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NEW RESULTS ON CAPUTO FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. This article investigates the existence and uniqueness of solutions for a nonlocal initial condition of the Caputo fractional Volterra-Fredholm integro-differential equation in a Banach space. We shall prove the existence and uniqueness of the results by using the Banach and Krasnoselskii fixed-point theorems. A number of illustrative examples will be given to further the understanding of our main conclusions.

Keywords: Volterra-Fredholm integro-differential equation, Caputo fractional derivative, Fixed point method.

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1. Introduction

Fractional calculus extends the concepts of derivatives and integrals for functions to non-integer orders. The exploration of equations involving fractional differentiation and integration is a relatively recent pursuit. Prominent contributions in this field include the works of Kilbas et al. [1], Podlubny [2], and other researchers. Integro-differential equations link the derivatives and function values over a domain, making them broadly useful models across science and engineering. By relating rates of change and net quantities, these equations describe conserved physical properties. For example, integro-differential formulations represent total mass in transport phenomena and moment integrals in mechanics, as established in several studies [3, 4, 5, 6, 7, 8, 9, 10]. The distinctive fusion of local and non-local theories empowers integro-differential equations as adaptable representations. Much research has thus focused on developing solutions for such formulations. Especially active investigation surrounds fractional order integro-differential equations,

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seeking conditions to guarantee solution existence and uniqueness. Multiple studies have presented proofs and numerical evidence certifying well-posed solutions to an array of fractional integro-differential systems [11, 12, 13, 14, 15, 16, 17, 31]. Elucidating requisite constraints for robust fractional integro-differential solution theories remains an open challenge with profound implications, commanding wide attention.

In [18], the local and global existence of the Cauchy problem is studied. In [19], the existence and uniqueness of the fractional integro-differential issue were examined.

The primary objective of this paper is to establish multiple standards that can be used to determine if a solution exists and is unique for a specified fractional Volterra-Fredholm integro-differential equation and corresponding initial value problem. In order to accomplish this goal, we first converted our main problem into an equivalent fixed point problem. We then utilized the fixed point theorems of Banach and Krasnoselskii to prove the existence and uniqueness results regarding solutions to our problem within a well-defined Banach space.

In this paper, we considered a new fractional class of Volterra-Fredholm integro-differential equation in the context of the standard Caputo fractional derivative:

$$ {}^{C}\mathfrak{D}^{\mathfrak{w}}_{0^+}\varphi(\mathfrak{r}) = \Phi(\varphi(\mathfrak{r})) + \mathfrak{F}(\mathfrak{r},\varphi(\mathfrak{r})) + \int_0^{\mathfrak{r}} \hbar_0(\mathfrak{r},\sigma,\varphi(\sigma))d\sigma + \int_0^1 \hbar_1(\mathfrak{r},\sigma,\varphi(\sigma))d\sigma, \ \mathfrak{r}\in\psi \quad (1) $$

$$ \varphi(0) = \xi\int_0^{\zeta}\varphi(\varpi)d\varpi, \ 0<\zeta<1, \ \psi=[0,1], \quad (2) $$

where ${}^{C}\mathfrak{D}^{\mathfrak{w}}$ is the Caputo fractional derivative of order $\mathfrak{w}$, $\mathfrak{F}:\psi\times\mathbb{R}\to\mathbb{R}$ and $\hbar_0,\hbar_1:\psi\times\psi\times\mathbb{R}\to\mathbb{R}$, $\Phi\in(\psi,\psi)$ are appropriate functions satisfying some conditions which will be stated later.

2. Auxiliary Results

Before presenting our primary results, we offer the essential definitions, preliminary details, and assumptions that will be employed in our subsequent discourse [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Definition 2.1. [1, 2] *The Riemann-Liouville fractional integral of order* $\mathfrak{w}$ *, is defined by*

$$ \mathfrak{I}^{\mathfrak{w}}_{0^+}\varphi(\mathfrak{r}) = \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\varphi(\sigma)d\sigma, \ \mathfrak{w}>0. \quad (3) $$

Definition 2.2. [1, 2] *The Caputo fractional derivative of order* $\mathfrak{w}$ $(\upsilon-1<\mathfrak{w}<\upsilon)$ *is defined as*

$$ {}^{C}\mathfrak{D}^{\mathfrak{w}}_{0^+}\varphi(\mathfrak{r}) = \frac{1}{\Gamma(\upsilon-\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\upsilon-\mathfrak{w}-1}\varphi^{(\upsilon)}(\sigma)d\sigma, \ \mathfrak{w}>0. \quad (4) $$

Lemma 2.1. [1]*, For real numbers* $\mathfrak{w},\mathfrak{q}>0$ *and appropriate function* φ*; we have for all*

(1) $\mathfrak{I}^{\mathfrak{w}}_{0^+}\mathfrak{I}^{\mathfrak{q}}_{0^+}\varphi(\mathfrak{r}) = \mathfrak{I}^{\mathfrak{q}}_{0^+}\mathfrak{I}^{\mathfrak{w}}_{0^+}\varphi(\mathfrak{r}) = \mathfrak{I}^{\mathfrak{w}+\mathfrak{q}}_{0^+}\varphi(\mathfrak{r})$

(2) $\mathfrak{I}^{\mathfrak{w}}_{0^+}\,{}^{C}\mathfrak{D}^{\mathfrak{w}}_{0^+}\varphi(\mathfrak{r}) = \varphi(\mathfrak{r})-\varphi(0), \ 0<\mathfrak{w}<1$

(3) ${}^{C}\mathfrak{D}^{\mathfrak{w}}_{0^+}\mathfrak{I}^{\mathfrak{w}}_{0^+}\varphi(\mathfrak{r}) = \varphi(\mathfrak{r})$.

Theorem 2.1. *(Banach's fixed point theorem)*[8]. *Suppose* Ω *be a non-empty complete metric space and* $\Im:\Omega\to\Omega$*, is contraction mapping. Then, there exists a unique point* $\sigma\in\Omega$ *such that* $\Phi(\sigma)=\sigma$.

Theorem 2.2. *(Krasnoselskii's fixed point theorem)*[8]. *Suppose ϖ be a closed convex and nonempty subset of a Banach space κ. If $\Im, \Im^*$ be two operators such that:*
1. $\Im\kappa + \Im^*\kappa^* \in \varpi$ whenever $\kappa, \kappa^* \in \varpi$.
2. $\Im$ is compact and continuous.
3. $\Im^*$ is a contraction mapping.
Then there exists $\varpi_0 \in \varpi$ such that $\Im^* = \Im\varpi_0 + \Im^*\varpi_0$.

3. Principal Findings

Using Banach's fixed point theorem, we will demonstrate the existence and uniqueness of the solution to the problem (1)-(2) in $C(\psi, \mathbb{R})$. We will require some presumptions about this fact.

(∇_1) $\Phi : \psi \to \mathbb{R}$ is continuous and there exists $0 < \mathfrak{K} < 1$ such that

$$|\Phi(\varphi(\mathfrak{r})) - \Phi(\varphi_0(\mathfrak{r}))| \le \mathfrak{K}\|\varphi - \varphi_0\|$$

(∇_2) $\Im : \psi \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\Theta \in \mathfrak{L}^\infty(\psi, \mathbb{R}^+)$ such that

$$|\Im(\mathfrak{r}, \varphi(\mathfrak{r})) - \Im(\mathfrak{r}, \varphi_0(\mathfrak{r})| \le \Theta(\mathfrak{r})\|\varphi - \varphi_0\|,$$

(∇_3) $\hbar_0, \hbar_1 : \mathcal{G} \times \psi \to \mathbb{R}$ are continuous on $\mathcal{G}$ and there exist $\mathfrak{H}_i, i = 1, 2 \in \mathfrak{L}^1(\psi, \mathbb{R}^+)$ such that

$$|\hbar_0(\mathfrak{r}, \sigma, \varphi(\sigma)) - \hbar_0(\mathfrak{r}, \sigma, \varphi_0(\sigma))| \le \mathfrak{H}_1(\mathfrak{r})\|\varphi - \varphi_0\|$$
$$|\hbar_1(\mathfrak{r}, \sigma, \varphi(\sigma)) - \hbar_1(\mathfrak{r}, \sigma, \varphi_0(\sigma))| \le \mathfrak{H}_2(\mathfrak{r})\|\varphi - \varphi_0\|.$$

Lemma 3.1. *Suppose $0 < \mathfrak{w} < 1$ and $\xi \neq \frac{1}{\zeta}$. Assume that $\Phi, \Im, \hbar_0$ and $, \hbar_1$ are continuous functions. If $\varphi \in C(\psi, \mathbb{R})$ then φ is solution of (1)-(2) if and only if φ satisfies the integral equation*

$$\begin{aligned} \varphi(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r} - \rho)^{\mathfrak{w}-1}\Big[\Phi(\varphi(\rho)) + \Im(\rho, \varphi(\rho)) + \int_0^\rho \hbar_0(\rho, \sigma, \varphi(\sigma))d\sigma \\ &+ \int_0^1 \hbar_1(\rho, \sigma, \varphi(\sigma))d\sigma\Big]d\rho + \frac{\xi}{(1 - \xi\zeta)\Gamma(\mathfrak{w} + 1)}\int_0^\zeta (\zeta - \mu)^{\mathfrak{w}}\Big[\Phi(\varphi(\mu)) + \Im(\mu, \varphi(\mu)) \\ &+ \int_0^\mu \hbar_0(\mu, \omega, \varphi(\omega))d\omega + \int_0^1 \hbar_1(\mu, \omega, \varphi(\omega))d\omega\Big]d\mu. \end{aligned}$$

Proof. Suppose $\varphi \in C(\psi, \mathbb{R})$ be a solution of (1)-(2). Firstly, we show that φ is solution of integral equation (5). By Lemma 2.1, we obtain

$$\mathfrak{I}^{\mathfrak{w}}_{0+}\,{}^C\mathfrak{D}^{\mathfrak{w}}_{0+}\varphi(\mathfrak{r}) = \varphi(\mathfrak{r}) - \varphi(0). \tag{5}$$

In addition, from equation in (1)-(2) and Definition 2.1, we have

$$\begin{aligned} \mathfrak{I}^{\mathfrak{w}}_{0+}\,{}^C\mathfrak{D}^{\mathfrak{w}}_{0+}\varphi(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r} - \rho)^{\mathfrak{w}-1}\Big[\Phi(\varphi(\rho)) + \Im(\rho, \varphi(\rho)) + \int_0^\rho \hbar_0(\rho, \sigma, \varphi(\sigma))d\sigma \\ &+ \int_0^1 \hbar_1(\rho, \sigma, \varphi(\sigma))d\sigma\Big]d\rho. \end{aligned} \tag{6}$$

By substituting (6) in (5) with nonlocal condition in problem (2), we get

$$\begin{aligned} \varphi(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r} - \rho)^{\mathfrak{w}-1}\Big[\Phi(\varphi(\rho)) + \Im(\rho, \varphi(\rho)) + \int_0^\rho \hbar_0(\rho, \sigma, \varphi(\sigma))d\sigma \\ &+ \int_0^1 \hbar_1(\rho, \sigma, \varphi(\sigma))d\sigma\Big]d\rho + \varphi(0) \end{aligned} \tag{7}$$

but, we have

$$\begin{aligned}
\varphi(0) &= \xi \int_0^{\zeta} \varphi(\varpi) d\varpi \\
&= \frac{\xi}{\Gamma(\mathfrak{w})} \int_0^{\zeta} \bigg[\int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \Big[\Phi(\varphi(\mu)) + \Im(\mu, \varphi(\mu)) \\
&+ \int_0^{\mu} \hbar_0(\mu, \omega, \varphi(\omega)) d\omega + \int_0^1 \hbar_1(\mu, \omega, \varphi(\omega)) d\omega \Big] d\mu \bigg] d\varpi + \xi\zeta\varphi(0) \\
&= \frac{\xi}{\Gamma(\mathfrak{w})} \bigg[\int_0^{\zeta} \int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \Phi(\varphi(\mu)) d\mu d\varpi \\
&+ \int_0^{\zeta} \int_0^{\varpi} (\sigma-\mu)^{\mathfrak{w}-1} \Im(\mu, \varphi(\mu)) d\mu d\varpi \\
&+ \int_0^{\zeta} \int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \int_0^{\mu} \hbar_0(\mu, \omega, \varphi(\omega)) d\omega d\mu d\varpi \\
&+ \int_0^{\zeta} \int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \int_0^{1} \hbar_1(\mu, \omega, \varphi(\omega)) d\omega d\mu d\varpi \bigg] + \xi\zeta\varphi(0).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\varphi(0) &= \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w})} \bigg[\int_0^{\zeta} \int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \Phi(\varphi(\mu)) d\mu d\varpi \\
&+ \int_0^{\zeta} \int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \Im(\mu, \varphi(\mu)) d\mu d\varpi \\
&+ \int_0^{\zeta} \int_0^{\varpi} (\varpi-\mu)^{\mathfrak{w}-1} \int_0^{\mu} \hbar_0(\mu, \omega, \varphi(\omega)) d\omega d\mu d\varpi \\
&+ \int_0^{\zeta} \int_0^{\sigma} (\sigma-\mu)^{\mathfrak{w}-1} \int_0^{1} \hbar_1(\mu, \omega, \varphi(\omega)) d\omega d\mu d\varpi \bigg].
\end{aligned}$$

Using some manipulations we obtain:

$$\begin{aligned}
\varphi(0) &= \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)} \int_0^{\zeta} (\zeta-\mu)^{\mathfrak{w}} \bigg[\Phi(\varphi(\mu)) + \Im(\mu, \varphi(\mu)) + \int_0^{\mu} \hbar_0(\mu, \omega, \varphi(\omega)) d\omega \\
&+ \int_0^{1} \hbar_1(\mu, \omega, \varphi(\omega)) d\omega \bigg] d\mu.
\end{aligned}$$

Now, by substituting the last value of $\varphi(0)$ in (7) we find (5). Conversely, in view of Lemma 2.1 and by applying the operator ${}^{C}\mathfrak{D}_{0+}^{\mathfrak{w}}$ on both sides of (5), we get

$$\begin{aligned}
{}^{C}\mathfrak{D}_{0+}^{\mathfrak{w}} \varphi(\mathfrak{r}) &= {}^{C}\mathfrak{D}_{0+}^{\mathfrak{w}} \mathfrak{I}_{0+}^{\mathfrak{w}} \Phi(\varphi(\mathfrak{r})) + {}^{C}\mathfrak{D}_{0+}^{\mathfrak{w}} \mathfrak{I}_{0+}^{\mathfrak{w}} \Im(z, \varphi(\mathfrak{r})) \\
&+ {}^{C}\mathfrak{D}_{0+}^{\mathfrak{w}} \mathfrak{I}_{0+}^{\mathfrak{w}} \Big(\int_0^{\mathfrak{r}} \hbar_0(\mathfrak{r}, \sigma, \varphi(\sigma)) d\sigma + \int_0^{1} \hbar_1(\mathfrak{r}, \sigma, \varphi(\sigma)) d\sigma \Big) \\
&= \Phi(\varphi(\mathfrak{r})) + \Im(\mathfrak{r}, \varphi(\mathfrak{r})) + \int_0^{\mathfrak{r}} \hbar_0(\mathfrak{r}, \sigma, \varphi(\sigma)) d\sigma \\
&+ \int_0^{1} \hbar_1(\mathfrak{r}, \sigma, \varphi(\sigma)) d\sigma. \qquad (8)
\end{aligned}$$

We will establish the problem's existence and uniqueness by verifying the fulfillment of the given equation (1)-(2) through the function φ. Furthermore, upon substituting $\mathfrak{r}$ with 0 in the integral equation (5), it becomes evident that the nonlocal condition specified in (1)-(2) is satisfied. Consequently, φ constitutes a solution for problem (1)-(2), thereby concluding the proof. □

Theorem 3.1. *Assume that*

(Δ_1) $\|\Phi(\varphi(\mathfrak{r})) - \Phi(\varphi_0(\mathfrak{r}))\| \leq \delta\|\varphi - \varphi_0\|, \ \varphi, \ \varphi_0 \in \mathbb{R}$
(Δ_2) $\|\Im(\mathfrak{r}, \varphi(\mathfrak{r})) - \Im(\mathfrak{r}, \varphi_0(\mathfrak{r})\| \leq \delta^*\|\varphi - \varphi_0\|, \mathfrak{r} \in \psi, \ \varphi, \ \varphi_0 \in \mathbb{R}$
(Δ_3) $\|\hbar_0(\mathfrak{r}, \sigma, \varphi(\sigma)) - \hbar_0(\mathfrak{r}, \sigma, \varphi_0(\sigma))\| \leq \delta_0\|\varphi - \varphi_0\|$
$\|\hbar_1(\mathfrak{r}, \sigma, \varphi(\sigma)) - \hbar_1(\mathfrak{r}, \sigma, \varphi_0(\sigma))\| \leq \delta_0^*\|\varphi - \varphi_0\|, \ (\mathfrak{r}, \sigma) \in \mathcal{G}, \ \varphi, \ \varphi_0 \in \mathbb{R}$

$$\mathfrak{Be} = \left[\frac{[\delta + \delta^* + \delta_0^*]}{\Gamma(\mathfrak{w}+1)} + \frac{\delta_0}{\Gamma(\mathfrak{w}+2)} + \frac{\left[|\xi|\delta + |\xi|\delta^* + |\xi|\delta_0^*\right]\zeta^{\mathfrak{w}+1}}{|1 - \xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{|\xi|\delta_0\zeta^{\mathfrak{w}+2}}{|1 - \xi\zeta|\Gamma(\mathfrak{w}+3)}\right] < 1. \quad (9)$$

In such a case, the fractional Volterra-Fredholm integro-differential problem (1)-(2) *possesses a solitary solution that is unique.*

Proof. To begin with, we establish the definition of an operator $\aleph : \mathcal{C}(\psi, \mathbb{R}) \to \mathcal{C}(\psi, \mathbb{R})$ by

$$\begin{aligned}(\aleph\varphi)(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r} - \rho)^{\mathfrak{w}-1}\Big[\Phi(\varphi(\rho)) + \Im(\rho, \varphi(\rho)) + \int_0^{\rho} \hbar_0(\rho, \sigma, \varphi(\sigma))d\sigma \\ &+ \int_0^1 \hbar_1(\rho, \sigma, \varphi(\sigma))d\sigma\Big]d\rho + \frac{\xi}{(1 - \xi\zeta)\Gamma(\mathfrak{w}+1)}\int_0^{\zeta} (\zeta - \mu)^{\mathfrak{w}}\Big[\Phi(\varphi(\mu)) + \Im(\mu, \varphi(\mu)) \\ &+ \int_0^{\mu} \hbar_0(\mu, \omega, \varphi(\omega))d\omega + \int_0^1 \hbar_1(\mu, \omega, \varphi(\omega))d\omega\Big]d\mu\end{aligned}$$

and we consider the subset $\mathcal{M}_\ell$ of $\mathcal{C}(\psi, \mathbb{R})$ defined by

$$\mathcal{M}_\ell = \{\varphi \in \mathcal{C}(\psi, \mathbb{R}) : \|\varphi\| \leq \ell\}$$

where ℓ is carefully selected as a strictly positive real number in order to ensure that

$$\ell \geq \frac{\frac{[\Phi^* + \Im^* + \hbar_1^*]}{\Gamma(\mathfrak{w}+1)} + \frac{\hbar_0^*}{\Gamma(\mathfrak{w}+2)} + \frac{\left[|\xi|\Phi^* + |\xi|\Im^* + |\xi|\hbar_1^*\right]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{|\xi|\hbar_0^*\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)}}{1 - \frac{[\delta + \delta^* + \delta_0^*]}{\Gamma(\mathfrak{w}+1)} - \frac{\delta_0}{\Gamma(\mathfrak{w}+2)} - \frac{\left[|\xi|\delta + |\xi|\delta^* + |\xi|\delta_0^*\right]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} - \frac{|\xi|\delta_0\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)}}$$

with

$$\Phi^* = |\Phi(0)|, \ \Im^* = \sup_{\rho \in J} |\Im(\rho, 0)|, \ \hbar_0^* = \sup_{\rho.\sigma \in D} |\hbar_0(\rho, \sigma, 0)|, \ \hbar_1^* = \sup_{\rho.\sigma \in D} |\hbar_1(\rho, \sigma, 0)|.$$

Subsequently, we proceed to demonstrate that the operator $\aleph$ has a unique fixed point within the range of $\mathcal{M}_\ell$ which is identical to the unique solution to the equations (1)- (2).

Our proof entails two steps.

$$
\begin{aligned}
& |(\aleph\varphi)(\mathfrak{r})| \\
= \; & \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\Big[|\Phi(\varphi(\rho))| + |\Im(\rho,\varphi(\rho))| + \int_0^{\rho}|\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma \\
+ \; & \int_0^1 |\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma\Big]d\rho + \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\Big[|\Phi(\varphi(\mu))| \\
+ \; & |\Im(\mu,\varphi(\mu))| + \int_0^{\mu}|\hbar_0(\mu,\omega,\varphi(\omega))|d\omega + \int_0^1|\hbar_1(\mu,\omega,\varphi(\omega))|d\omega\Big]d\mu \\
\leq \; & \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\big[|\Phi(\varphi(\rho))-\Phi(0)| + |\Phi(0)|\big]d\rho \\
+ \; & \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\big[|\Im(\rho,\varphi(\rho))-\Im(\rho,0)| + |\Im(\rho,0)|\big]d\rho \\
+ \; & \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\int_0^{\rho}\Big[|\hbar_0(\rho,\sigma,\varphi(\sigma))-\hbar_0(\rho,\sigma,0)| + |\hbar_0(\rho,\sigma,0)|\Big]d\sigma d\rho \\
+ \; & \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\int_0^{1}\Big[|\hbar_1(\rho,\sigma,\varphi(\sigma))-\hbar_1(\rho,\sigma,0)| + |\hbar_1(\rho,\sigma,0)|\Big]d\sigma d\rho \\
+ \; & \frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\big[|\Phi(\varphi(\mu))-\Phi(0)| + |\Phi(0)|\big]d\mu \\
+ \; & \frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\big[|\Im(\mu,\varphi(\mu))-\Im(\mu,0)| + |\Im(\mu,0)|\big]d\mu \\
+ \; & \frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\int_0^{\mu}\Big[|\hbar_0(\mu,\omega,\varphi(\omega))-\hbar_0(\mu,\omega,0)| \\
+ \; & |\hbar_0(\mu,\omega,0)|\Big]d\omega d\mu + \frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\int_0^{1}\Big[|\hbar_1(\mu,\omega,\varphi(\omega)) \\
- \; & \hbar_1(\mu,\omega,0)| + |\hbar_1(\mu,\omega,0)|\Big]d\omega d\mu \\
\leq \; & \frac{[\delta\|\varphi\| + \Phi^*]\mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} + \frac{[\delta^*\|\varphi\| + \Im^*]\mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} + \frac{[\delta_0\|\varphi\| + \hbar_0^*]\mathfrak{r}^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)} + \frac{[\delta_0^*\|\varphi\| + \hbar_1^*]\mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} \\
+ \; & \frac{|\xi|[\delta\|\varphi\| + \Phi^*]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{|\xi|[\delta^*\|\varphi\| + \Im^*]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{|\xi|[\delta_0\|\varphi\| + \hbar_0^*]\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)} \\
+ \; & \frac{|\xi|[\delta_0^*\|\varphi\| + \hbar_1^*]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} \\
\leq \; & \left[\frac{[\delta+\delta^*+\delta_0^*]}{\Gamma(\mathfrak{w}+1)} + \frac{\delta_0}{\Gamma(\mathfrak{w}+2)} + \frac{\big[|\xi|\delta + |\xi|\delta^* + |\xi|\delta_0^*\big]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{|\xi|\delta_0\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)}\right]\ell \\
+ \; & \frac{[\Phi^* + \Im^* + \hbar_1^*]}{\Gamma(\mathfrak{w}+1)} + \frac{\hbar_0^*}{\Gamma(\mathfrak{w}+2)} + \frac{\big[|\xi|\Phi^* + |\xi|\Im^* + |\xi|\hbar_1^*\big]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{|\xi|\hbar_0^*\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)} \\
\leq \; & \ell.
\end{aligned}
$$

Therefore $\|\aleph\varphi\| \leq \ell$, which means that $\aleph\mathcal{M}_\ell \subset \mathcal{M}_\ell$.

Step II: We shall show that $\aleph : \mathcal{M}_\ell \to \mathcal{M}_\ell$ is a contraction. In view of the assumption (∇_1), we have for any $\varphi, \varphi_0 \in \mathcal{M}_\ell$ and for each $\mathfrak{r} \in \psi$

$$\begin{aligned}
&|(\aleph\varphi)(\mathfrak{r})-(\aleph\varphi_0)(\mathfrak{r})|\\
&\leq \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\big[|\Phi(\varphi(\rho))-\Phi(\varphi_0(\rho))|\big]d\rho+\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\big[|\Im(\rho,\varphi(\rho))\\
&-\Im(\rho,\varphi_0(\rho))|\big]d\rho+\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\int_0^{\rho}\Big[|\hbar_0(\rho,\sigma,\varphi(\sigma))-\hbar_0(\rho,\sigma,\varphi_0(\sigma))|\Big]d\sigma d\rho\\
&+\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\int_0^{1}\Big[|\hbar_1(\rho,\sigma,\varphi(\sigma))-\hbar_1(\rho,\sigma,\varphi_0(\sigma))|\Big]d\sigma d\rho\\
&+\frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\big[|\Phi(\varphi(\mu))-\Phi(\varphi_0(\mu))|\big]d\mu\\
&+\frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\big[|\Im(\mu,\varphi(\mu))-\Im(\mu,\varphi_0(\mu))\big]d\mu\\
&+\frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\int_0^{\mu}\Big[|\hbar_0(\mu,\omega,\varphi(\omega))-\hbar_0(\mu,\omega,\varphi_0(\omega))|\Big]d\omega d\mu\\
&+\frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\int_0^{1}\Big[|\hbar_1(\mu,\omega,\varphi(\omega))-\hbar_1(\mu,\omega,\varphi_0(\omega))|\Big]d\omega d\mu\\
&\leq\left[\frac{[\delta+\delta^*+\delta_0^*]}{\Gamma(\mathfrak{w}+1)}+\frac{\delta_0}{\Gamma(\mathfrak{w}+2)}+\frac{\big[|\xi|\delta+|\xi|\delta^*+|\xi|\delta_0^*\big]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)}+\frac{|\xi|\delta_0\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)}\right]\\
&\times\|\varphi-\varphi_0\|.
\end{aligned}$$

Using Banach's fixed point theorem, we will demonstrate the existence and uniqueness estimation (9), it can be said that $\aleph$ is a contraction. When all of the requirements of Theorem 2.1 are satisfied, the proof of Theorem 3.1 is finished, and in that case, there exists $\varphi\in\mathcal{C}(\psi,\mathbb{R})$ such that $\aleph\varphi=\varphi$, which is the only solution to Problem (1)- (2) in $C(\psi,\mathbb{R})$. We will require some presumptions about this fact.

□

Theorem 3.2. *Assuming the conditions* (∇_1), (∇_2) *and* (∇_3) *are satisfied, if*

$$\frac{|\xi|}{|1-\xi\gamma|\Gamma(\mathfrak{w}+2)}\Big[\mathfrak{K}+\|\Theta\|_{\mathfrak{L}^\infty}+\|\mathfrak{H}_1\|_{\mathfrak{L}^1}+\|\mathfrak{H}_2\|_{\mathfrak{L}^1}\Big]\zeta^{\mathfrak{w}+1}<1. \tag{10}$$

Consequently, $\mathcal{C}(\psi,\mathbb{R})$ *on* ψ *has at least one solution to the fractional Volterra-Fredholm integro-dierential problem* (1)-(2).

Proof. Initially, we convert the problem (1)-(2) into a fixed point problem by defining the operator $\aleph:\mathcal{C}(\psi,\mathbb{R})\to\mathcal{C}(\psi,\mathbb{R})$ by

$$\begin{aligned}
|(\aleph\varphi)(\mathfrak{r})| &= \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\Big[|\Phi(\varphi(\rho))|+|\Im(\rho,\varphi(\rho))|+\int_0^{\rho}|\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma\\
&+\int_0^{1}|\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma\Big]d\rho+\frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\Big[|\Phi(\varphi(\mu))|\\
&+|\Im(\mu,\varphi(\mu))|+\int_0^{\mu}|\hbar_0(\mu,\omega,\varphi(\omega))|d\omega+\int_0^{1}|\hbar_1(\mu,\omega,\varphi(\omega))|d\omega\Big]d\mu\,.
\end{aligned}$$

Prior to commencing the proof of our theorem, we break down the operator $\aleph$ into the combination of two operators $\mho$ and $\eth$, where,

$$\begin{aligned}(\mho\varphi)(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\Big[\Phi(\varphi(\rho))+\Im(\rho,\varphi(\rho))+\int_0^{\rho}\hbar_0(\rho,\sigma,\varphi(\sigma))d\sigma\\ &+ \int_0^1\hbar_1(\rho,\sigma,\varphi(\sigma))d\sigma\Big]d\rho\end{aligned}$$

and

$$\begin{aligned}(\eth\varphi)(\mathfrak{r}) &= \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)}\int_0^{\zeta}(\zeta-\mu)^{\mathfrak{w}}\Big[\Phi(\varphi(\mu))+\Im(\mu,\varphi(\mu))+\int_0^{\mu}\hbar_0(\mu,\omega,\varphi(\omega))d\omega\\ &+ \int_0^1\hbar_1(\mu,\omega,\varphi(\omega))d\omega\Big]d\mu.\end{aligned}$$

For any function $\varphi\in\mathcal{C}(\psi,\mathbb{R})$, we define the norm

$$\|\varphi\|=\sup\{|\varphi(\mathfrak{r})|:\mathfrak{r}\in\psi\}.$$

The outcome of our existence will now be examined in various steps:

Step I :
Put

$$\varpi_1=\sup_{\varphi\in\mathfrak{q}_\ell}|\Phi(\varphi)|,\ \varpi_2=\sup_{(\rho,\varphi)\in\psi\times\mathfrak{q}_\ell}|\Im(\rho,\varphi)|,\ ,$$

$$\varpi_3=\sup_{(\rho,\sigma,\varphi)\in\mathcal{G}\times\mathfrak{q}_\ell}\int_0^{\rho}|\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma,\ \varpi_4=\sup_{(\rho,\sigma,\varphi)\in\mathcal{G}\times\mathfrak{q}_\ell}\int_0^1|\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma.$$

We establish the set. $\mathfrak{q}_\ell=\{\varphi\in\mathcal{C}(\psi,\mathbb{R}):\|\varphi\|\le\ell\}$ as the An assembly of components, with ℓ representing a positive real constant.

$$\ell\ge\left[\frac{|\xi|\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)}+\frac{1}{\Gamma(\mathfrak{w}+1)}\right](\varpi_1+\varpi_2+\varpi_3+\varpi_4), \tag{11}$$

and prove that $\mho\varphi+\eth\varphi_0\in\mathfrak{q}_\rho\subset\mathcal{C}(\psi,\mathbb{R})$. For each $\varphi,\varphi_0\in\mathfrak{q}_\ell$, and $\mathfrak{r}\in\psi$. It follows that

$$\begin{aligned}|\mho\varphi(\mathfrak{r})| &\le \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\Big[\|\Phi(\varphi(\rho))|+|\Im(\rho,\varphi(\rho))|+\int_0^{\rho}|\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma\\ &+ \int_0^1|\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma\Big]d\rho\\ &\le \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\rho)^{\mathfrak{w}-1}\Big[\sup_{\varphi\in\mathfrak{q}_\ell}|\Phi(\varphi)|+\sup_{(\rho,\varphi)\in\psi\times\mathfrak{q}_\ell}|\Im(\rho,\varphi)|\\ &+ \sup_{(\rho,\sigma,\varphi)\in\mathcal{G}\times\mathfrak{q}_\ell}\int_0^{\rho}|\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma+\sup_{(\rho,\sigma,\varphi)\in\mathcal{G}\times\mathfrak{q}_\ell}\int_0^1|\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma\Big]d\rho\\ &\le \frac{\varpi_1+\varpi_2+\varpi_3+\varpi_4}{\Gamma(\mathfrak{w}+1)}.\end{aligned}$$

Thus,

$$\|\mho\varphi(\mathfrak{r})\|\le\frac{\varpi_1+\varpi_2+\varpi_3+\varpi_4}{\Gamma(\mathfrak{w}+1)}. \tag{12}$$

In a similar way, for $\varphi_0 \in \mathfrak{q}_\ell$ and $\mathfrak{r} \in \psi$, we find

$$\begin{aligned}
&|\eth\varphi_0(\mathfrak{r})| \leq \\
&\frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)} \int_0^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[|\Phi(\varphi_0(\mu))| + |\Im(\mu,\varphi_0(\mu))| + \int_0^\mu |\hbar_0(\mu,\omega,\varphi_0(\omega))| d\omega \\
&+ \int_0^1 |\hbar_1(\mu,\omega,\varphi_0(\omega))| d\omega\Big] d\mu \\
&\leq \frac{|\xi|}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)} \int_0^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[\sup_{\varphi_0\in\mathfrak{q}_\ell} |\Phi(\varphi_0)| + \sup_{(\mu,\varphi_0)\in\psi\times\mathfrak{q}_\ell} |\Im(\mu,\varphi_0)| \\
&+ \sup_{(\mu,\omega,\varphi_0(\omega))\in\mathcal{G}\times\mathfrak{q}_\rho} \int_0^\mu |\hbar_0(\mu,\omega,\varphi_0(\omega))| d\omega + \sup_{(\mu,\omega,\varphi_0(\omega))\in\mathcal{G}\times\mathfrak{q}_\ell} \int_0^1 |\hbar_1(\mu,\omega,\varphi_0(\omega))| d\omega\Big] d\mu \\
&\leq \frac{|\xi|[\varpi_1+\varpi_2+\varpi_3+\varpi_4]}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} \zeta^{\mathfrak{w}+1}.
\end{aligned}$$

Therefore,

$$\|\eth\varphi_0(\mathfrak{r})\| \leq \frac{|\xi|[\varpi_1+\varpi_2+\varpi_3+\varpi_4]}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} \zeta^{\mathfrak{w}+1}. \tag{13}$$

As a result, considering the inequalities (12)- (13), we obtain

$$\begin{aligned}
\|\mho\varphi + \eth\varphi_0\| &\leq \|\mho\varphi\| + \|\eth\varphi_0\| \\
&\leq \frac{\varpi_1+\varpi_2+\varpi_3+\varpi_4}{\Gamma(\mathfrak{w}+1)} + \frac{|\xi|[\varpi_1+\varpi_2+\varpi_3+\varpi_4]}{|1-\xi\zeta|\Gamma(\mathfrak{w}+1)} \zeta^{\mathfrak{w}+1} \\
&\leq \left[\frac{|\xi|\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)} + \frac{1}{\Gamma(\mathfrak{w}+1)}\right](\varpi_1+\varpi_2+\varpi_3+\varpi_4) \\
&\leq \ell.
\end{aligned}$$

This indicates that $\mho\varphi + \eth\varphi_0 \in \mathfrak{q}_\ell$.

Step II :

We establish that $\eth$ functions as a contraction mapping on $\mathfrak{q}_\ell$. By referring to the operator's definition and employing Fubini's theorem, we can express this as follows:

$$\begin{aligned}
\eth\varphi(\mathfrak{r}) &= \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)} \int_0^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[\Phi(\varphi(\mu)) + \Im(\mu,\varphi(\mu)) \\
&+ \int_0^\mu \hbar_0(\mu,\omega,\varphi(\omega)) d\omega + \int_0^1 \hbar_1(\mu,\omega,\varphi(\omega)) d\omega\Big] d\mu \\
&= \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)} \int_0^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[\Phi(\varphi(\mu)) + \Im(\mu,\varphi(\mu))\Big] d\mu \\
&+ \frac{\xi}{(1-\xi\zeta)\Gamma(\mathfrak{w}+1)} \int_0^\zeta \int_\omega^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[\hbar_0(\mu,\omega,\varphi(\omega)) + \hbar_1(\mu,\omega,\varphi(\omega))\Big] d\mu d\omega.
\end{aligned}$$

Hence, for any $\varphi, \varphi_0 \in \mathfrak{q}_\ell$ and $\mathfrak{r} \in \psi$ we can find

$$\begin{aligned}
|\eth\varphi(\mathfrak{r}) - \eth\varphi_0(\mathfrak{r})| &\leq \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+1)} \int_0^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[|\Phi(\varphi(\mu)) - \Phi(\varphi_0(\mu))| \\
&+ |\Im(\mu,\varphi(\mu)) - \Im(\mu,\varphi_0(\mu))|\Big] d\mu \\
&+ \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+1)} \int_0^\zeta \int_\omega^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[|\hbar_0(\mu,\omega,\varphi(\omega)) \\
&- \hbar_0(\mu,\omega,\varphi_0(\omega))| + |\hbar_1(\mu,\omega,\varphi(\omega)) - \hbar_1(\mu,\omega,\varphi_0(\omega))|\Big] d\mu d\omega \\
&\leq \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+1)} \int_0^\zeta (\zeta-\mu)^{\mathfrak{w}} \Big[\mathfrak{K}|\varphi(\mu) - \varphi_0(\mu)| \\
&+ \Theta(\mu)|\varphi(\mu) - \varphi_0(\mu)|\Big] d\mu \\
&+ \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+1)} \int_0^\zeta \int_\omega^\zeta (\zeta-\omega)^{\mathfrak{w}} \Big[\mathfrak{H}_1(\mu)|\varphi(\omega) - \varphi_0(\omega)| \\
&+ \mathfrak{H}_1(\mu)|\varphi(\omega) - \varphi_0(\omega)|\Big] d\mu d\omega \\
&\leq \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+2)} \Bigg[\mathfrak{K}\|\varphi - \varphi_0\| + \|\Theta\|_{\mathfrak{L}^\infty}\|\varphi - \varphi_0\| \\
&+ (\|\mathfrak{H}_1\|_{\mathfrak{L}^1} + \|\mathfrak{H}_2\|_{\mathfrak{L}^1})\|\varphi - \varphi_0\|\Bigg] \zeta^{\mathfrak{w}+1} \\
&\leq \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+2)} \Bigg[\mathfrak{K} + \|\Theta\|_{\mathfrak{L}^\infty} + (\|\mathfrak{H}_1\|_{\mathfrak{L}^1} + \|\mathfrak{H}_2\|_{\mathfrak{L}^1})\Bigg] \zeta^{\mathfrak{w}+1} \Omega.
\end{aligned}$$

Thus,

$$|\eth\varphi - \eth\varphi_0| \leq \frac{|\xi|}{(|1-\xi\zeta|)\Gamma(\mathfrak{w}+2)} \Bigg[\mathfrak{K} + \|\Theta\|_{\mathfrak{L}^\infty}\| + \|\mathfrak{H}_1\|_{\mathfrak{L}^1} + \|\mathfrak{H}_2\|_{\mathfrak{L}^1}|\Bigg] \zeta^{\mathfrak{w}+1}\|\varphi - \varphi_0\|.$$

Thus, utilizing equation (10), we can deduce that $\eth$ functions as a contractions on a $\mathfrak{q}_\ell$.

Step III:

To demonstrate that $\mho$ is a compact operator, we claim that $\mho(\bar{\mathfrak{q}}_\ell)$ is a compact subset of $\mathcal{C}(\psi,\mathbb{R})$. We merely need to demonstrate this to do so. $\mho(\bar{\mathfrak{q}}_\ell)$ is a uniformly bounded and equicontinuous subset of $\mathcal{C}(\psi,\mathbb{R})$.. To start, it is evident by inequality (12) that $\mho(\bar{\mathfrak{q}}_\ell)$ is uniformly bounded. Next, we will prove that $\mho(\bar{\mathfrak{q}}_\ell)$ is an equicontinuous subset of $\mathcal{C}(\psi,\mathbb{R})$. We have for any of these. $\varphi \in \mathfrak{q}_\ell$ and for each $\mathfrak{r}_1, \mathfrak{r}_2 \in \psi$ where $\mathfrak{r}_1 \leq \mathfrak{r}_2$:

$$\begin{aligned}
&|\mho\varphi(\mathfrak{r}_1) - \mho\varphi(\mathfrak{r}_2)| \\
&= \frac{1}{\Gamma(\mathfrak{w})} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} |(\mathfrak{r}_1-\rho)^{\mathfrak{w}-1}| \Big[|\Phi(\varphi(\rho))| + |\Im(\rho,\varphi(\rho))| \\
&+ \int_0^\rho |\hbar_0(\rho,\sigma,\varphi(\sigma))| d\sigma + \int_0^1 |\hbar_1(\rho,\sigma,\varphi(\sigma))| d\sigma\Big] d\rho \\
&+ \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_1} |(\mathfrak{r}_2-\rho)^{\mathfrak{w}-1} - (\mathfrak{r}_1-\rho)^{\mathfrak{w}-1}| \Big[|\Phi(\varphi(\rho))| + |\Im(\rho,\varphi(\rho))|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\rho |\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma+\int_0^1 |\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma\Big]d\rho\\
&\leq \frac{1}{\Gamma(\mathfrak{w})}\Big[\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}|(\mathfrak{r}_1-\rho)^{\mathfrak{w}-1}|+\int_0^{\mathfrak{r}_1}|(\mathfrak{r}_2-\rho)^{\mathfrak{w}-1}-(\mathfrak{r}_1-\rho)^{\mathfrak{w}-1}|d\rho\Big]\\
&\times\Big[\sup_{\varphi\in\mathfrak{q}_\ell}|\Phi(\varphi)|+\sup_{(\rho,\varphi)\in\psi\times\mathfrak{q}_\ell}|\Im(\rho,\varphi)|\\
&+\sup_{(\rho,\sigma,\varphi)\in\mathcal{G}\times\mathfrak{q}_\ell}\int_0^\rho|\hbar_0(\rho,\sigma,\varphi(\sigma))|d\sigma+\sup_{(\rho,\sigma,\varphi)\in\mathcal{G}\times\mathfrak{q}_\ell}\int_0^1|\hbar_1(\rho,\sigma,\varphi(\sigma))|d\sigma\Big]\\
&\leq \frac{\varpi_1+\varpi_2+\varpi_3+\varpi_4}{\Gamma(\mathfrak{w}+1)}[2(\mathfrak{r}_2-\mathfrak{r}_1)^{\mathfrak{w}}+\mathfrak{r}_2^{\mathfrak{w}}-\mathfrak{r}_1^{\mathfrak{w}}]\\
&\leq \frac{\varpi_1+\varpi_2+\varpi_3+\varpi_4}{\Gamma(\mathfrak{w}+1)}[(\mathfrak{r}_2-\mathfrak{r}_1)^{\mathfrak{w}}]\\
&\longrightarrow 0\ as\ \mathfrak{r}_1\to\mathfrak{r}_2.
\end{aligned}
$$

Then

$$\|\mho\varphi(\mathfrak{r}_1)-\mho\varphi(\mathfrak{r}_2)\|\to 0,$$

which means that $\mho(\mathfrak{q}_\ell)$ is equicontinuous. □

Lastly, considering the continuity of $\Phi,\Im,\hbar_0$ and $\hbar_1$, it can be inferred that the operator $\mho:\mathfrak{q}_\ell\longrightarrow\mathfrak{q}_\ell$ is continuous. Consequently, the operator $\mho$ is compact on $\mathfrak{q}_\ell$. With this, all the conditions of Theorem 2.2 are satisfied. Thus, the operator $\aleph=\mho+\eth$ possesses a fixed point in $\mathfrak{q}_\ell$. As a result, the fractional Volterra-Fredholm integro-differential problem (2) has a solution $\varphi\in\mathcal{C}(\psi,\mathbb{R})$. This concludes the proof of Theorem 3.2.

4. An Application

Example 1. Consider the following nonlocal fractional integro-differential problem:

$$ {}^{C}\mathfrak{D}_{0^+}^{\frac{1}{4}}\varphi(\mathfrak{r})=\frac{1}{20}\cos(\varphi(\mathfrak{r}))+\frac{\varphi(\mathfrak{r})}{30+e^{-2\mathfrak{r}}}+\int_0^{\mathfrak{r}}\frac{e^{2(\sigma-\mathfrak{r})}}{45+e^{\mathfrak{r}-1}}\varphi(\sigma)d\sigma+\int_0^1\frac{e^{\sigma\mathfrak{r}}}{25}\varphi(\sigma)d\sigma \tag{14}$$

$$\varphi(0)=\frac{1}{15}\int_0^{0.5}\varphi(\sigma)d\sigma. \tag{15}$$

For $\varphi,\varphi_0\in\mathbb{R}^+$ and $\mathfrak{r}\in[0,1]$ we have:

$$|\Phi(\varphi)-\Phi(\varphi_0)|\leq\frac{1}{20}\|\varphi-\varphi_0\|$$

$$|\Im(\mathfrak{r},\varphi)-\Im(\mathfrak{r},\varphi_0)|\leq\frac{1}{30}\|\varphi-\varphi_0\|$$

$$|\hbar_0(\mathfrak{r},\sigma,\varphi(\sigma))-\hbar_0(\mathfrak{r},\sigma,\varphi_0(\sigma))|\leq\frac{1}{45}\|\varphi-\varphi_0\|$$

and

$$|\hbar_1(\mathfrak{r},\sigma,\varphi(\sigma))-\hbar_1(\mathfrak{r},\sigma,\varphi_0(\sigma))|\leq\frac{1}{25}\|\varphi-\varphi_0\|.$$

Currently, the conditions $(\Delta_1)-(\Delta_3)$ are fulfilled, given that: $\delta=\frac{1}{20}$, $\delta^*=\frac{1}{30}$, $\delta_0=\frac{1}{45}$ and $\delta_0^*=\frac{1}{25}$, subsequently, through a series of calculations, it is determined that.

$$\begin{aligned}\mathfrak{B}\mathfrak{e} &= \left[\frac{[\delta+\delta^*+\delta_0^*]}{\Gamma(\mathfrak{w}+1)}+\frac{\delta_0}{\Gamma(\mathfrak{w}+2)}+\frac{\left[|\xi|\delta+|\xi|\delta^*+|\xi|\delta_0^*\right]\zeta^{\mathfrak{w}+1}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+2)}+\frac{|\xi|\delta_0\zeta^{\mathfrak{w}+2}}{|1-\xi\zeta|\Gamma(\mathfrak{w}+3)}\right]\\ &= 0.33<1.\end{aligned}$$

Therefore, by applying Theorem 3.1 the problem (14)-(15) has a unique solution on ψ.

Example 2. Consider the following nonlocal fractional Volterra-Fredholm integro-differential problem:

$$^{C}\mathfrak{D}^{\frac{1}{3}}_{0^+}\varphi(\mathfrak{r})=\frac{1}{30}\cos^2(\varphi(\mathfrak{r}))+\frac{2\varphi(\mathfrak{r})}{15+3e^{2\mathfrak{r}}}+\int_0^{\mathfrak{r}}\frac{2e^{3\mathfrak{r}}}{7+e^{\sigma}}\varphi(\sigma)d\sigma+\int_0^1\frac{e^{\mathfrak{r}+1}}{1+7e^{-\sigma}}\varphi(\sigma)d\sigma \quad (16)$$

$$\varphi(0)=\frac{1}{12}\int_0^{0.3}\varphi(\sigma)d\sigma. \quad (17)$$

Then for $\varphi,\varphi_0\in\mathbb{R}^+$ and $\mathfrak{r}\in[0,1]$ we have:

$$|\Phi(\varphi)-\Phi(\varphi_0)|\le\frac{1}{15}\|\varphi-\varphi_0\|$$

$$|\phi(z,\varphi)-\phi(z,\varphi_0)|\le\frac{2}{15+3e^{2\mathfrak{r}}}\|\varphi-\varphi_0\|$$

$$|\hbar_0(\mathfrak{r},\sigma,\varphi(\sigma))-\hbar_0(\mathfrak{r},\sigma,\varphi_0(\sigma))|\le\frac{1}{5}e^{3\mathfrak{r}}\|\varphi-\varphi_0\|$$

and

$$|\hbar_1(\mathfrak{r},\sigma,\varphi(\sigma))-\hbar_1(\mathfrak{r},\sigma,\varphi_0(\sigma))|\le\frac{1}{8}e^{\mathfrak{r}+1}\|\varphi-\varphi_0\|.$$

Currently, the conditions $(\nabla_1),(\nabla_2)$ and (∇_3) are fulfilled, given that $\mathfrak{K}=\frac{1}{15}$, $\Theta(\mathfrak{r})=\frac{2}{15+3e^{2\mathfrak{r}}}$, $\mathfrak{H}_1(z)=\frac{1}{5}e^{3\mathfrak{r}}$ and $\mathfrak{H}_2(z)=\frac{1}{8}e^{z+1}$, where $\|\Theta\|_{\mathfrak{L}^\infty}=\frac{1}{9}$, $\|\mathfrak{H}_1\|_{\mathfrak{L}^1}=\frac{1}{15}(e^3-1)$, $\|\mathfrak{H}_2\|_{\mathfrak{L}^1}=\frac{e}{8}(e-1)$ subsequently, through a series of calculations, it is determined that.

$$\frac{|\xi|}{|1-\xi\gamma|\Gamma(\mathfrak{w}+2)}\left[\mathfrak{K}+\|\Theta\|_{\mathfrak{L}^\infty}+\|\mathfrak{H}_1\|_{\mathfrak{L}^1}+\|\mathfrak{H}_2\|_{\mathfrak{L}^1}\right]\zeta^{\mathfrak{w}+1}=0.1461<1.$$

The confirms that the (10) condition is true. This leads us to conclude that the nonlocal (FVFIDE) problem (16)-(17) has a solution on ψ by using theorem 3.2.

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