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SOME TOPOLOGICAL PROPERTIES OF A GENERALIZED S-METRIC SPACE TOGETHER WITH SOME FIXED POINT RESULTS AND THEIR APPLICATIONS

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Abstract. In this paper, the concept of $S_b^{(p,q)}$-metric space is introduced as a generalization of S-metric space, S_b-metric space, (p,q)-metric space and $QM(n,b)$-metric space. A topology is formed with the help of $S_b^{(p,q)}$-metric and some topological properties are studied to establish Cantor's intersection theorem. Sehgal-Guseman, Reich and Akram type fixed point theorems are proved over such spaces. Several examples are given in support of our results. Moreover, the proven fixed point theorems are applied to well-posedness and Ulam-Hyers stability of fixed point problems.

Keywords: Fixed point, $S_b^{(p,q)}$-metric space, Ulam-Hyers stability and well-posedness of fixed point problems

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1. Introduction and Preliminaries

In the year 2012, Sedghi et al. [17] introduced the concept of S-metric spaces. A good number of research works have been done over such spaces for establishing various types of fixed point theorems (see [7]). Generalization of S-metric structure is an interesting work in the field of fixed point theory. Several new structures have been introduced with the help of such concept (see [19, 1, 20, 5, 4, 13, 3, 16]).

In 2016, N. Souayah and N. Mlaiki [19] initiated the notion of S_b-metric spaces combining the concept of b-metric space and the concept of S-metric space. The class of S_b-metric spaces is much bigger than the class of S-metric spaces. However there are several differences between the properties of an S-metric space and an S_b-metric space, one of which is that an S-metric is always symmetric but an S_b-metric may not so. Here we recall the definitions of S-metric space and S_b-metric space.

Definition 1.1. **(S-metric space)**[17] *Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

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(S1) $S(x,y,z) = 0$, *if and only if* $x = y = z$,
(S2) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.
The pair (X,S) *is called an* S*-metric space.*

Example 1.1. [17, 7] *(a) Let* $\mathbb{R}$ *be the real line. Then* $S : \mathbb{R}^3 \to [0,\infty)$ *defined by* $S(x,y,z) = |x-y| + |y-z| + |x-z|$ *for all* $x,y,z \in \mathbb{R}$ *is an* S*-metric on* $\mathbb{R}$.
(b) Let $\mathbb{R}$ *be the real line. Then* $S : \mathbb{R}^3 \to [0,\infty)$ *defined by* $S(x,y,z) = |2x-y-z|+|y-z|$ *for all* $x,y,z \in \mathbb{R}$ *is an* S*-metric on* $\mathbb{R}$.

Definition 1.2. **(S_b-metric space)**[19] *Let* X *be a nonempty set and let* $s \geq 1$ *be a given number. A function* $S_b : X^3 \to [0,\infty)$ *is said to be* S_b-*metric if and only if for all* $x,y,z,t \in X$, *the following conditions holds:*
(S_b*1)* $S_b(x,y,z) = 0$ *if and only if* $x = y = z$,
(S_b*2)* $S_b(x,y,z) \leq s[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)]$.
The pair (X,S_b) *is called an* S_b-*metric space.*

Example 1.2. *Let us consider* $X = \mathbb{R}$ *and let* $S^* : X^3 \to [0,\infty)$ *be defined by* $S^*(x,y,z) = (|2x-y-z| + |y-z|)^2$ *for all* $x,y,z \in X$. *Then* X *is an* S_b*-metric space for* $s = 4$ *but not an* S*-metric space.*

In 2015, Abbas et al. [1] have introduced an interesting concept of $n-$tuple ($n \geq 2$) metric space known as $A-$metric space as a generalization of metric spaces and $S-$metric spaces [17].

Definition 1.3. **(A-metric space)**[1] *Let* X *be a nonempty set,* $n \geq 2$ *and* $A : X^n \to [0,\infty)$ *be a mapping. Then function* A *is said to be an* $A-$*metric if it satisfies the following conditions:*
$(A1)$ $A(x_1,x_2,...,x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n$,
$(A2)$ $A(x_1,x_2,...,x_n) \leq \sum_{i=1}^n A(x_i,x_i,...,(x_i)_{n-1},a)$ *for all* $x_1,x_2,...,x_n,a \in X$.

Example 1.3. [1] *Let* $X = \mathbb{R}$. *Define* $A : X^n \to [0,\infty)$ *by*

$$\begin{aligned} A(x_1,x_2,...,x_n) &= |\sum_{i=2}^n x_i - (n-1)x_1| + |\sum_{i=3}^n x_i - (n-2)x_2| + ... \\ &\quad +|\sum_{i=n-1}^n x_i - 2x_{n-2}| + |x_n - x_{n-1}| \end{aligned} \tag{1}$$

for all $x_1,x_2,...,x_n \in X$. *Then* (X,A) *is an* $A-$*metric space.*

In the year 2017, motivated from the definition of $A-$metric space Ughade et al. [20] established the concept of A_b-metric space with the help of $b-$triangle inequality.

Recently Dey et al. [5] have introduced the concept of $QM(n,b)$-metric space which generalizes several known metric type structures including S-metric, S_b-metric and A-metric spaces. This metric structure is an extension of the quasi metric space.

Definition 1.4. ***($QM(n,b)$-metric space)***[5] *Let* X *be a nonempty set,* $n \geq 2$, $b \geq 1$ *and* $u : X^n \to [0,\infty)$ *be a mapping. Then function* u *is said to be an* $QM(n,b)$*-metric if it satisfies the following conditions:*
$(u1)$ $u(x_1,x_2,...,x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n$,
$(u2)$ $u(x_1,x_2,...,x_n) \leq b\sum_{i=1}^n u(x_i,x_i,...,(x_i)_{n-1},a)$ *for all* $x_1,x_2,...,x_n,a \in X$.

Example 1.4. [5] *Let* (X,A) *be an* n*-variable* A*-metric space and let* $u : X^n \to [0,\infty)$ *be defined by* $u(x_1,x_2,...,x_n) = A(x_1,x_2,...,x_n)^p$, $p > 1$. *Then* u *is a* $QM(n,b)$*-metric space for* $b = n^{p-1}$.

Example 1.5.

Bendahmane et al. [4] recently developed the concept of (p, q)-metric space as a generalization of metric space and S-metric space. This space also extends the concept of A-metric space.

Definition 1.5. ***((p, q)-metric space)***[4] *Let X be a nonempty set and p, q be two positive integers. Let $S^{(p,q)} : X^{p+q} \to [0, \infty)$ be a mapping which satisfies the following conditions:*
(i) $S^{(p,q)}(x_1, x_2, ..., x_{p+q}) = 0$ if and only if $x_1 = x_2 = ... = x_{p+q}$;
(ii) for all $(x_1, x_2, ..., x_{p+q}) \in X^{p+q}$ and $(a_1, a_2, ..., a_q) \in X^q$

$$S^{(p,q)}(x_1, x_2, ..., x_{p+q}) \leq \sum_{i=1}^{q} \sum_{j=1}^{p+q} S^{(p,q)}([x_j]_p, [a_i]_q),$$

where $[x_j]_p = (x_j, x_j, ..., x_j)_p$ and $[a_i]_q = (a_i, a_i, ..., a_i)_q$ for all $j = 1$ to $p + q$ and $i = 1$ to q. Then $S^{(p,q)}$ is called a (p, q)-metric and the pair $(X, S^{(p,q)})$ a (p, q)-metric space.

Example 1.6. [4] *(a) Let X be any nonempty set and $n \geq 2$ be a positive integer. Define $S^{(p,q)} : X^n \to [0, \infty)$ as*

$$S^{(p,q)}(x_1, x_2, ..., x_n) = \begin{cases} 0, & \text{if } x_1 = x_2 = ... = x_n; \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

Then $S^{(p,q)}$ is a (p, q)-metric for any $p, q \geq 1$ satisfying $p + q = n$.
(b) Let (X, D) be a metric space. Then the mapping $S^{(p,q)} : X^4 \to [0, \infty)$ is defined by $S^{(p,q)}(t, u, v, w) = D(t, u) + D(t, v) + D(v, w)$ for all $t, u, v, w \in X$ is a $(2, 2)$-metric on X.

Several authors have generalized the fixed point theorem due to Banach using different types of contraction type mappings (see [11]). The following fixed point theorems are some of them.
In 1969, Sehgal [18] proved the following interesting fixed point theorem.

Theorem 1.1. [18] *Let (X, d) be a complete metric space, $l \in [0, 1)$ and $T : X \to X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d(T^{k(x)}x, T^{k(x)}y) \leq ld(x, y) \text{ for all } y \in X, \tag{3}$$

then T has a unique fixed point v in X. Moreover, for any $x \in X$, $\lim_{n\to\infty} T^n x = v$.

Later in 1970, Guseman [8] extended the result of Sehgal in a unique way, where he exclude the continuity condition of T from the above theorem.
In 1971, Reich [12] proved the following theorem.

Theorem 1.2. [12] *In a complete metric space (X, d), a map $T : X \to X$ satisfying*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty), \tag{4}$$

where a, b, c are all non-negative reals with $a + b + c < 1$ then T has a unique fixed point in X.

A Banach contraction mapping is always continuous on the underlying space but a mapping T satisfying the contractive condition of the above fixed point theorem need not be continuous.
Akram et al.(see [2]) proved a fixed point theorem for a class of generalized contraction type mappings, which is given below.
Let A be a class of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying

(A_1) : α is continuous on the set $\mathbb{R}_+^3$ of all triplets of nonnegative reals(with respect to the Euclidean metric on $\mathbb{R}^3$).

(A_2) : $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$, for all a, b.

Theorem 1.3. [2] *If a self mapping T on a metric space (X,d) satisfies the condition*

$$d(Tx,Ty) \leq \alpha(d(x,y), d(x,Tx), d(y,Ty)) \tag{5}$$

for all $x, y \in X$ and for some α in A, then T has a unique fixed point in X.

Motivated on the survey of literature of the above generalized metric spaces and several generalized contraction type mappings, we introduce a new extension of S-metric space and investigate fixed points of various generalized contraction type mappings on such spaces.

2. Introduction to $S_b^{(p,q)}$-metric space

We now start this section with a new type of metric structure. The definition is given as follows.

Definition 2.1. *Let X be a nonempty set and $p, q \geq 1$ be two positive integers. A mapping $\rho : X^{p+q} \to [0,\infty)$ is said to be an $S_b^{(p,q)}$-metric space if there exists $b \geq 1$ such that*

(i) $\rho(x_1, x_2, ..., x_{p+q}) = 0$ if and only if $x_1 = x_2 = ... = x_{p+q}$;

(ii) for all $(x_1, x_2, ..., x_{p+q}) \in X^{p+q}$ and $(a_1, a_2, ..., a_q) \in X^q$

$$\rho(x_1, x_2, ..., x_{p+q}) \leq b \sum_{i=1}^{q} \sum_{j=1}^{p+q} \rho([x_j]_p, [a_i]_q),$$

where $[x_j]_p = (x_j, x_j, ..., x_j)_p$ and $[a_i]_q = (a_i, a_i, ..., a_i)_q$ for all $j = 1$ to $p+q$ and $i = 1$ to q.

Definition 2.2. *An $S_b^{(p,q)}$-metric space (X,ρ) is called symmetric if $\rho([x]_p, [y]_q) = \rho([y]_p, [x]_q)$ for all $x, y \in X$.*

Remark 2.1. *(i) Any metric space is an $S_b^{(p,q)}$-metric space for $p = 1$, $q = 1$ and $b = 1$;*

(ii) Any b-metric space is an $S_b^{(p,q)}$-metric space for $p = 1$ and $q = 1$;

(iii) Any S-metric space is an $S_b^{(p,q)}$-metric space for $p = 2$, $q = 1$ and $b = 1$;

(iv) Any S_b-metric space is an $S_b^{(p,q)}$-metric space for $p = 2$ and $q = 1$;

(v) Any A-metric space is an $S_b^{(p,q)}$-metric space for $q = 1$ and $b = 1$;

(vi) Any A_b-metric space is an $S_b^{(p,q)}$-metric space for $q = 1$;

(vii) Any $QM(n,b)$-metric space is an $S_b^{(p,q)}$-metric space for $q = 1$;

(viii) Any (p,q)-metric space is an $S_b^{(p,q)}$-metric space for $b = 1$.

Example 2.1. *(i) Let $X = \mathbb{R}$ and $\rho : X^4 \to [0,\infty)$ be defined by*

$$\rho(x,y,z,w) = |x-y|^2 + |y-z|^2 + |z-w|^2 \text{ for all } x,y,z,w \in X. \tag{6}$$

Then (X,ρ) is a symmetric $S_4^{(3,1)}$-metric space.

(ii) Let $X = \mathbb{R}$ and $\rho : X^4 \to [0,\infty)$ be defined by

$$\rho(x,y,z,w) = |x-y|^2 + |x-z|^2 + |z-w|^2 \text{ for all } x,y,z,w \in X. \tag{7}$$

Then (X,ρ) is a symmetric $S_2^{(2,2)}$-metric space.

(iii) Let $X = \mathbb{R}$ and $\rho : X^4 \to [0, \infty)$ be defined by

$$\begin{cases} \rho(x, y, z, w) = 0, & \text{if } x = y = z = w; \\ \rho(0, 0, 1, 1) = 10; \\ \rho(1, 1, 0, 0) = 3; \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

Then (X, ρ) is an $S_1^{(2,2)}$-metric space which is not symmetric.

Definition 2.3. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ is said to converge to x if $\rho([x]_p, [x_n]_q) \to 0$ as $n \to \infty$;
(ii) $\{x_n\}$ is said to be Cauchy if $\rho([x_n]_p, [x_m]_q) \to 0$ as $n, m \to \infty$;
(iii) X is called complete if any Cauchy sequence in X is convergent.

Now we give some propositions which will be used in our main results.

Proposition 2.1. *In an $S_b^{(p,q)}$-metric space (X, ρ) we have*
(a) $(bq^2)^{-1}\rho([y]_p, [x]_q) \leq \rho([x]_p, [y]_q) \leq (bq^2)\rho([y]_p, [x]_q)$ for all $x, y \in X$,
(b) $\rho([x]_p, [y]_q) \leq bpq\rho([x]_p, [a]_q) + bq^2\rho([y]_p, [a]_q)$ for all x, y and $a \in X$.

Proof. (a) Using the inequality (ii) of Definition 2.1 for any $x, y \in X$ if we choose $x_1 = x_2 = ... = x_p = x$, $x_{p+1} = x_{p+2} = ... = x_{p+q} = y$ and $a_1 = a_2 = ... = a_q = x$ then we get

$$\begin{aligned} \rho([x]_p, [y]_q) &\leq bq(p\rho([x]_p, [x]_q) + q\rho([y]_p, [x]_q)) \\ &= bpq.0 + bq^2\rho([y]_p, [x]_q) \\ &= bq^2\rho([y]_p, [x]_q). \end{aligned} \tag{9}$$

Interchanging x and y in (9) we get $\rho([y]_p, [x]_q) \leq bq^2\rho([x]_p, [y]_q)$ i.e. $(bq^2)^{-1}\rho([y]_p, [x]_q) \leq \rho([x]_p, [y]_q)$. Combining these two we get the required result.

(b) For any $x, y, a \in X$ if we take $x_1 = x_2 = ... = x_p = x$, $x_{p+1} = x_{p+2} = ... = x_{p+q} = y$ and $a_1 = a_2 = ... = a_q = a$ then we get

$$\begin{aligned} \rho([x]_p, [y]_q) &\leq bq(p\rho([x]_p, [a]_q) + q\rho([y]_p, [a]_q)) \\ &= bpq\rho([x]_p, [a]_q) + bq^2\rho([y]_p, [x]_q). \end{aligned} \tag{10}$$

□

Remark 2.2. *From (a) of Proposition 2.1 it follows that a sequence $\{x_n\}$ is convergent to some $x \in X$ if $\rho([x_n]_p, [x]_q) \to 0$ as $n \to \infty$.*

Proposition 2.2. *In an $S_b^{(p,q)}$-metric space (X, ρ) if a sequence $\{x_n\}$ converges to $x, y \in X$ then $x = y$.*

Proof. From Proposition 2.1, we have

$$\rho([x]_p, [y]_q) \leq bpq\rho([x]_p, [x_n]_q) + bq^2\rho([y]_p, [x_n]_q) \text{ for all } n \geq 1. \tag{11}$$

Letting $n \to \infty$ we get $\rho([x]_p, [y]_q) = 0$ and thus $x = y$. □

Proposition 2.3. *In an $S_b^{(p,q)}$-metric space (X, ρ) a convergent sequence $\{x_n\}$ is also Cauchy.*

Proof. Let $\{x_n\}$ be a sequence in X which converges to some $x \in X$. Then from Proposition 2.1, we get

$$\rho([x_n]_p, [x_m]_q) \leq bpq\rho([x_n]_p, [x]_q) + bq^2\rho([x_m]_p, [x]_q) \text{ for all } n, m \geq 1. \tag{12}$$

Taking $n, m \to \infty$, from (12) we conclude that $\rho([x_n]_p, [x_m]_q) \to 0$ and therefore $\{x_n\}$ is Cauchy in X. □

Proposition 2.4. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $\{x_n\}$ be a sequence in X converges to some $x \in X$. Then for all $y \in X$ we have*

$$(b^2q^4)^{-1}\rho([x]_p, [y]_q) \leq \liminf_{n\to\infty} \rho([x_n]_p, [y]_q) \leq \limsup_{n\to\infty} \rho([x_n]_p, [y]_q) \leq (b^2q^4)\rho([x]_p, [y]_q). \tag{13}$$

Proof. From Proposition 2.1, we have

$$\begin{aligned}\rho([x_n]_p, [y]_q) &\leq bpq\rho([x_n]_p, [x]_q) + bq^2\rho([y]_p, [x]_q)\\ &\leq bpq\rho([x_n]_p, [x]_q) + b^2q^4\rho([x]_p, [y]_q) \text{ for all } n \geq 1 \text{ and } y \in X.\end{aligned} \tag{14}$$

Letting $n \to \infty$ we get $\limsup_{n\to\infty} \rho([x_n]_p, [y]_q) \leq b^2q^4\rho([x]_p, [y]_q)$. Interchanging the roles of x and x_n in (14) we get

$$\rho([x]_p, [y]_q) \leq bpq\rho([x]_p, [x_n]_q) + b^2q^4\rho([x_n]_p, [y]_q) \text{ for all } n \geq 1 \text{ and } y \in X. \tag{15}$$

Therefore by taking $n \to \infty$ we see that $(b^2q^4)^{-1}\rho([x]_p, [y]_q) \leq \liminf_{n\to\infty} \rho([x_n]_p, [y]_q)$. Hence combining these two

$$(b^2q^4)^{-1}\rho([x]_p, [y]_q) \leq \liminf_{n\to\infty} \rho([x_n]_p, [y]_q) \leq \limsup_{n\to\infty} \rho([x_n]_p, [y]_q) \leq (b^2q^4)\rho([x]_p, [y]_q). \tag{16}$$

□

3. A study on topological properties of $S_b^{(p,q)}$- metric space

Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $x \in X$, $r > 0$. Then we define open ball and closed ball with center at $x \in X$ and radius $r > 0$ respectively as below:

$$\begin{aligned}B^\rho(x, r) &= \{y \in X : \rho([y]_p, [x]_q) < r\},\\ \mathbf{B}^\rho(x, r) &= \{y \in X : \rho([y]_p, [x]_q) \leq r\}.\end{aligned} \tag{17}$$

Example 3.1. *Let us cobsider the $S_1^{(2,2)}$-metric space (X, ρ) given in Example 2.1. Then*

$$\begin{aligned}B^\rho(0, 1) &= \{y \in X : \rho([y]_2, [0]_2) < 1\} = \{0\},\\ \mathbf{B}^\rho(0, 1) &= \{y \in X : \rho([y]_2, [0]_2) \leq 1\} = \mathbb{R} \setminus \{1\}.\end{aligned} \tag{18}$$

Remark 3.1. *The collection $\tau_\rho = \{\emptyset\} \cup \{U_\rho(\neq \emptyset) \subset X :$ there exists $\epsilon_x > 0$ for every $x \in U_\rho$ such that $B^\rho(x, \epsilon_x) \subset U_\rho\}$ forms a topology on X.*

Definition 3.1. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $F \subset X$. Then F is said to be closed if there exists an open set $U_\rho \subset X$ such that $F = U_\rho^c$.*

Proposition 3.1. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $F \subset X$ be closed. Let $\{x_n\} \subset F$ be such that $x_n \to z$ as $n \to \infty$, then $z \in F$.*

Proof. If possible let $z \notin F$. Then $z \in F^c = U_\rho$, where U_ρ is open due to the Definition 3.1. So there exists $r > 0$ such that $B^\rho(z, r) \subset U_\rho$. Now $\lim_{n\to\infty} \rho([x_n]_p, [z]_q) = 0$ so for $r > 0$ there exists $N \in \mathbb{N}$ such that $\rho([x_n]_p, [z]_q) < r$ whenever $n \geq N$. Thus $x_n \in B^\rho(z, r) \subset U_\rho$ for all $n \geq N$, a contradiction. Hence, $z \in F$. □

Definition 3.2. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $B \subset X$. Then diam(B) = $\sup\{\rho([x]_p, [y]_q) : x, y \in B\}$.*

Definition 3.3. *In an $S_b^{(p,q)}$-metric space (X, ρ), a sequence $\{F_n\}$ of subsets of X is said to be nested sequence if $F_1 \supset F_2 \supset F_3 \supset ...$.*

Theorem 3.1. *(Analogue of Cantor's intersection theorem) Let (X, ρ) be a complete $S_b^{(p,q)}$-metric space and $\{F_n\}$ be a nested sequence of non-empty closed subsets of X such that $diam(F_n) \to 0$ as $n \to \infty$. Then the intersection $\cap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. Let $x_n \in F_n$ be arbitrary for all $n \in \mathbb{N}$. Since $\{F_n\}$ is decreasing, we have $\{x_n, x_{n+1}, ...\} \subset F_n$ for all $n \in \mathbb{N}$.

Now for any $n, m \in \mathbb{N}$ with $n, m \geq k$ we have $\rho([x_n]_p, [x_m]_q) \leq diam(F_k)$, $k \geq 1$. Let $\epsilon > 0$ be given. Then there exists some $l \in \mathbb{N}$ such that $diam(F_l) < \epsilon$ since $diam(F_n) \to 0$ as $n \to \infty$. From this it follows that $\rho([x_n]_p, [x_m]_q) < \epsilon$ whenever $n, m \geq l$. Therefore $\{x_n\}$ is Cauchy in X. By the completeness of X there exists $z \in X$ such that $\{x_n\}$ converges to z. Since $\{x_n, x_{n+1}, ...\} \subset F_n$ and F_n is closed for each $n \in \mathbb{N}$, using Proposition 3.1 we have $z \in \cap_{n=1}^{\infty} F_n$.

Next we prove the uniqueness of z. Let $y \in \cap_{n=1}^{\infty} F_n$ be another point, then $\rho([z]_p, [y]_q) > 0$. As $diam(F_n) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$diam(F_n) < \rho([z]_p, [y]_q) \leq diam(F_n)$$

for all $n \geq N_0$, a contradiction. Hence $\cap_{n=1}^{\infty} F_n = \{z\}$ and this completes the proof of our theorem. □

4. Some fixed point theorems

Lemma 4.1. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ satisfies*

$$\rho([x_i]_p, [x_{i+1}]_q) \leq c^i \Lambda \text{ for all } i \geq 1 \tag{19}$$

and for $0 \leq c < \frac{1}{b^2 q^4}$, $0 < \Lambda < \infty$ then $\{x_n\}$ is a Cauchy sequence in X.

Proof. If $c = 0$ then $\{x_n\}$ is clearly Cauchy in X. So let us assume that $c > 0$. Then for $1 \leq n < m$ we have,

$$\begin{aligned}
\rho([x_n]_p, [x_m]_q) &\leq bpq\rho([x_n]_p, [x_{n+1}]_q) + b^2q^4\rho([x_{n+1}]_p, [x_m]_q)[\text{Using Proposition 2.1}] \\
&\leq bpq\rho([x_n]_p, [x_{n+1}]_q) + b^2q^4\{bpq\rho([x_{n+1}]_p, [x_{n+2}]_q) + b^2q^4\rho([x_{n+2}]_p, [x_m]_q)\} \\
&= bpq\{\rho([x_n]_p, [x_{n+1}]_q) + b^2q^4\rho([x_{n+1}]_p, [x_{n+2}]_q)\}+ \\
&\quad (b^2q^4)^2\{bpq\rho([x_{n+2}]_p, [x_{n+3}]_q) + b^2q^4\rho([x_{n+3}]_p, [x_m]_q)\} \\
&= bpq\{\rho([x_n]_p, [x_{n+1}]_q) + b^2q^4\rho([x_{n+1}]_p, [x_{n+2}]_q) + ...+ \\
&\quad (b^2q^4)^{m-n-2}\rho([x_{m-2}]_p, [x_{m-1}]_q)\} + (b^2q^4)^{m-n-1}\rho([x_{m-1}]_p, [x_m]_q) \\
&\leq bpq\sum_{i=1}^{m-n}(b^2q^4)^{i-1}\rho([x_{n+i-1}]_p, [x_{n+i}]_q) \\
&\leq bpq\sum_{i=1}^{m-n}(b^2q^4)^{i-1}c^{n+i-1}\Lambda \\
&\leq bpq\sum_{i=1}^{m-n}(b^2q^4c)^{n+i-1}\Lambda \\
&\leq bpq\Lambda(b^2q^4c)^n\sum_{i=1}^{\infty}(b^2q^4c)^{i-1} \\
&= \frac{bpq\Lambda}{1 - b^2q^4c}(b^2q^4c)^n. \qquad (20)
\end{aligned}$$

Therefore by letting $n \to \infty$ we see that $\{x_n\}$ is a Cauchy sequence in X. □

Theorem 4.1. *(Analogue to Sehgal-Guseman fixed point theorem) Let (X, ρ) be a complete $S_b^{(p,q)}$-metric space and $T : X \to X$ be a mapping such that for each $y \in X$ there exists $n(y) \in \mathbb{N}$ such that*

$$\rho([T^{n(y)}x]_p, [T^{n(y)}y]_q) \leq \lambda\rho([x]_p, [y]_q) \text{ for all } x \in X, \qquad (21)$$

where $0 < \lambda < \frac{1}{b^2q^4}$. Then T has a unique fixed point in X.

Proof. Let $y \in X$ be chosen arbitrarily. Then there exists $n_0 \equiv n(y) \in \mathbb{N}$ such that (21) satisfied for all $x \in X$. Let $n \geq n_0$ then $n = n_0 s + t$, where $s \geq 1$ and $0 \leq t < n_0$. Therefore

$$\begin{aligned}
\rho([y]_p, [T^n y]_q) &\leq bpq\rho([y]_p, [T^{n_0} y]_q) + bq^2\rho([T^n y]_p, [T^{n_0} y]_q)[\text{Using Proposition 2.1}] \\
&\leq bpq\rho([y]_p, [T^{n_0} y]_q) + bq^2\lambda\rho([T^{n-n_0} y]_p, [y]_q) \\
&\leq bpq\rho([y]_p, [T^{n_0} y]_q) + b^2q^4\lambda\rho([y]_p, [T^{n-n_0} y]_q)[\text{Using Proposition 2.1}] \\
&\leq bpq\rho([y]_p, [T^{n_0} y]_q) + \\
&\quad b^2q^4\lambda\{bpq\rho([y]_p, [T^{n_0} y]_q) + b^2q^4\lambda\rho([y]_p, [T^{n-2n_0} y]_q)\} \\
&= bpq(1 + b^2q^4\lambda)\rho([y]_p, [T^{n_0} y]_q) + (b^2q^4\lambda)^2\rho([y]_p, [T^{n-2n_0} y]_q) \\
&\leq bpq[1 + b^2q^4\lambda + (b^2q^4\lambda)^2 + ... + (b^2q^4\lambda)^{s-1}]\rho([y]_p, [T^{n_0} y]_q) + \\
&\quad (b^2q^4\lambda)^s\rho([y]_p, [T^t y]_q) \\
&\leq bpq[1 + b^2q^4\lambda + (b^2q^4\lambda)^2 + ... + (b^2q^4\lambda)^{s-1} + (b^2q^4\lambda)^s]R(y) \\
&\leq \frac{bpq}{1 - b^2q^4\lambda}R(y) < \infty, \qquad (22)
\end{aligned}$$

where $R(y) = \{\rho([y]_p, [T^i y]_q) : 1 \leq i \leq n_0\}$. Thus $\sup_{n\geq 1}\rho([y]_p, [T^n y]_q) \leq \frac{bpq}{1-b^2q^4\lambda}R(y)$.

Let $x_0 \in X$ be taken as arbitrary. Then there exists $m_0 \equiv n(x_0) \in \mathbb{N}$ such that (21) satisfied for all $x \in X$.. Let us take $x_1 = T^{m_0}x_0$. Then we can get $m_1 \equiv n(x_1) \in \mathbb{N}$ such that (21) satisfied for all $x \in X$. Let us put $x_2 = T^{m_1}x_1$. Proceeding in a similar way we get a sequence $\{x_i\}$, where $x_i = T^{m_{i-1}}x_{i-1}$, $m_{i-1} \equiv n(x_{i-1})$ for all $i \geq 1$. Now for any $i \in \mathbb{N}$ by Proposition 2.1 we get,

$$\begin{aligned}
\rho([x_i]_p, [x_{i+1}]_q) &\leq bq^2\rho([x_{i+1}]_p, [x_i]_q) \\
&= bq^2\rho([T^{m_i}x_i]_p, [x_i]_q) \\
&= bq^2\rho([T^{m_{i-1}}(T^{m_i}x_{i-1})]_p, [T^{m_{i-1}}x_{i-1}]_q) \\
&\leq bq^2\lambda\rho([T^{m_i}x_{i-1}]_p, [x_{i-1}]_q) \\
&\ldots \\
&\leq bq^2\lambda^i\rho([T^{m_i}x_0]_p, [x_0]_q) \\
&\leq b^2q^4\lambda^i\rho([x_0]_p, [T^{m_i}x_0]_q) \\
&\leq \lambda^i\frac{b^3pq^5}{1 - b^2q^4\lambda}R(x_0) = \lambda^i r(x_0), \text{ where } r(x_0) = \frac{b^3pq^5}{1 - b^2q^4\lambda}R(x_0). \qquad (23)
\end{aligned}$$

Therefore by Lemma 4.1 it follows that $\{x_i\}$ is Cauchy sequence in X and due to the completeness of X there exists $z \in X$ such that $x_i \to z$ as $i \to \infty$. Since $z \in X$ then from the contractive condition (21) we get $N \equiv n(z) \geq 1$ such that

$$\rho([T^N x]_p, [T^N z]_q) \leq \lambda\rho([x]_p, [z]_q) \text{ for all } x \in X. \qquad (24)$$

Thus, $\rho([T^N x_i]_p, [T^N z]_q) \leq \lambda\rho([x_i]_p, [z]_q) \to 0$ as $i \to \infty$ i.e. $T^N x_i \to T^N z$ as $i \to \infty$. Now,

$$\begin{aligned}
\rho([z]_p, [T^N z]_q) &\leq bpq\rho([z]_p, [T^N x_i]_q) + b^2q^4\rho([T^N x_i]_p, [T^N z]_q) \\
&\leq bpq\{\rho([z]_p, [x_i]_q) + b^2q^4\rho([x_i]_p, [T^N x_i]_q)\} + b^2q^4\rho([T^N x_i]_p, [T^N z]_q). \qquad (25)
\end{aligned}$$

Also,

$$\begin{aligned}\rho([x_i]_p,[T^N x_i]_q) &\leq bq^2\rho([T^N x_i]_p,[x_i]_q)\\ &= bq^2\rho([T^{m_{i-1}}(T^N x_{i-1})]_p,[T^{m_{i-1}}x_{i-1}]_q)\\ &\leq bq^2\lambda\rho([T^N x_{i-1}]_p,[x_{i-1}]_q)\\ &\dots\\ &\leq bq^2\lambda^i\rho([T^N x_0]_p,[x_0]_q)\to 0 \text{ as } i\to\infty. \end{aligned}\tag{26}$$

Therefore from (25) it follows that $\rho([z]_p,[T^N z]_q)=0$ and therefore $T^N z=z$. If w is another fixed point of T^N then

$$\begin{aligned}\rho([w]_p,[z]_q) &= \rho([T^N w]_p,[T^N z]_q)\\ &\leq \lambda\rho([w]_p,[z]_q). \end{aligned}\tag{27}$$

Hence $w=z$ and T^N has a unique fixed point in X. So it is clear that T has a fixed point in X and it is also unique. □

Corollary 4.1. *Let (X,ρ) be a complete $S_b^{(p,q)}$-metric space and $T:X\to X$ be a mapping. If T satisfies*

$$\rho([Tx]_p,[Ty]_q)\leq \mu\rho([x]_p,[y]_q) \textit{ for all } x,y\in X, \tag{28}$$

for $0<\mu<\frac{1}{b^2q^4}$ then T has a unique fixed point in X.

Proof. If we choose $n(y)=1$ for all $y\in X$ in Theorem 4.1 then we get our required result. □

Theorem 4.2. *(Analogue to Reich fixed point theorem) Let (X,ρ) be a complete $S_b^{(p,q)}$-metric space and $T:X\to X$ be a mapping. If T satisfies*

$$\rho([Tx]_p,[Ty]_q)\leq \alpha\rho([x]_p,[y]_q)+\beta\rho([x]_p,[Tx]_q)+\gamma\rho([y]_p,[Ty]_q) \textit{ for all } x,y\in X, \tag{29}$$

for $0<\alpha+\beta+\gamma<1$ with $0\leq\alpha,\gamma<\frac{1}{b^2q^4}$ and $0\leq\beta<\frac{1}{bq^2}$ then T has a unique fixed point in X.

Proof. Let $x_0\in X$ be chosen arbitrarily and let us construct the Picard iterating sequence $\{x_n\}$ by $x_n=Tx_{n-1}=T^n x_0$ for all $n\in\mathbb{N}$. Then

$$\begin{aligned}&\rho([x_n]_p,[x_{n+1}]_q)\leq \alpha\rho([x_{n-1}]_p,[x_n]_q)+\beta\rho([x_{n-1}]_p,[x_n]_q)+\gamma\rho([x_n]_p,[x_{n+1}]_q)\\ \Rightarrow\ &\rho([x_n]_p,[x_{n+1}]_q)\leq \frac{\alpha+\beta}{1-\gamma}\rho([x_{n-1}]_p,[x_n]_q) \text{ for all } n\geq 1. \end{aligned}\tag{30}$$

Now,

$$\begin{aligned}\rho([x_n]_p,[x_m]_q) &\leq \alpha\rho([x_{n-1}]_p,[x_{m-1}]_q)+\beta\rho([x_{n-1}]_p,[x_n]_q)+\gamma\rho([x_{m-1}]_p,[x_m]_q)\\ &\leq \alpha\{bpq\rho([x_{n-1}]_p,[x_n]_q)+bq^2\rho([x_{m-1}]_p,[x_n]_q)\}+\beta\rho([x_{n-1}]_p,[x_n]_q)+\\ &\quad \gamma\rho([x_{m-1}]_p,[x_m]_q)\\ &\leq \alpha bpq\rho([x_{n-1}]_p,[x_n]_q)+\alpha bq^2\{bpq\rho([x_{m-1}]_p,[x_m]_q)+bq^2\rho([x_n]_p,[x_m]_q)\}\\ &\quad +\beta\rho([x_{n-1}]_p,[x_n]_q)+\gamma\rho([x_{m-1}]_p,[x_m]_q) \text{ for all } n,m\geq 1. \end{aligned}\tag{31}$$

Therefore from (31) we have

$$\begin{aligned}\rho([x_n]_p,[x_m]_q) &\le \frac{\alpha bpq+\beta}{1-b^2q^4\alpha}\rho([x_{n-1}]_p,[x_n]_q)+\frac{\alpha b^2pq^3+\gamma}{1-b^2q^4\alpha}\rho([x_{m-1}]_p,[x_m]_q)\\ &\le \left[\frac{\alpha bpq+\beta}{1-b^2q^4\alpha}\left(\frac{\alpha+\beta}{1-\gamma}\right)^n+\frac{\alpha b^2pq^3+\gamma}{1-b^2q^4\alpha}\left(\frac{\alpha+\beta}{1-\gamma}\right)^m\right]\rho([x_0]_p,[x_1]_q), \quad (32)\end{aligned}$$

for all $n, m \in \mathbb{N}$. Taking $n, m \to \infty$ we see that $\{x_n\}$ is Cauchy in X and due to the completeness of X there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now from the contractive condition (29) we get

$$\begin{aligned}\rho([Tu]_p,[x_n]_q) &\le \alpha\rho([u]_p,[x_{n-1}]_q)+\beta\rho([u]_p,[Tu]_q)+\gamma\rho([x_{n-1}]_p,[x_n]_q)\\ &\le \alpha\rho([u]_p,[x_{n-1}]_q)+\beta\{bpq\rho([u]_p,[x_n]_q)+bq^2\rho([Tu]_p,[x_n]_q)\}+\\ &\quad \gamma\rho([x_{n-1}]_p,[x_n]_q) \text{ for all } n\ge 1. \quad (33)\end{aligned}$$

Which implies that

$$\begin{aligned}\rho([Tu]_p,[x_n]_q) &\le \frac{\alpha}{1-bq^2\beta}\rho([u]_p,[x_{n-1}]_q)+\frac{\beta bpq}{1-bq^2\beta}\rho([u]_p,[x_n]_q)+\\ &\quad \frac{\gamma}{1-bq^2\beta}\rho([x_{n-1}]_p,[x_n]_q)\to 0 \text{ as } n\to\infty. \quad (34)\end{aligned}$$

Thus from (34) it follows that $Tu = u$. If v be a fixed point of T in X then

$$\begin{aligned}\rho([u]_p,[v]_q) &= \rho([Tu]_p,[Tv]_q)\\ &\le \alpha\rho([u]_p,[v]_q)+\beta\rho([u]_p,[Tu]_q)+\gamma\rho([v]_p,[Tv]_q)\\ &= \alpha\rho([u]_p,[v]_q). \quad (35)\end{aligned}$$

Hence $u = v$ and T has a unique fixed point in X. □

Corollary 4.2. *Let (X, ρ) be a complete $S_b^{(p,q)}$-metric space and $T : X \to X$ be a mapping. If T satisfies*

$$\rho([Tx]_p,[Ty]_q) \le \nu[\rho([x]_p,[Tx]_q)+\rho([y]_p,[Ty]_q)] \text{ for all } x,y\in X, \quad (36)$$

for $0 < \nu < \frac{1}{2b^2q^4}$ then T has a unique fixed point in X.

Proof. The corollary follows if we take $\alpha = 0$ and $\beta = \nu = \gamma$ in Theorem 4.2. □

Let $\mathcal{A}$ be the set of all functions $\zeta : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying

($\mathcal{A}$ 1) ζ is continuous on the set $\mathbb{R}_+^3$ of all triplets of nonnegative reals (with respect to the Euclidean metric on $\mathbb{R}_+^3$).

($\mathcal{A}$ 2) $u \le \delta v$ for some $\delta \in (0, \frac{1}{b^2q^4})$ whenever $u \le bq^2\zeta(u, v, v)$ or $u \le bq^2\zeta(v, u, v)$ or $u \le bq^2\zeta(v, v, u)$, for all $u, v \in \mathbb{R}_+$.

Definition 4.1. *A mapping $T : X \to X$ is said to be an Akram contraction (shortly A-contraction) in an $S_b^{(p,q)}$-metric space (X, ρ) if it satisfies the condition*

$$\rho([Tx]_p,[Ty]_q) \le \zeta\left(\rho([x]_p,[y]_q),\rho([x]_p,[Tx]_q),\rho([y]_p,[Ty]_q)\right) \text{ for all } x,y\in X, \quad (37)$$

for some $\zeta \in \mathcal{A}$.

Theorem 4.3. *(Analogue to Akram fixed point theorem) Let (X, ρ) be a complete $S_b^{(p,q)}$-metric space and $T : X \to X$ be an A-contraction mapping. Then T has a unique fixed point in X.*

Proof. Let us consider the Picard iterating sequence $\{x_n\}$ defined by $x_n = Tx_{n-1} = T^n x_0$ for all $n \geq 1$. Then

$$\rho([x_n]_p, [x_{n+1}]_q) \leq \zeta(\rho([x_{n-1}]_p, [x_n]_q), \rho([x_{n-1}]_p, [x_n]_q), \rho([x_n]_p, [x_{n+1}]_q)) \text{ for all } n \in \mathbb{N}, \tag{38}$$

implies that $\rho([x_n]_p, [x_{n+1}]_q) \leq \delta\rho([x_{n-1}]_p, [x_n]_q) \leq ... \leq \delta^n \rho([x_0]_p, [x_1]_q)$ for all $n \geq 1$. Therefore by Lemma 4.1 it follows that $\{x_n\}$ is Cauchy sequence in X and due to the completeness of X there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now,

$$\begin{aligned}\rho([z]_p, [Tz]_q) &\leq bpq\rho([z]_p, [x_{n+1}]_q) + bq^2\rho([Tz]_p, [x_{n+1}]_q) \\ &= bpq\rho([z]_p, [x_{n+1}]_q) + bq^2\rho([Tz]_p, [Tx_n]_q) \\ &\leq bpq\rho([z]_p, [x_{n+1}]_q) + \\ &\qquad bq^2\zeta(\rho([z]_p, [x_n]_q), \rho([z]_p, [Tz]_q), \rho([x_n]_p, [x_{n+1}]_q)) \text{ for all } n \geq 1. \end{aligned} \tag{39}$$

Taking $n \to \infty$ in (39) we get,

$$\rho([z]_p, [Tz]_q) \leq bq^2\zeta(0, \rho([z]_p, [Tz]_q), 0) \tag{40}$$

implying that $\rho([z]_p, [Tz]_q) \leq \delta.0$ i.e. $Tz = z$. Let w be a fixed point of T then

$$\begin{aligned}\rho([z]_p, [w]_q) = \rho([Tz]_p, [Tw]_q) &\leq \zeta(\rho([z]_p, [w]_q), \rho([z]_p, [Tz]_q), \rho([w]_p, [Tw]_q)) \\ &= \zeta(\rho([z]_p, [w]_q), 0, 0) \end{aligned} \tag{41}$$

that is $\rho([z]_p, [w]_q) \leq \delta.0$ and therefore $z = w$. Hence T has a unique fixed point in X. □

Example 4.1. *(i) Let $X = \mathbb{R}_+^0$ and $\rho : X^4 \to [0, \infty)$ is defined by*

$$\rho(x, y, z, w) = |x + y - 2z|^2 + |x + y - 2w|^2 + |x - y|^2 \text{ for all } x, y, z, w \in X. \tag{42}$$

Then (X, ρ) is a complete symmetric $S_3^{(3,1)}$-metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} 0, & \text{if } x = 0 \text{ or } x > 2; \\ x + 1, & \text{if } x \in (0, 1]; \\ 2x, & \text{if } x \in (1, 2]. \end{cases} \tag{43}$$

Then T satisfies the contractive condition (21) *for $n(y) = 3$ for all $y \in X$ but does not satisfy the contractive condition* (28)*. Therefore all the conditions of Theorem* 4.1 *are satisfied and* 0 *is the unique fixed point of T in X.*

(ii) Let $X = \mathbb{R}_+^0$ and $\rho : X^4 \to [0, \infty)$ be taken as in (7)*. Then (X, ρ) is a complete symmetric $S_2^{(2,2)}$-metric space. Let $T : X \to X$ be given by*

$$Tx = \begin{cases} \frac{x}{15}, & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{x}{16}, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases} \tag{44}$$

Then T satisfies the contractive condition (29) *for $\alpha = \frac{1}{225}$, $\beta = \frac{1}{98}$ and $\gamma = \frac{2}{225}$. Therefore all the conditions of Theorem* 4.2 *are satisfied and* 0 *is the unique fixed point of T in X.*

(iii) Let $X = \{0, 1, 2, 3\}$ and $\rho : X^3 \to [0, \infty)$ is defined by

$$\rho(x, y, z) = |x - y| + |x - z| \text{ for all } x, y, z \in X. \tag{45}$$

Then (X, ρ) is a complete symmetric $S_1^{(2,1)}$-metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} 0, & \text{if } x = 2; \\ 1, & \text{otherwise}. \end{cases} \tag{46}$$

Then T satisfies the contractive condition (37) *for the function $\zeta(u,v,w) = \max\{u+v, v+w, w+u\}$ for all $(u,v,w) \in \mathbb{R}^3_+$. Therefore all the conditions of Theorem* 4.3 *are satisfied and* 1 *is the unique fixed point of T in X.*

Corollary 4.3. *If we consider $b = 1 = p = q$ then we get the metric versions of Theorem* 4.1*, Theorem* 4.2 *and Theorem* 4.3 *(See* [8], [18], [10], [12] *and* [2]*).*

5. Well-posedness and Ulam-Hyers stability of fixed point problems

In algebra, the stability of functional equation for a homomorphism has great importance. In 1940, Ulam have raised an open problem concerning the stability of homomorphisms. Hyers was the first Mathematician who gave an answer for the stability of functional equations in the setting of Banach spaces (See [9]). The result of Hyers is given below.

Theorem 5.1. *Let $h : B_1 \to B_2$ be a function between two Banach spaces B_1 and B_2 such that*

$$||h(x+y) - h(x) - h(y)|| \leq \delta \tag{47}$$

for some $\delta > 0$ and for all $x, y \in B_1$. Then the limit

$$H(x) = \lim_{n\to\infty} 2^{-n} h(2^n x) \tag{48}$$

exists for each $x \in B_1$, and $H : B_1 \to B_2$ is the unique additive function such that $||h(x) - H(x)|| \leq \delta$ for every $x \in B_1$. Also, if $h(tx)$ is continuous function of t for each fixed $x \in B_1$, then the function H is linear.

The additive Cauchy equation $h(x+y) = h(x) + h(y)$ is said to be Ulam-Hyers stable on (B_1, B_2) if for every function $h : B_1 \to B_2$ satisfying the inequality (47) for some $\delta \geq 0$ and for all $x, y \in B_1$, there exists an additive function $H : B_1 \to B_2$ such that $h - H$ is bounded on B_1.

In this section, we first discuss Ulam-Hyers stability of fixed point problem. One can find such applications in the paper [14, 15]. Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $T : X \to X$ be a given mapping. Let us consider the fixed point equation

$$Tx = x \tag{49}$$

and for some $\epsilon > 0$

$$\rho([v]_p, [Tv]_q) < \epsilon. \tag{50}$$

Any point $v \in X$ which satisfies the above equation (50) is called an $\epsilon-$solution of the mapping T. We say that the fixed point problem (49) is Ulam-Hyers stable in an $S_b^{(p,q)}$-metric space if there exists a constant $c > 0$ such that for each $\epsilon > 0$ and an $\epsilon-$solution $v \in X$, there exists a solution u of the fixed point equation (49) such that

$$\rho([v]_p, [u]_q) < c\,\epsilon. \tag{51}$$

Theorem 5.2. *Let (X, ρ) be a complete $S_b^{(p,q)}$-metric space and $T : X \to X$ be a mapping. If T satisfies*

$$\rho([Tx]_p, [Ty]_q) \leq \alpha\rho([x]_p, [y]_q) + \beta\rho([x]_p, [Tx]_q) + \gamma\rho([y]_p, [Ty]_q) \text{ for all } x, y \in X, \tag{52}$$

for $0 < \alpha + \beta + \gamma < 1$ with $0 \leq \alpha, \gamma < \frac{1}{b^2 q^4}$ and $0 \leq \beta < \frac{1}{bq^2}$ then the fixed point equation (49) *of T is Ulam-Hyers stable.*

Proof. From Theorem 4.2 we see that T has a unique fixed point in X, that is the fixed point equation (49) of T has a unique solution say u. Let $\epsilon > 0$ be arbitrary and v be an $\epsilon-$solution that is $\rho([v]_p, [Tv]_q) < \epsilon$.

Since T satisfies the contractive condition (52) therefore

$$\begin{aligned}\rho([v]_p, [u]_q) &\leq bpq\rho([v]_p, [Tv]_q) + bq^2\rho([u]_p, [Tv]_q)\\ &= bpq\rho([v]_p, [Tv]_q) + bq^2\rho([Tu]_p, [Tv]_q)\\ &\leq bpq\rho([v]_p, [Tv]_q) + bq^2\{\alpha\rho([u]_p, [v]_q)+\\ &\quad \beta\rho([u]_p, [Tu]_q) + \gamma\rho([v]_p, [Tv]_q)\}\\ &\leq bpq\rho([v]_p, [Tv]_q) + b^2q^4\alpha\rho([v]_p, [u]_q) + bq^2\gamma\rho([v]_p, [Tv]_q). \qquad (53)\end{aligned}$$

From (53) we get $\rho([v]_p, [u]_q) \leq \frac{bpq+bq^2\gamma}{1-b^2q^4\alpha}\rho([v]_p, [Tv]_q) < \frac{bpq+bq^2\gamma}{1-b^2q^4\alpha}\epsilon$ and hence the fixed point problem of T is Ulam-Hyers stable. □

Well-posedness of fixed point problem is an interesting portion of the study in metric fixed point theory. The definition of well posedness of fixed point problem over metric spaces is given as follows:

Definition 5.1. [6] *Let (X, d) be a metric space and $F : (X, d) \to (X, d)$ be a mapping. The fixed point problem of F is said to be well-posed if (i) F has a unique fixed point $z \in X$, (ii) for any sequence $\{x_n\}$ in X with $d(x_n, F(x_n)) \to 0$ as $n \to \infty$ we have $d(z, x_n) \to 0$ as $n \to \infty$.*

Now we give the definition of well-posedness of fixed point problem in the setting of $S_b^{(p,q)}$-metric spaces which runs as follows:

Definition 5.2. *Let (X, ρ) be an $S_b^{(p,q)}$-metric space and $T : (X, \rho) \to (X, \rho)$ be a mapping. The fixed point problem of T is said to be well posed if*
(i) T has a unique fixed point $u \in X$;
(ii) for any sequence $\{y_n\}$ in X with $\rho([y_n]_p, [Ty_n]_q) \to 0$ as $n \to \infty$ we have $\rho([y_n]_p, [u]_q) \to 0$ as $n \to \infty$.

Theorem 5.3. *Let (X, ρ) be a complete $S_b^{(p,q)}$-metric space and $T : X \to X$ be a mapping. If T satisfies*

$$\rho([Tx]_p, [Ty]_q) \leq \alpha\rho([x]_p, [y]_q) + \beta\rho([x]_p, [Tx]_q) + \gamma\rho([y]_p, [Ty]_q) \text{ for all } x, y \in X, \qquad (54)$$

for $0 < \alpha + \beta + \gamma < 1$ with $0 \leq \alpha, \gamma < \frac{1}{b^2q^4}$ and $0 \leq \beta < \frac{1}{bq^2}$ then the fixed point problem of T is well-posed.

Proof. Theorem 4.2 shows that T has a unique fixed point in X, that is the fixed point problem of T has a unique solution u (say). Let $\{y_n\}$ be a sequence in X with $\rho([y_n]_p, [Ty_n]_q) \to 0$ as $n \to \infty$, then we have

$$\begin{aligned}\rho([y_n]_p, [u]_q) &\leq bpq\rho([y_n]_p, [Ty_n]_q) + bq^2\rho([u]_p, [Ty_n]_q)\\ &= bpq\rho([y_n]_p, [Ty_n]_q) + bq^2\rho([Tu]_p, [Ty_n]_q)\\ &\leq bpq\rho([y_n]_p, [Ty_n]_q) + bq^2\{\alpha\rho([u]_p, [y_n]_q)+\\ &\quad \beta\rho([u]_p, [Tu]_q) + \gamma\rho([y_n]_p, [Ty_n]_q)\}\\ &\leq bpq\rho([y_n]_p, [Ty_n]_q) + b^2q^4\alpha\rho([y_n]_p, [u]_q) + bq^2\gamma\rho([y_n]_p, [Ty_n]_q). \qquad (55)\end{aligned}$$

From (55) it follows that $\rho([y_n]_p, [u]_q) \leq \frac{bpq+bq^2\gamma}{1-b^2q^4\alpha}\rho([y_n]_p, [Ty_n]_q) \to 0$ as $n \to \infty$ and hence the fixed point problem of T is well-posed. □

6. Conclusion and future works

If we increase the number of variables in a distance function then some metrical as well as topological properties may differ. One of such is the lack of continuity of distance function with respect to all of it's variables. There are several three variables and n-variables metric structures which have different interesting properties. Due to the complicated nature of multi-variable distance function it is quite difficult to discuss topology and to establish fixed point theorems on such spaces. We take this challenge and introduce a new generalized metric structure which is multi-variable in nature. We have given a topology on such space and discussed several related properties. With the help of build metric-like structure and the induced topology some fixed point results have been proved. Finally our results have been successfully applied to well-posedness and Ulam-Hyers stability of fixed point problems. In future by considering our metric-like space one can prove different fixed point theorems and apply them to several areas of mathematics. Moreover one can generalize our metric structure by weakening either condition (i) or (ii). It is our belief that our proposed metric-type structure gives a advanced direction of research in the area of fixed point theory.

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