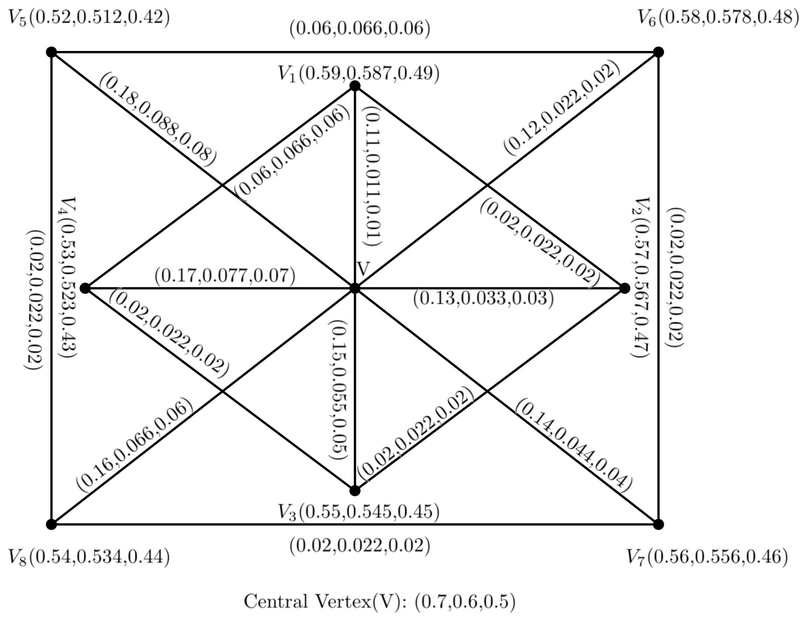

FIGURE 6. Neutrosophic vertex graceful DW_4 graph

Theorem 4.7.
Every neutrosophic helm graph $H_n, n \geq 3$ admits a neutrosophic edge-vertex graceful labeling.

Proof. Consider a neutrosophic helm graph $H_n, n \geq 3$, which has the vertex set $V(v) = \{V\} \cup \{V_i/1 \leq i \leq 2n\}$ and edge set $E(e) = \{E_j/1 \leq j \leq 3n\} = \{VV_i/1 \leq i \leq n\} \cup \{V_iV_{i+1}/1 \leq i \leq n-1\} \cup \{V_nV_1\} \cup \{V_iV_{i+3}/1 \leq i \leq n\}$ such that these edges are greater than zero and $\mu(V_{n+1}, V_{n+2}) = 0$. Let V be the central vertex and V_i denotes the pendent vertices and vertices of the cycle in H_n. In a neutrosophic helm graph $H_n, n \geq 3$, the central vertex(V) labeling $\sigma : V \to [0,1]$ satisfies the condition that if each membership value of V starts from $\frac{n+2}{10^k}, \frac{n+3}{10^k}, \frac{n+4}{10^k}$, etc., where $k \in \mathbb{N}$. The neutrosophic helm graph is said to have neutrosophic vertex graceful labeling if it bears the following edge labels: For $1 \leq i \leq n$ and $l \in \mathbb{W}$, define $\mu(V_i, V_{i+1}) = (i \times \frac{0.02}{10^l}, i \times \frac{0.03}{10^l}, i \times \frac{0.04}{10^l})$, $\mu(V_i, V_{n+i}) = ((n+i) \times \frac{0.02}{10^l}, (n+i) \times \frac{0.03}{10^l}, (n+i) \times \frac{0.04}{10^l})$. Fix $\mu(V, V_1) = (\frac{0.01}{10^l}, \frac{0.02}{10^l}, \frac{0.03}{10^l})$, $l \in \mathbb{W}$ and using this value, generate the other edges connected to central vertex by $\mu(V, V_{i+i}) = \mu(V_i, V_{i+i}) + \mu(V, V_i)$. The labels for vertices can be obtained by using $\sigma(V_i) = \sigma(V) - \mu(V, V_i)$ and $\sigma(V_{i+n}) = \sigma(V_i) - \mu(V_i, V_{n+i})$, for $1 \leq i \leq n$. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every neutrosophic double helm graph $H_n, n \geq 3$ is a neutrosophic vertex graceful labeling graph.

Example :

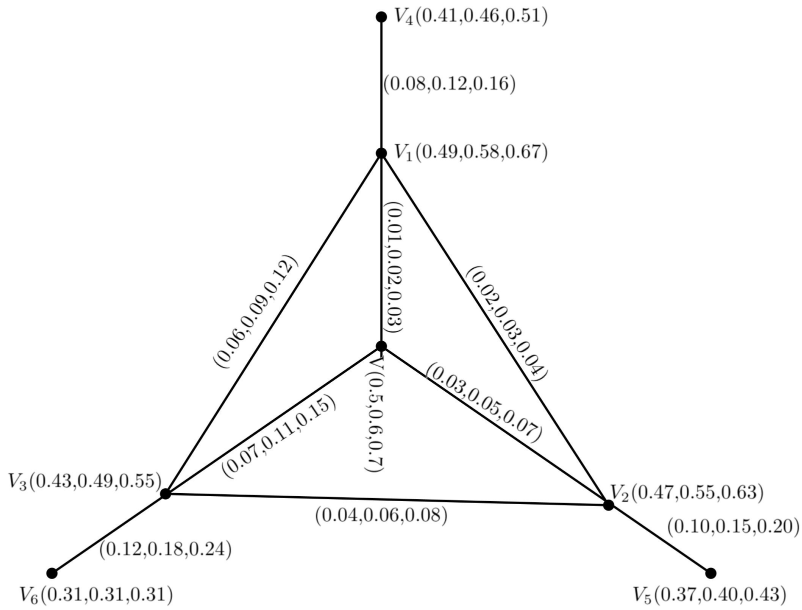

FIGURE 7. Neutrosophic vertex graceful H_3 graph

Theorem 4.8.
Every neutrosophic bistar graph $B_{m,n}, m, n \geq 1$ admits a neutrosophic edge-vertex graceful labeling.

Proof. Let $B_{m,n}, m, n \geq 1$ be a neutrosophic bistar graph, which consists of the vertex set $V(v) = \{U, V\} \cup \{U_i/1 \leq i \leq m\} \cup \{V_j/1 \leq j \leq n\}$ and edge set $E(e) = \{E_k/1 \leq k \leq 2n+1\} = \{UV\} \cup \{UU_i/1 \leq i \leq m\} \cup \{VV_j/1 \leq j \leq n\}$ such that these edges are greater than zero and $\mu(U_i, U_{i+1}) = \mu(V_j, V_{j+1}) = 0$ for $1 \leq i \leq m$, $1 \leq i \leq n$. Let U, V be the central vertices and U_i, V_j denotes the pendent vertices from U & V respectively. In a neutrosophic bistar graph $B_{m,n}, m, n \geq 1$, let $\sigma : U \to [0,1]$ & $\sigma : V \to [0,1]$ be the labeling for central vertices U and V respectively, which satisfies the condition that if each membership value of U starts from $\frac{m+n+1}{100^k}, \frac{m+n+3}{100^k}, \frac{m+n+5}{100^k}$, etc., where $k \in \mathbb{N}$ and V starts from $\frac{2(m+n+1)}{100^k}, \frac{2(m+n+3)}{100^k}, \frac{2(m+n+5)}{100^k}$, etc., where $k \in \mathbb{N}$. The neutrosophic vertex graceful labeling of neutrosophic bistar graph is assured by the following edge labels: For $1 \leq i \leq m$ & $k \in \mathbb{N}$, define $\mu(U, U_i) = (\frac{i}{100^k}, \frac{i+1}{100^k}, \frac{i+2}{100^k})$ and for $1 \leq j \leq n$ & $k \in \mathbb{N}$, we have $\mu(V, V_j) = (\frac{i}{100^k}, \frac{i+1}{100^k}, \frac{i+2}{100^k})$ where $i = m+1, m+2, \cdots$. The edge $\mu(U, V)$ connecting the central vertices U and V is obtained by the absolute difference between these vertices.

The label value for pendent vertices is given by $\sigma(U_i) = \sigma(U) - \mu(U, U_i)$, for $1 \leq i \leq m$ and $\sigma(V_j) = \sigma(V) - \mu(V, V_j)$, for $1 \leq i \leq n$. Since, each edge membership value are less than the corresponding membership value of vertices and all the vertices are distinct by each membership, we conclude that every neutrosophic bistar graph $B_{m,n}, m, n \geq 1$ is a neutrosophic vertex graceful labeling graph.

Example :

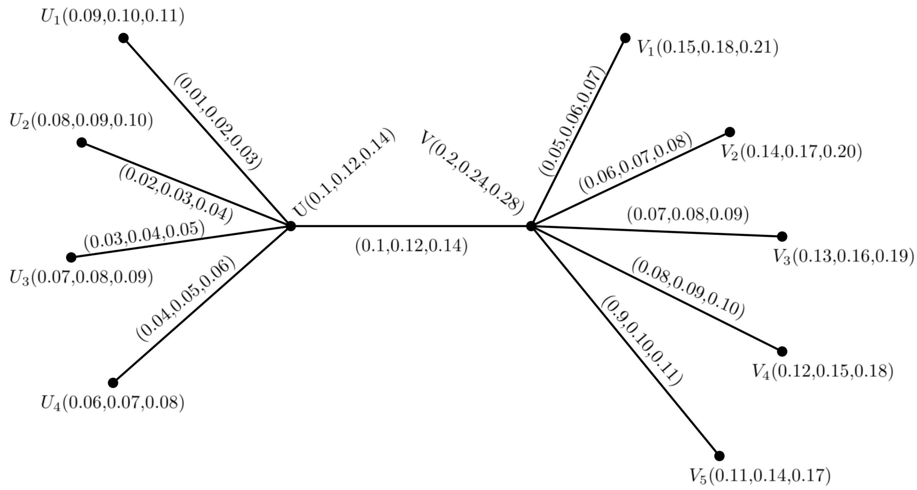

FIGURE 8. Neutrosophic edge-vertex graceful $BF_{4,5}$ graph

5. Application

Energy System: Among the power generation systems, thermal power plants produce a high range of electricity by burning fossil fuels such as coal, natural gas, etc. The generated electricity is transmitted and distributed to industries, transport systems, and household consumers through HT & LT lines, substations, and transformers. Globally, it is the most centralized and primary source of electric power generation. Here, we consider the central vertex as the thermal power plant in a town or city and the other vertices as the cities that need the significant electric power generated through power plants. The edges connecting the central vertex and the other vertices are considered to be the villages, which lie in the route between the power plant and the city. The edges between the adjacent vertices are taken as villages, which receive the difference amount of electricity(through the cities) from the villages situated in the pathway between these cities and the power plant. The truth membership of the central vertex is assumed to be the percentage of electricity that is equally transmitted to all cities. The indeterminacy membership of the central vertex is taken to be the percentage of electricity that is equally shared but not correctly received due to fluctuations, cable failure, climatic influence, etc. The central vertex's false membership is thought to be the same amount of electricity that is not sent because of shutdowns and not enough fossil fuel sources. The memberships of each edge (village) between the central vertex and the city are considered to be the amount of electricity consumed, uncertain electricity, and unsupplied electricity, respectively. Likewise, the membership of each city can be taken, and the membership of each edge (village) between the cities is assumed to be the same as the edges between the power plant and the city, with a difference amount of electricity from villages situated between these cities and the power plant. Through this model, a high range of electricity can be equally distributed among the cities and an adequate amount to villages. This model resembles the structure of a wheel graph, and it can be matched with Figure 2.

6. Conclusion

In this paper, the study of vertex and edge-vertex graceful labeling on some intuitionistic and neutrosophic graphs has been carried out by introducing the definition of vertex and edge-vertex graceful labeling in the intuitionistic and neutrosophic types of graphs. In the future, extended work can be done by applying graceful labeling to other kinds of neutrosophic graphs like, pythagorean neutrosophic graphs, fermatean neutrosophic graphs, etc. Further, we have planned to execute different types of labeling on neutrosophic graphs.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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