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A NEW VARIANT OF KIKKAWA-SUZUKI TYPE FIXED POINT THEOREM FOR MULTI-VALUED MAPPINGS WITH STABILITY ANALYSIS AND APPLICATION TO VOLTERRA INTEGRAL INCLUSION

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Abstract. This paper aims to present a new variant of Kikkawa-Suzuki type common fixed point theorem for multi-valued mappings in the framework of partial metric space. This result is followed by the establishment of a Reich type common fixed point theorem applicable to multi-valued mappings. Some illustrative examples are provided to demonstrate our findings. Moreover, we analyse the data dependence and stability of fixed point sets for such mappings. To show the practical significance of the derived results, an application is shown to a system of Volterra integral inclusions.

Keywords: Fixed point, multi-valued mapping, partial metric space, integral inclusion.

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1. Introduction

Fixed point theory deals with fundamental results in nonlinear analysis that establish the existence of points that remain unchanged under certain mappings or functions. Over the years, the most well-known Banach's fixed point theorem[6] has been extended and generalized by many researchers, leading to a wide range of powerful fixed point results applicable to various settings. Fixed point theory for multi-valued mappings generalize the concept of fixed point from single valued to set-valued mappings. Multi-valued mappings are used to model situations where a single input can lead to multiple feasible solution with potential outcomes. In 1969, S.B. Nadler [17] initially proved some fixed point theorems for multi-valued contraction mapping on complete metric space. Thereafter, several researchers have done a rigorous study in this area (refer to [7], [11]).

Partial metric spaces are generalization of metric spaces in the sense of defining a non-zero self distance between points of a metric space. The concept of partial metric was introduced by S.G. Matthews in 1994 [15]. In [3],[14], Altun et al. and Masiha et al. developed fixed point theory in partial metric space considering contractive type

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mappings and weakly contractive type mappings respectively. Different interesting fixed point results in partial metric spaces can be found in [1],[5],[9],[10],[12].

In [13], Kikkawa and Suzuki derived a fixed point theorem for generalized contractions with constants in a complete metric space. Their findings represented a broader extension of a fixed point theorem previously demonstrated by Suzuki [18] ,which itself was a new generalization of the Banach contraction principle, offering insights into the characterization of metric completeness. Motivated by these works, here we derive a new variant of Kikkawa-Suzuki type common fixed point theorem considering multi-valued mapping in partial metric space along with a Reich type common fixed point theorem. As application, we provide an analysis of data dependence and stability of fixed point sets for such mappings. Another application is given for showing the existence of solution to a system of Volterra integral inclusions.

2. Preliminaries and Definitions

Before going to the main findings, we present some basic definitions and related results.

Definition 2.1. [15] *Consider a non-empty set X, and let* $p : X \times X \to [0, \infty)$ *be a mapping that satisfies the following axioms for all* $x, y, z \in X$*:*
$P_0 :\ 0 \leq p(x,x) \leq p(x,y)$
$P_1 :\ p(x,x) = p(x,y) = p(y,y)$ *if and only if* $x = y$
$P_2 :\ p(x,y) = p(y,x)$
$P_3 :\ p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
Then p is called the partial metric and the pair (X,p) *forms a partial metric space.*

For example, on the set of non-negative real numbers, $p_1(x,y) = \max\{x,y\}$*,* $p_2(x,y) = 1 + |x-y|$ *are some partial metrics. Some other examples are as follows.*
(i) If $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ *and* $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ *then* (X,p) *is a partial metric space (refer to* [15]*).*
(ii) Let $X = \{0,2,4\}$ *and* $p(x,y) = \frac{1}{4}|e^x - e^y| + \frac{1}{2}\max\{e^x, e^y\}$ *for all* $x, y \in X$*. Then p is a partial metric on X.*

Definition 2.2. [15] *Let* (X,p) *be a partial metric space.*

(i) *A sequence* $\{x_n\}$ *in* (X,p) *is said to be Cauchy if and only if* $\lim_{n,m\to\infty} p(x_n,x_m)$ *exists and is finite.*
(ii) *A sequence* $\{x_n\}$ *in (X,p) converges to y in X if and only if* $\lim_{n\to\infty} p(x_n,y) = \lim_{n\to\infty} p(x_n,x_n) = p(y,y)$.

Definition 2.3. [15] *A partial metric space (X,p) is said to be complete if and only if every Cauchy sequence* $\{x_n\}$ *in* (X,p) *converges to a point y in X, that is,* $p(y,y) = \lim_{n,m\to\infty} p(x_n,x_m)$.

For a partial metric space (X,p), the function $p^s : X \times X \to \mathbb{R}$, defined by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x, y \in X$ is a metric on X and it is known as induced metric (refer to [8]).

Lemma 2.1. [8] *In a partial metric space* (X,p)*,*

(i) *A sequence* $\{x_n\}$ *is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space* (X,p^s)*.*

(ii) *A partial metric space (X,p) is complete if and only if the metric space* (X, p^s) *is complete. Moreover,* $\lim_{n\to\infty} p^s(x_n, z) = 0$ *if and only if*

$$p(z,z) = \lim_{n\to\infty} p(x_n, z) = \lim_{m,n\to\infty} p(x_n, x_m).$$

Remark 2.1. *In a partial metric space* (X, p)*, the uniqueness of the limit of a sequence is not guaranteed. Furthermore, even if two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* (X, p) *converge to* x *and* y *respectively, there is no assurance that* $p(x_n, y_n)$ *will converge to* $p(x, y)$*, i.e.,* p *need not be continuous.*

For a partial metric space (X, p), let $P(X)$ denote the set of all subsets of X and $CB^p(X)$ denote the set containing all non-empty closed and bounded subsets of X with respect to the partial metric p. For any $A, B \in CB^p(X)$, the partial Hausdorff metric H_p is defined by

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where $p(a, A) = \inf\{p(a, x) : x \in A\}$, $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$, and $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$ (refer to [5]).

Lemma 2.2. [5] *Consider* (X, p) *a partial metric space and let* $h > 1$*. For any* $a \in A$*, there exists* $b = b(a) \in B$*, where* $A, B \in CB^p(X)$ *such that* $p(a, b) \leq hH_p(A, B)$*.*

Definition 2.4. *Let* (X, p) *be a partial metric space. A point* $x \in X$ *is called a fixed point of a multi-valued mapping* $T : X \to P(X)$ *if* $x \in Tx$*.*

In [13], Kikkawa and Suzuki proved the following multi-valued fixed point theorem.

Theorem 2.1. [13] *Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$*. Define a strictly decreasing function* η *from* $[0, 1)$ *onto* $(\frac{1}{2}, 1]$ *by* $\eta(r) = \frac{1}{1+r}$*. Assume that there exists* $r \in [0, 1)$ *such that*

$$\eta(r)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq rd(x, y) \text{ for all } x, y \in X.$$

Then there exists $z \in X$ *such that* $z \in Tz$*.*

3. Main Results

Considering partial metric space, we prove the following new variant of Kikkawa-Suzuki type common fixed point theorem. Here $CF(S, T)$ denotes the set of all common fixed points of S and T.

Theorem 3.1. *Let* (X, p) *be a complete partial metric space and* $S, T : X \to CB^p(x)$ *be multi-valued mappings. Let* θ *be a strictly decreasing function from* $[0, \frac{1}{2})$ *onto* $(\frac{2}{3}, 1]$ *given by* $\theta(r) = \frac{1}{1+r}$*. Assume that there exist* r_1 *and* r_2 *in* $[0, \frac{1}{2})$ *with* $r_2 \leq r_1$ *satisfying*

$$\theta(r_1)\min\{p(x, Sx), p(z, Sz)\} + \theta(r_2)\min\{p(y, Ty), p(z, Tz)\} \leq p(x, z) + p(y, z)$$
$$\Rightarrow \max\{H_p(Sx, Tz), H_p(Sz, Ty)\} \leq r_1 p(x, z) + r_2 p(y, z) \text{ for all } x, y, z \in X. \quad (1)$$

Then there exists a unique $u \in X$ *such that* $u \in CF(S, T)$*.*

Proof. Let $x_0 \in X$ be arbitrary. For $x_1 \in Sx_0$, we have,

$$\theta(r_1)\min\{p(x_0, Sx_0), p(x_1, Sx_1\} \leq \theta(r_1)p(x_0, Sx_0) \leq p(x_0, Sx_0) \leq p(x_0, x_1).$$

Let $h \in (1, \frac{1}{2r_1})$. By Lemma 2.2, there exists $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq hH_p(Sx_0, Tx_1). \quad (2)$$

Now, $\theta(r_1)\min\{p(x_0,Sx_0),p(x_1,Sx_1\}+\theta(r_2)\min\{p(x_2,Tx_2),p(x_1,Tx_1)\}$
$$\le p(x_0,x_1)+p(x_1,x_2).$$

By (1), $\max\{H_p(Sx_0,Tx_1),H_p(Sx_1,Tx_2)\}\le r_1p(x_0,x_1)+r_2p(x_1,x_2)$.
So, $H_p(Sx_0,Tx_1)\le r_1\{p(x_0,x_1)+p(x_1,x_2)\}$ (since $r_2\le r_1$), (3)

and by (2), $p(x_1,x_2)\le hH_p(Sx_0,Tx_1)\le hr_1\{p(x_0,x_1)+p(x_1,x_2)\}$,
i.e., $p(x_1,x_2)\le \frac{hr_1}{1-hr_1}p(x_0,x_1)$.
Again, for $x_2\in Tx_1$, there exists $x_3\in Sx_2$ such that
$$p(x_2,x_3)\le hH_p(Tx_1,Sx_2), \tag{4}$$
and $\theta(r_1)\min\{p(x_2,Sx_2),p(x_3,Sx_3)\}+\theta(r_2)\min\{p(x_1,Tx_1),p(x_2,Tx_2\}$
$$\le p(x_3,x_2)+p(x_2,x_1).$$

Applying (1), $\max\{H_p(Sx_3,Tx_2),H_p(Sx_2,Tx_1)\}\le r_1p(x_3,x_2)+r_2p(x_2,x_1)$,
i.e., $H_p(Sx_2,Tx_1)\le r_1\{p(x_3,x_2)+p(x_2,x_1)\}$. (5)
Using (5) in (4), we get,
$$p(x_2,x_3)\le hr_1\{p(x_3,x_2)+p(x_2,x_1)\}=\frac{hr_1}{1-hr_1}p(x_1,x_2)\le (\frac{hr_1}{1-hr_1})^2p(x_0,x_1).$$
Continuing in this way, we generate a sequence $\{x_n\}$ in X with
$$x_{2n+1}\in Sx_{2n},\ x_{2n+2}\in Tx_{2n+1},\ x_{2n+3}\in Sx_{2n+2} \text{ such that}$$
$$p(x_{2n+1},x_{2n+2})\le (\frac{hr_1}{1-hr_1})^{2n+1}p(x_0,x_1),$$
$$\text{and}\ \ p(x_{2n+2},x_{2n+3})\le (\frac{hr_1}{1-hr_1})^{2n+2}p(x_0,x_1)\ \text{ for all } n\in\mathbb{N}.$$
Thus, for each $n\in\mathbb{N}$,
$$p(x_n,x_{n+1})\le k^np(x_0,x_1),\ \text{where } k=\frac{hr_1}{1-hr_1}<1. \tag{6}$$
Now, we show that $\{x_n\}$ is a Cauchy sequence in (X,p). For all $n,m\in\mathbb{N}$,
$$\begin{aligned}p(x_n,x_{n+m})&\le p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\dots\dots\dots+p(x_{n+m-1},x_{n+m})\\ &\le (k^n+k^{n+1}+\dots\dots+k^{n+m-1}+\dots\dots)p(x_0,x_1)\\ &=\frac{k^n}{1-k}p(x_0,x_1)\to 0 \quad \text{as } n\to\infty \quad (\text{since } 0<k<1).\end{aligned}$$
Now, $p^s(x_n,x_{n+m})\le 2p(x_n,x_{n+m})\to 0$ as $n\to\infty$.
Thus $\{x_n\}$ is a Cauchy sequence in (X,p^s). Using Lemma (2.1), (X,p^s) is complete, since (X,p) is complete. So, there exists some $u\in X$ such that $\lim_{n\to\infty}p^s(x_n,u)=0$.
By Lemma 2.1,
$$p(u,u)=\lim_{n\to\infty}p(x_n,u)=\lim_{n,m\to\infty}p(x_n,x_m)=0. \tag{7}$$

Now, $\theta(r_1)\min\{p(x_{2n},Sx_{2n}),p(x_{2n+1},Sx_{2n+1})\}+\theta(r_2)\min\{p(x_{2n+2},Tx_{2n+2}),$
$$p(x_{2n+1},Tx_{2n+1})\}\le p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+2}).$$

So by (1),

$$\max\{H_p(Sx_{2n}, Tx_{2n+1}), H_p(Sx_{2n+1}, Tx_{2n+2})\} \leq r_1 p(x_{2n}, x_{2n+1}) + r_2 p(x_{2n+1}, x_{2n+2}),$$

and hence, $H_p(Sx_{2n}, Tx_{2n+1}) \leq r_1\{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})\}$.

Since $p(u, Sx_{2n}) \leq p(u, x_{2n+2}) + H_p(Tx_{2n+1}, Sx_{2n})$
$\leq p(u, x_{2n+2}) + r_1\{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})\}$,

taking $n \to \infty$ and using (7), we get, $\lim_{n\to\infty} p(u, Sx_{2n}) = 0$.
So, for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $p(u, Sx_{2n}) < \epsilon$ for all $n \geq n_0$.
Taking $\epsilon = r_1 p(u, x)$ with $x \neq u$, for $k \geq n_0$, we get,

$$p(u, Sx_{2k}) < r_1 p(u, x). \tag{8}$$

For each $n \in \mathbb{N}$, we choose $v_n \in Sx_{2k}$ such that

$$p(u, v_n) \leq p(u, Sx_{2k}) + \frac{1}{n} p(x, u). \tag{9}$$

Now, $p(x, Sx_{2k}) \leq p(x, v_n) \leq p(x, u) + p(u, Sx_{2k}) + \frac{1}{n} p(x, u)$ (using (9))
$< (1 + r_1 + \frac{1}{n}) p(x, u)$, for all $n \in \mathbb{N}$ (using (8)).

In the limit as $n \to \infty$,

$$\frac{1}{1 + r_1} p(x, Sx_{2k}) \leq p(x, u), \quad \text{i.e., } \theta(r_1) p(x, Sx_{2k}) \leq p(x, u).$$

So, $\theta(r_1) \min\{p(x, Sx_{2k}), p(u, Su)\} \leq p(x, u)$ for all $x \neq u$.
Since $k \geq n_0$ is arbitrary, so,

$$\theta(r_1) \min\{p(x, Sx_{2n}), p(u, Su)\} \leq p(x, u) \quad \text{for all } n \geq n_0 \text{ and for all } x \neq u.$$

In particular, for $x = x_{2n}$,

$$\theta(r_1) \min\{p(x_{2n}, Sx_{2n}), p(u, Su)\} \leq p(x_{2n}, u) \quad \text{for all } n \geq n_0. \tag{10}$$

Again, $p(u, Tx_{2n+1}) \leq p(u, x_{2n+1}) + H_p(Sx_{2n}, Tx_{2n+1})$
$\leq p(u, x_{2n+1}) + r_1\{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})\}$.

Taking $n \to \infty$ and using (9), we get, $\lim_{n\to\infty} p(u, Tx_{2n+1}) = 0$.
In a similar manner as in the previous case, we can show that

$$\theta(r_2) \min\{p(x_{2n+1}, Tx_{2n+1}), p(u, Tu)\} \leq p(x_{2n+1}, u) \text{ for all } n \geq n_1. \tag{11}$$

Let $n_2 = \max\{n_0, n_1\}$ and adding (10) and (11), we get,

$$\theta(r_1) \min\{p(x_{2n}, Sx_{2n}), p(u, Su)\} + \theta(r_2) \min\{p(x_{2n+1}, Tx_{2n+1}), p(u, Tu)\}$$
$$\leq p(x_{2n}, u) + p(x_{2n+1}, u) \text{ for all } n \geq n_2.$$

Using (1),

$$\max\{H_p(Sx_{2n}, Tu), H_p(Tx_{2n+1}, Su)\} \leq r_1 p(x_{2n}, u) + r_2 p(x_{2n+1}, u) \quad \text{for all } n \geq n_2. \tag{12}$$

Now, $p(u, Tu) \leq p(u, x_{2n+1}) + H_p(Sx_{2n}, Tu)$
$\leq p(u, x_{2n+1}) + r_1\{p(x_{2n}, u) + p(x_{2n+1}, u)\}$ for all $n \geq n_2$. (by (12))

As $n \to \infty$, $p(u, Tu) = 0$, i.e., $u \in Tu$.
Similarly, $p(u, Su) = 0$, and thus, $u \in CF(S,T)$.
To prove the uniqueness, let $u_1, u_2 \in CF(S,T)$ such that $u_1 \neq u_2$.

$$\text{Now, } \theta(r_1)\min\{p(u_1,Su_1),p(u_2,Su_2)\}+\theta(r_2)\min\{p(u_2,Tu_2),p(u_1,Tu_1)\} \leq p(u_1,u_2)+p(u_2,u_1),$$

and so, $\max\{H_p(Su_1,Tu_2),H_p(Su_2,Tu_1)\} \leq r_1p(u_1,u_2)+r_2p(u_1,u_2) \leq 2r_1p(u_1,u_2)$.
Again, $p(u_1,u_2) \leq hH_p(Su_1,Tu_2)$ and $p(u_2,u_1) \leq hH_p(Su_2,Tu_1)$. So,

$$p(u_1,u_2) \leq h\max\{H_p(Su_1,Tu_2),H_p(Su_2,Tu_1)\} \leq 2hr_1p(u_1,u_2), \text{ i.e., } p(u_1,u_2)=0.$$

Similarly, $p(u_1,u_1) = 0 = p(u_2,u_2)$, and thus, $u_1 = u_2$. □

Corollary 3.1. *Let (X,p) be a complete partial metric space and $S : X \to CB^p(x)$ be a multi-valued mapping. Let θ be a strictly decreasing function from $[0,\frac{1}{2})$ onto $(\frac{2}{3},1]$ given by $\theta(r) = \frac{1}{1+r}$. Assume that there exist r_1 and r_2 in $[0,\frac{1}{2})$ with $r_2 \leq r_1$ satisfying*

$$\begin{aligned}&\theta(r_1)\min\{p(x,Sx),p(z,Sz)\}+\theta(r_2)\min\{p(y,Sy),p(z,Sz)\} \leq p(x,z)+p(y,z)\\ &\Rightarrow \max\{H_p(Sx,Sz),H_p(Sz,Sy)\} \leq r_1p(x,z)+r_2p(y,z) \quad \text{for all } x,y,z \in X. \qquad (13)\end{aligned}$$

Then there exists $u \in X$ such that $u \in F(S)$. (Here, $F(S)$ denotes the set of all fixed points of S.)

Remark 3.1. *For $S = T$, $x = y = z$ and taking $r_1 = r_2 = r \in [0,1)$ and θ from $[0,1)$ onto $(\frac{1}{2},1]$, Corollary 3.1 reduces to Theorem 2 of [2].*

Remark 3.2. *It can be seen that Theorem 3.1 does not hold in general for an incomplete partial metric space. For example, let $X = (0,\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = \max\{x,y\}$, $x,y \in X$, which is not complete. Consider $S,T : X \to CB^p(X)$ defined by*

$$Sx = Tx = \{\frac{x}{5}\} \text{ for all } x \in X.$$

Then S and T satisfy condition (1) of Theorem 3.1. But S and T do not have a common fixed point.

The following example exhibits Theorem 3.1.

Example 3.1. *Let $X = \{1,2,3\}$ and define a complete partial metric p on X by*
$p(1,1) = 0 = p(2,2)$, $p(3,3) = \frac{2}{3}$,
$p(1,2) = p(2,1) = \frac{1}{3}$, $p(2,3) = p(3,2) = \frac{14}{17}$, $p(1,3) = p(3,1) = \frac{9}{11}$.
Let S, $T : X \to CB^p(X)$ be defined by

$$Sx = \{1\} \text{ for all } x \in X \quad \text{and } Tx = \begin{cases}\{1\}, & \text{if } x \in \{1,2\}\\ \{1,2\}, & \text{otherwise}\end{cases}.$$

Obviously {1} and {1,2} are closed in X with respect to the partial metric p.
Let $r_1 = \frac{12}{25}$, and $r_2 = \frac{9}{20}$, so that $\theta(r_1) = \frac{25}{37}$ and $\theta(r_2) = \frac{20}{29}$.

Now, $p(1,S1) = 0 = p(1,T1)$, $p(2,S2) = \dfrac{1}{3} = p(2,T2)$, $p(3,S3) = \dfrac{9}{11} = p(3,T3)$,
$H_p(S1,T1) = 0 = H_p(S1,T2) = H_p(S2,T2) = H_p(S2,T1) = H_p(S3,T1) = H_p(S3,T2)$.
$H_p(S1,T3) = \dfrac{1}{3} = H_p(S2,T3) = H_p(S3,T3)$.

For convenience, we take

$\eta_1(x,y,z) = \theta(r_1)\min\{p(x,Sx),p(z,Sz)\} + \theta(r_2)\min\{p(y,Ty),p(z,Tz)\}$
$\eta_2(x,y,z) = \max\{H_p(Sx,Tz),H_p(Sz,Ty)\}$, $\xi_1(x,y,z) = p(x,z) + p(y,z)$
and $\xi_2(x,y,z) = r_1p(x,z) + r_2p(y,z)$.

The values of η_i, ξ_i $(i = 1,2)$ for different values of x, y and z are depicted in the following table.

x	y	z	η_1	ξ_1	η_2	ξ_2	x	y	z	η_1	ξ_1	η_2	ξ_2	x	y	z	η_1	ξ_1	η_2	ξ_2
1	1	1	0	0	0	0	2	1	1	0	0.33	0	0.16	3	1	1	0	0.82	0	0.39
1	1	2	0	0.67	0	0.31	2	1	2	0.22	0.33	0	0.15	3	1	2	0.22	1.15	0	0.54
1	1	3	0	1.63	0.33	0.76	2	1	3	0.22	1.64	0.33	0.76	3	1	3	0.55	1.48	0.33	0.68
1	2	1	0	0.33	0	0.15	2	2	1	0	0.67	0	0.31	3	2	1	0	1.15	0	0.54
1	2	2	0.23	0.33	0	0.16	2	2	2	0.45	0	–	–	3	2	2	0.45	0.82	0	0.39
1	2	3	0.23	1.64	0.33	0.76	2	2	3	0.45	1.64	0.33	0.76	3	2	3	0.78	1.49	0.33	0.69
1	3	1	0	0.82	0.33	0.36	2	3	1	0	1.15	0.33	0.52	3	3	1	0	1.64	0.33	0.76
1	3	2	0.23	1.15	0.33	0.53	2	3	2	0.45	0.82	0.33	0.37	3	3	2	0.45	1.64	0.33	0.76
1	3	3	0.23	1.48	0.33	0.69	2	3	3	0.78	1.49	0.33	0.69	3	3	3	1.11	1.33	0.33	0.62

From the table, it is clear that if $\eta_1(x,y,z) \le \xi_1(x,y,z)$ then $\eta_2(x,y,z) \le \xi_2(x,y,z)$ for all $x,y,z \in X$, that is, the hypothesis (1) of Theorem 3.1 is satisfied by the mappings S and T. Hence S and T have a unique common fixed point, which is clearly 1 here.

Our next result is a Reich type common fixed point theorem.

Theorem 3.2. *Let (X,p) be a complete partial metric space and $S,T : X \to CB^p(x)$ be multi-valued mappings. Let θ be a strictly decreasing function from $[0,1)^3$ onto $(-\infty,1]$ given by $\theta(x,y,z) = \frac{1-y-z}{1+x}$, $x,y,z \in [0,1)$. If for some non-negative real numbers k_1,k_2,k_3 and h_1,h_2,h_3 with $k_3 \le k_2$, $h_3 \le h_2$, $,k_1+k_2+k_3 = h_1+h_2+h_3 = \mu$ and $2k_1+3k_2+k_3, 2h_1+3h_2+h_3 \in [0,\frac{1}{2})$, S and T satisfy the following conditions:*

(i) $\theta(k_1,k_2,k_3)\min\{p(x,Sx),p(z,Sz)\} + \theta(h_1,h_2,h_3)\min\{p(y,Ty),p(z,Tz)\}$
$$\le p(x,z) + p(y,z)$$

$$\Rightarrow \max\{H_p(Sx,Tz),H_p(Sz,Ty)\} \le k_1p(x,z) + k_2p(x,Sx) + k_3p(z,Tz) + h_1p(y,z) + h_2p(z,Sz) + h_3p(y,Ty) \text{ for all } x,y,z \in X, \quad (14)$$

(ii) $p(x,Sx) + p(y,Ty) \le 2p(x,y)$ *for all* $x,y \in X$,
then there exists $u \in X$ such that $u \in CF(S,T)$.

Proof. Let $x_0 \in X$ be arbitrarily chosen. For $x_1 \in Sx_0$, we have,

$$\theta(k_1,k_2,k_3)\min\{p(x_0,Sx_0),p(x_1,Sx_1\} \le \theta(k_1,k_2,k_3)p(x_0,Sx_0) \le p(x_0,x_1).$$

Let $\lambda \in (1, \frac{1}{\mu+k_1+2k_2})$. By Lemma 2.2, there exists $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \le \lambda H_p(Sx_0,Tx_1). \quad (15)$$

Now, $\theta(k_1,k_2,k_3)\min\{p(x_0,Sx_0),p(x_1,Sx_1\} + \theta(h_1,h_2,h_3)$
$$\min\{p(x_2,Tx_2),p(x_1,Tx_1)\} \le p(x_0,x_1) + p(x_1,x_2).$$

By (14), we get,

$$\max\{H_p(Sx_0,Tx_1),H_p(Sx_1,Tx_2)\} \le k_1p(x_0,x_1) + k_2p(x_0,Sx_0) + k_3p(x_1,Tx_1) + h_1p(x_1,x_2) + h_2p(x_1,Sx_1) + h_3p(x_2,Tx_2),$$

and so, $$H_p(Sx_0,Tx_1) \le k_1p(x_0,x_1) + k_2p(x_0,x_1) + k_3p(x_1,Tx_1) + h_1p(x_1,x_2) + h_2p(x_1,Sx_1) + h_3p(x_2,Tx_2). \quad (16)$$

From (15) and (16), we get,
$p(x_1, x_2) \leq \lambda H_p(Sx_0, Tx_1)$

$$\leq \lambda[(k_1 + k_2)p(x_0, x_1) + (k_3 + h_1)p(x_1, x_2) + h_2\{p(x_1, Sx_1) + p(x_2, Tx_2)\}] \quad \text{(since } h_2 \geq h_3\text{)}$$

$$\leq \lambda\{(k_1 + k_2)p(x_0, x_1) + (k_3 + h_1)p(x_1, x_2) + 2h_2 p(x_1, x_2)\}$$

$$\leq \frac{\lambda(k_1 + k_2)}{1 - \lambda(h_1 + 2h_2 + k_3)} p(x_0, x_1) \tag{17}$$

$$\leq ap(x_0, x_1), \quad \text{where } \ a = \frac{\lambda(k_1 + k_2)}{1 - \lambda(h_1 + 2h_2 + k_3)}. \tag{18}$$

Again, for $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$, such that $p(x_2, x_3) \leq \lambda H_p(Tx_1, Sx_2)$.
Similar to the previous case, we get,

$$p(x_2, x_3) \leq bp(x_1, x_2), \quad \text{where } b = \frac{\lambda(h_1 + h_3)}{1 - \lambda(k_1 + 2k_2 + h_2)},$$

and by (18), $\;p(x_2, x_3) \leq abp(x_0, x_1)$.
Again, for $x_3 \in Sx_2$, there exists $x_4 \in Tx_3$, such that $p(x_3, x_4) \leq \lambda H_p(Sx_2, Tx_3)$, and as above, $p(x_3, x_4) \leq a^2 bp(x_0, x_1)$.
Continuing in this way, we generate a sequence $\{x_n\}$ in X with $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$ such that

$$p(x_{2n+1}, x_{2n+2}) \leq a^{n+1}b^n p(x_0, x_1), \tag{19}$$

$$\text{and } \ p(x_{2n+2}, x_{2n+3}) \leq a^{n+1}b^{n+1}p(x_0, x_1) \ \text{ for all } n \in \mathbb{N}. \tag{20}$$

For all $n, m \in \mathbb{N}$, we have,

$$\begin{aligned} p(x_{2n+1}, x_{2n+m}) &\leq p(x_{2n+1}, x_{2n+2}) + p(x_{2n+2}, x_{2n+3}) + ... + p(x_{2n+m-1}, x_{2n+m}) \\ &\leq p(x_{2n+1}, x_{2n+2}) + p(x_{2n+2}, x_{2n+3}) + \ldots + p(x_{2n+m-1}, x_{2n+m}) + \ldots \\ &= a^n b^n(a + ab + a^2b + a^2b^2 + a^3b^2 + a^3b^3 + \ldots + \ldots)p(x_0, x_1) \\ &= a^n b^n\{(a + a^2b + a^3b^2 + \ldots) + (ab + a^2b^2 + a^3b^3 + \ldots)\}p(x_0, x_1) \\ &= a^n b^n(\frac{a}{1 - ab} + \frac{ab}{1 - ab})p(x_0, x_1) \ \to 0 \text{ as } n \to \infty \text{ (since } 0 < a, b < 1). \end{aligned}$$

Similarly, we have, $\;p(x_{2n+2}, x_{2n+m}) \to 0 \quad$ as $n \to \infty$.
Therefore, $\;p(x_n, x_{n+m}) \to 0$ as $n \to \infty$. Since $p^s(x_n, x_{n+m}) \leq 2p(x_n, x_{n+m})$ for all $n, m \in \mathbb{N}$, so, $\{x_n\}$ is a Cauchy sequence in (X, p^s). Also, (X, p^s) is complete, so, there exists $u \in X$ such that $\lim_{n\to\infty} p^s(x_n, u) = 0$.
Again by Lemma 2.1,

$$p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{21}$$

Now,

$$\theta(k_1, k_2, k_3)\min\{p(x_{2n}, Sx_{2n}), p(x_{2n+1}, Sx_{2n+1})\} + \theta(h_1, h_2, h_3)\min\{p(x_{2n+2}, Tx_{2n+2}), p(x_{2n+1}, Tx_{2n+1})\} \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}).$$

By (14),

$$\begin{aligned}
&\max\{H_p(Sx_{2n},Tx_{2n+1}),H_p(Sx_{2n+1},Tx_{2n+2})\}\le k_1p(x_{2n},x_{2n+1})+k_2p(x_{2n},Sx_{2n})\\
&+k_3p(x_{2n+1,Tx_{2n+1}})+h_1p(x_{2n+1},x_{2n+2})+h_2p(x_{2n+1},Sx_{2n+1})+h_3p(x_{2n+2},Tx_{2n+2})\\
&\text{and so, }\ H_p(Sx_{2n},Tx_{2n+1})\le k_1p(x_{2n},x_{2n+1})+k_2p(x_{2n},x_{2n+1})+k_3p(x_{2n+1},x_{2n+2})\\
&\qquad+h_1p(x_{2n+1},x_{2n+2})+h_2\{p(x_{2n+2},Sx_{2n+1})+p(x_{2n+2},Tx_{2n+2})\}\\
&\text{i.e., }\ H_p(Sx_{2n},Tx_{2n+1})\le(k_1+k_2)p(x_{2n},x_{2n+1})+(k_3+h_1+2h_2)p(x_{2n+1},x_{2n+2}).
\end{aligned}$$

Since $p(u,Sx_{2n})\le p(u,x_{2n+2})+H_p(Tx_{2n+1},Sx_{2n})$
$$\le p(u,x_{2n+2})+(k_1+k_2)p(x_{2n},x_{2n+1})+(k_3+h_1+2h_2)p(x_{2n+1},x_{2n+2}),$$

taking $n\to\infty$, we get, $\lim_{n\to\infty}p(u,Sx_{2n})=0$.
So, for given $\epsilon>0$, there exists $n_0\in\mathbb{N}$, such that $p(u,Sx_{2n})<\epsilon$ for all $n\ge n_0$.
Taking $\epsilon=(k_1+\frac{k_2}{\theta_1})p(u,x)$, where $x\ne u$ and $\theta_1=\theta(k_1,k_2,k_3)$, for $k\ge n_0$, we get,

$$p(u,Sx_{2k})<(k_1+\frac{k_2}{\theta_1})p(u,x). \tag{22}$$

For all $n\in\mathbb{N}$, we choose $v_n\in Sx_{2k}$ such that

$$p(u,v_n)\le p(u,Sx_{2k})+\frac{1}{n}p(x,u). \tag{23}$$

$$\begin{aligned}
\text{Therefore, }\ p(x,Sx_{2n_0})&\le p(x,v_n)\le p(x,u)+p(u,v_n)-p(u,u)\\
&\le p(x,u)+p(u,Sx_{2n_0})+\frac{1}{n}p(x,u)\ \text{(using (23))}\\
&\le(1+k_1+\frac{k_2}{\theta_1}+\frac{1}{n})p(u,x)+k_3p(x,Sx_{2n_0}).\ \ \text{(using (22))}
\end{aligned}$$

As $n\to\infty$, $(1-k_3)p(x,Sx_{2n_0})\le(1+k_1+\frac{k_2}{\theta_1})p(x,u)$,

$$\text{i.e., } p(x,Sx_{2n_0})\le\frac{(1+k_1+\frac{k_2}{\theta_1})}{(1-k_3)}p(x,u).$$

This gives $\theta(k_1,k_2,k_3)p(x,Sx_{2n_0})\le p(x,u)$,
and so, $\theta(k_1,k_2,k_3)\min\{p(x,Sx_{2n_0}),p(u,Su)\}\le p(x,u)$ for all $x\ne u$.
This is satisfied for all $k\ge n_0$.
So, $\theta(k_1,k_2,k_3)\min\{p(x,Sx_{2n}),p(u,Su)\}\le p(x,u)$ for all $n\ge n_0$.
In particular, for $x=x_{2n}$,

$$\theta(k_1,k_2,k_3)\min\{p(x_{2n},Sx_{2n}),p(u,Su)\}\le p(x_{2n},u)\quad\text{for all } n\ge n_0. \tag{24}$$

Proceeding as above, it can be shown that

$$\theta(h_1,h_2,h_3)\min\{p(x_{2n+1},Tx_{2n+1}),p(u,Tu)\}\le p(x_{2n+1},u)\ \text{ for all } n\ge n_1. \tag{25}$$

Let $n_2=\max\{n_0,n_1\}$ and adding (24) and (25), we get,

$$\begin{aligned}
&\theta(k_1,k_2,k_3)\min\{p(x_{2n},Sx_{2n}),p(u,Su)\}+\theta(h_1,h_2,h_3)\min\{p(x_{2n+1},Tx_{2n+1}),\\
&\qquad p(u,Tu)\}\le p(x_{2n},u)+p(x_{2n+1},u)\ \text{ for all } n\ge n_2.
\end{aligned}$$

By (14),

$$\begin{aligned}\max\{H_p(Sx_{2n},Tu),H_p(Su,Tx_{2n+1})\} &\le k_1p(x_{2n},u)+k_2\{p(x_{2n},Sx_{2n})+p(u,Tu)\}\\ &\quad + h_1p(u,x_{2n+1})+h_2\{p(u,Su)+p(x_{2n+1},Tx_{2n+1})\}\\ &\le k_1p(x_{2n},u)+2k_2p(x_{2n},u)+h_1p(u,x_{2n+1})+2h_2p(u,x_{2n+1}) \quad \text{for all } n\ge n_2.\end{aligned}$$

Now, $p(u,Tu) \le p(u,x_{2n+1}) + H_p(Sx_{2n},Tu)$

$$\le p(u,x_{2n+1}) + (k_1+2k_2)p(x_{2n},u) + (h_1+2h_2)p(u,x_{2n+1}) \quad \text{for all } n\ge n_2.$$

As $n\to\infty$, $p(u,Tu)=0 \Rightarrow u\in Tu$.
Similarly, $u\in Su$, and thus, u is a common fixed point of S and T. □

Example 3.2. *Let X={0,2,4} and define a complete partial metric p on X by*
$p(0,0)=0$, $p(2,2)=\frac{1}{5}$, $p(4,4)=\frac{19}{23}$,
$p(2,0)=p(0,2)=\frac{1}{5}$, $p(4,0)=p(0,4)=\frac{19}{23}$, $p(2,4)=p(4,2)=\frac{19}{23}$.
Let $S,\ T: X\to CB^p(X)$ *be defined by*

$$Sx=\{0\} \text{ for all } x\in X \quad \text{and } Tx=\begin{cases}\{2\}, & \text{if } x=4\\ \{0\}, & \text{otherwise}\end{cases}.$$

Obviously $\{0\}$ and $\{2\}$ are closed in X with respect to the partial metric p.
Let $k_1=\frac{7}{30}$, $k_2=\frac{2}{15}$, $k_3=\frac{1}{10}$ and $h_1=\frac{1}{6}$, $h_2=\frac{1}{6}$, $h_3=\frac{2}{15}$.
So, $\theta(k_1,k_2,k_3)=\frac{23}{37}$ and $\theta(h_1,h_2,h_3)=\frac{3}{5}$.

Now, $p(0,S0)=0=p(0,T0)$, $p(2,S2)=\dfrac{1}{5}=p(2,T2)$, $p(4,S4)=\dfrac{19}{23}=p(4,T4)$,
$H_p(S0,T0)=0=H_p(S0,T2)=H_p(S2,T2)=H_p(S2,T0)=H_p(S4,T0)=H_p(S4,T2)$,
$H_p(S0,T4)=\dfrac{1}{5}=H_p(S2,T4)=H_p(S4,T4)$. We take,
$\eta_1(x,y,z)=\theta(k_1,k_2,k_3)\min\{p(x,Sx),p(z,Sz)\}+\theta(h_1,h_2,h_3)\min\{p(y,Ty),p(z,Tz)\}$
$\eta_2(x,y,z)=\max\{H_p(Sx,Tz),H_p(Sz,Ty)\}$, $\xi_1(x,y,z)=p(x,z)+p(y,z)$,
$\xi_2(x,y,z)=k_1p(x,z)+k_2p(x,Sx)+k_3p(z,Tz)+h_1p(y,z)+h_2p(z,Sz)+h_3p(y,Ty)$.

A table illustrating the values of η_i,ξ_i $(i=1,2)$ corresponding to various combinations of x,y and z is presented below.

x	y	z	η_1	ξ_1	η_2	ξ_2	x	y	z	η_1	ξ_1	η_2	ξ_2	x	y	z	η_1	ξ_1	η_2	ξ_2
0	0	0	0	0	0	0	2	0	0	0	0.2	0	0.07	4	0	0	0	0.82	0	0.30
0	0	2	0	0.4	0	0.13	2	0	2	0.12	0.4	0	0.16	4	0	2	0.2	1.02	0	0.38
0	0	4	0	1.64	0.2	0.55	2	0	4	0.12	1.02	0.2	0.57	4	0	4	0.51	1.65	0.2	0.66
0	2	0	0	0.2	0	0.06	2	2	0	0	0.4	0	0.13	4	2	0	0	1.02	0	0.36
0	2	2	0.12	0.4	0	0.16	2	2	2	0.24	0.4	0	0.18	4	2	2	0.24	1.02	0	0.41
0	2	4	0.12	1.65	0.2	0.57	2	2	4	0.24	1.65	0.2	0.66	4	2	4	0.63	1.65	0.2	0.68
0	4	0	0	0.82	0.2	0.24	2	4	0	0	1.02	0.2	0.47	4	4	0	0	1.65	0.2	0.55
0	4	2	0.12	1.65	0.2	0.41	2	4	2	0.24	0.7	0.2	0.37	4	4	2	0.24	1.65	0.2	0.6
0	4	4	0.49	1.65	0.2	0.66	2	4	4	0.62	1.65	0.2	0.77	4	4	4	0.82	1.65	0.2	0.77

From the tabulated data, it is evident that whenever $\eta_1(x,y,z)\le\xi_1(x,y,z)$ the corresponding $\eta_2(x,y,z)\le\xi_2(x,y,z)$ for all $x,y,z\in X$. This observation validates condition (i) of Theorem 3.2 for the mappings S and T. Condition (ii) is also clearly satisfied. Consequently, S and T have a common fixed point, which is clearly 0 here.

4. Data Dependence and Stability

The study of data dependence of fixed point sets estimates the distance between the fixed point set of a mapping with that of another mapping. In the following, we obtain a data dependence result concerning our defined type of mapping in a complete partial metric space.

Theorem 4.1. *Let (X,p) be a complete partial metric space and $S,T : X \to CB^p(X)$ be two multi-valued mappings such that $H_p(Sx,Tx) \leq M$ for all $x \in X$, where $M > 0$ is a constant. For a strictly decreasing function θ from $[0,\frac{1}{2})$ onto $(\frac{2}{3},1]$ given by $\theta(r) = \frac{1}{1+r}$ and $r_1, r_2 \in [0,\frac{1}{2})$ with $r_2 \leq r_1$, suppose T satisfies Corollary 3.1 and $F(S) \neq \emptyset$. Then $\sup_{x\in F(S)} p(x,F(T)) \leq \beta M$, where $\beta > 0$.*

Proof. Since $F(S) \neq \emptyset$, let $y_0 \in Sy_0$. By Lemma 2.2, there exists $y_1 \in Ty_0$ such that $p(y_0,y_1) \leq hH_p(Sy_0,Ty_0) \leq hM$.
Again by Lemma 2.2 , for $y_1 \in Ty_0$, there exists $y_2 \in Ty_1$ such that

$$p(y_1,y_2) \leq hH_p(Ty_0,Ty_1).$$

As T satisfies (13), similar to the proof of Theorem 3.1, we can construct a sequence $\{y_n\}$ in X such that for all $n \in \mathbb{N}$,

$$y_{n+1} \in Ty_n \text{ and } p(y_n,y_{n+1}) \leq k^n p(y_0,y_1), \text{ where } k = \frac{hr_1}{1-hr_1} < 1.$$

Then $\{y_n\}$ is a Cauchy sequence in X, and so, there exists $u \in X$, such that $y_n \to u$ as $n \to \infty$, and u is a fixed point of T, i.e., $u \in Tu$.
Now, $p(y_0,u) \leq \sum_{i=0}^{n} p(y_i,y_{i+1}) + p(y_{n+1},u) \leq \sum_{i=0}^{n} k^i p(y_0,y_1) + p(y_{n+1},u)$.
Taking limit as $n \to \infty$, we have,

$$p(y_0,u) \leq \sum_{i=0}^{\infty} k^i p(y_0,y_1) \leq \frac{1}{1-k} p(y_0,y_1) \leq \frac{1-hr_1}{1-2hr_1} hM.$$

Therefore, for given $y_0 \in F(S)$, we have, $u \in F(T)$, such that $p(y_0,u) \leq \frac{1-hr_1}{1-2hr_1} hM$ holds.
Hence, $p(y_0,F(T)) \leq \frac{1-hr_1}{1-2hr_1} hM$.
Since $y_0 \in F(S)$ is arbitrary, we get,

$$\sup_{x\in F(S)} p(x,F(T)) \leq \frac{1-hr_1}{1-2hr_1} hM = \beta M, \quad \text{where } \beta = \frac{1-hr_1}{1-2hr_1} h > 0.$$

□

The following result gives the stability analysis of the fixed point sets for a sequence of uniformly convergent continuous multi-valued mappings.

Theorem 4.2. *Let (X,p) be a complete partial metric space with p continuous, and $\{T_n : X \to CB^p(X) : n \in \mathbb{N}\}$ be a sequence of continuous multi-valued mappings, which uniformly converges to $T : X \to CB^p(X)$. Suppose for a strictly decreasing function θ from $[0,\frac{1}{2})$ onto $(\frac{2}{3},1]$ given by $\theta(r) = \frac{1}{1+r}$ and $\{r_{1_n}\}, \{r_{2_n}\} \subset [0,\frac{1}{2})$ with $r_{2_n} \leq r_{1_n}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} r_{1_n} = r_1$, $\lim_{n\to\infty} r_{2_n} = r_2$ such that $r_1, r_2 \in [0,\frac{1}{2})$, each $T_n (n \in \mathbb{N})$ satisfies the condition (13). Then $F(T_n) \neq \emptyset$ for each $n \in \mathbb{N}$ and $F(T) \neq \emptyset$, $\lim_{n\to\infty} H_p(F(T_n),F(T)) = 0$, i.e., the fixed point sets of the sequence of mappings $\{T_n\}$ are stable.*

Proof. By Theorem 3.1, $F(T_n) \neq \emptyset$ for every $n \in \mathbb{N}$.
Let $x, y, z \in X$. Since each T_n satisfies the condition (13), so, for each $n \in \mathbb{N}$,

$$\theta(r_{1_n}) \min\{p(x, T_n x), p(z, T_n z)\} + \theta(r_{2_n}) \min\{p(y, T_n y), p(z, T_n z)\} \leq p(x, z) + p(y, z)$$
$$\Rightarrow \max\{H_p(T_n x, T_n z), H_p(T_n z, T_n y)\} \leq r_{1_n} p(x, z) + r_{2_n} p(y, z), \quad \text{for all } x, y, z \in X.$$

Taking limit as $n \to \infty$ and using the continuity of p and T_n, $(n \in \mathbb{N})$,

$$\theta(r_1) \min\{p(x, Tx), p(z, Tz)\} + \theta(r_2) \min\{p(y, Ty), p(z, Tz)\} \leq p(x, z) + p(y, z)$$
$$\Rightarrow \max\{H_p(Tx, Tz), H_p(Tz, Ty)\} \leq r_1 p(x, z) + r_2 p(y, z) \text{ for all } x, y, z \in X.$$

Thus, T satisfies the condition (13), and by Theorem 3.1, $F(T) \neq \emptyset$.
Now, let $M_n = \sup_{x \in X} H_p(T_n x, Tx)$, $n \in \mathbb{N}$. Since $T_n \to T$ uniformly, so,

$$\lim_{n \to \infty} M_n = \lim_{n \to \infty} \sup_{x \in X} H_p(T_n x, Tx) = 0.$$

By Theorem 4.1, we get,

$$\sup_{x \in F(T)} p(x, F(T_n)) \leq \beta M_n \text{ and } \sup_{x \in F(T_n)} p(x, F(T)) \leq \beta M_n \text{ for all } n \in \mathbb{N}.$$

Therefore, $H_p(F(T_n), F(T)) \leq \beta M_n$, for all $n \in \mathbb{N}$.
Thus, $\lim_{n \to \infty} H_p(F(T_n), F(T)) \leq \lim_{n \to \infty} \beta M_n = 0$, i.e., $\lim_{n \to \infty} H_p(F(T_n), F(T)) = 0$. □

5. Application to Volterra Type Integral Inclusion

In this section, we apply our derived result to prove the existence of a common solution for a system of integral inclusions of Volterra type. Let $X = C([0, 1), [1, \infty))$ be the space of all continuous functions from $I = [0, 1)$ into $[1, \infty)$. We consider the following system of Volterra type integral inclusions:

$$x(t) \in f(t) + \int_0^t K_1(t, s, x(s))\, ds, \ t \in I, \tag{26}$$
$$x(t) \in f(t) + \int_0^t K_2(t, s, x(s))\, ds, \ t \in I$$

where $f \in X$ and $K_1, K_2 : I \times I \times \mathbb{R} \to CB^p(\mathbb{R})$.
Define the partial metric $p : X \times X \to [0, \infty)$ as

$$p(x, y) = \sup_{t \in I} |x(t) - y(t)| \text{ for all } x, y \in X.$$

Clearly (X, p) is a complete partial metric space.

Theorem 5.1. *Suppose that the following conditions hold:*

(i) *For each $x \in X$, the multi-valued mappings $K_1, K_2 : I \times I \times \mathbb{R} \to CB^p(\mathbb{R})$ are such that $K_{1,x}(t, s) := K_1(t, s, x(s))$ and $K_{2,x}(t, s) := K_2(t, s, x(s))$ are lower semicontinuous in $I \times I$.*

(ii) *There exists a continuous function $\eta : I \times I \to [0, 1)$ such that for each $k_{1,x} : I \times I \to \mathbb{R}$ with $k_{1,x}(t, s) \in K_{1,x}(t, s)$, there exists a continuous function $k_{2,y}(t, s) \in K_{2,y}(t, s)$ satisfying $|k_{1,x}(t, s) - k_{2,y}(t, s)| \leq \eta(t, s)|x(s) - y(s)|$ for all $t, s \in I$ and for all $x, y \in X$ with $x \neq y$.*

(iii) *There exists $\lambda \in [0, 1)$ such that $\sup_{t \in I} \int_0^1 \eta(t, s) \leq \frac{\lambda}{2}$.*

Then the system of integral inclusions (26) has a solution in X.

Proof. Consider the multi-valued mappings $S, T : X \to CB^p(X)$ as follows:

$$Sx = \{u \in X : u(t) \in f(t) + \int_0^t K_1(t, s, x(s))\, ds,\ t \in I\}$$

$$Tx = \{v \in X : v(t) \in f(t) + \int_0^t K_2(t, s, x(s))\, ds,\ t \in I\}$$

for each $x \in X$. It is obvious that common fixed point of S and T is a solution of the system (26).

Now, we consider $K_{1,x}, K_{2,x} : I \times I \to CB^p(\mathbb{R})$. By Michael's selection theorem [16], for $x \in X$ there exist continuous mappings $k_{1,x}, k_{2,x} : I \times I \to \mathbb{R}^+$ such that $k_{1,x}(t, s) \in K_{1,x}(t, s)$ and $k_{2,x}(t, s) \in K_{2,x}(t, s)$ for all $t, s \in I$.

We take $u_1,\ u_2 \in X$, where

$$u_1(t) = f(t) + \int_0^t k_{1,x}(t, s)\, ds \quad \text{and} \quad u_2(t) = f(t) + \int_0^t k_{2,x}(t, s)\, ds, \quad t \in I.$$

Then $u_1 \in Sx$ and $u_2 \in Tx$ and so, $Sx \neq \emptyset$, $Tx \neq \emptyset$. Since f and $K_{1,x}, K_{2,x}$ are continuous on I and $I \times I$ respectively, so their ranges are bounded and thus, Sx and Tx are bounded. Also, Sx and Tx are closed in (X, p).

For $x, y, z \in X$ let $u \in Sx$ and $v \in Sz$.

Then $u(t) \in f(t) + \int_0^t K_{1,x}(t, s, x(s))\, ds,\ t \in I$.

This implies $u(t) = f(t) + \int_0^t k_{1,x}(t, s)\, ds,\ (t, s) \in I \times I,$ for some $k_{1,x}(t, s) \in K_{1,x}(t, s)$.

From (ii), there exists $g(t, s) \in K_{2,z}(t, s)$ such that

$$|k_{1,x}(t, s) - g(t, s)| \leq \eta(t, s)|x(s) - z(s)| \text{ for all } (t, s) \in I \times I.$$

Consider the multi-valued operator L defined by

$$L(t, s) = K_{2,z}(t, s) \cap \{w \in \mathbb{R} : |k_{1,x}(t, s) - w| \leq \eta(t, s)|x(s) - z(s)|\}, \text{ for all } (t, s) \in I \times I.$$

Since by (i), L is lower semicontinuous, there exists a continuous function $k_{2,z} : I \times I \to \mathbb{R}^+$ with $k_{2,z}(t, s) \in L(t, s)$ for $t, s \in I$, such that

$$m(t) = f(t) + \int_0^t k_{2,z}(t, s)\, ds \ \in f(t) + \int_0^t K_2(t, s, z(s))\, ds,\ t \in I.$$

$$\begin{aligned}
\text{Now, } |u(t) - m(t)| &= \left|\int_0^t k_{1,x}(t, s)\, ds - \int_0^t k_{2,z}(t, s)\, ds\right| \\
&\leq \int_0^t \eta(t, s)|x(s) - z(s)|\, ds \\
&\leq \int_o^t \eta(t, s) \sup_{t \in I}|x(s) - z(s)|\, ds \\
&= p(x, z) \int_0^t \eta(t, s)\, ds \\
&\leq p(x, z) \sup_{t \in I} \int_0^t \eta(t, s)\, ds \\
&\leq r_1 p(x, z), \quad \text{where } r_1 = \frac{\lambda}{2} \ \text{(using } (iii)).
\end{aligned}$$

Consequently, $p(u, m) \leq r_1 p(x, z)$.

Exchanging the roles of u and m, we have, $\quad H_p(Sx, Tz) \leq r_1 p(x, z)$

Now, for $v \in Sz$, we have, $v(t) \in f(t) + \int_0^t K_{1,z}(t,s,z(s))\,ds,\ t \in I$. So, for some $k_{1,z}(t,s) \in K_{1,z}(t,s)$,

$$v(t) = f(t) + \int_0^t k_{1,z}(t,s)\,ds,\ (t,s) \in I \times I.$$

From (ii), there exists $h(t,s) \in K_{2,y}(t,s)$ such that

$$|k_{1,z}(t,s) - h(t,s)| \leq \eta(t,s)|z(s) - y(s)| \text{ for all } (t,s) \in I \times I.$$

Similar to the previous case, we define a multi-valued operator M by

$$M(t,s) = K_{2,y}(t,s) \cap \{w \in \mathbb{R} : |k_{1,z}(t,s) - w| \leq \eta(t,s)|z(s) - y(s)|\}, \text{ for all } (t,s) \in I \times I,$$

and there exists a continuous function $k_{2,y}$ with $k_{2,y}(t,s) \in M(t,s)$ for $t, s \in I$, such that

$$n(t) = f(t) + \int_0^t k_{2,y}(t,s)\,ds \ \in f(t) + \int_0^t K_2(t,s,y(s))\,ds,\ t \in I.$$

$$\text{Now, } |v(t) - n(t)| \leq \sup_{t \in [0,1)} |z(s) - y(s)| \int_0^t \eta(t,s)\,ds$$
$$\leq r_2 p(x,z), \quad \text{where } r_2 = \frac{\lambda}{2}.$$

Therefore, $H_p(Sz, Ty) \leq r_2 p(y,z)$, and thus,

$$H_p(Sx, Tz) + H_p(Sz, Ty) \leq r_1 p(x,z) + r_2 p(y,z).$$

Clearly, $\max\{H_p(Sx,Tz), H_p(Sz,Ty) \leq r_1 p(x,z) + r_2 p(y,z)\}$.
Thus, S and T satisfy the hypothesis of Theorem 3.1. Hence, S and T have a unique common fixed point and so, the system of Volterra integral inclusions has a common solution. □

6. Conclusion

We have derived some common fixed point results concerning multi-valued mappings in partial metric spaces with analysis of data dependence and stability of fixed point sets. It is worth mentioning that the completeness property is a necessary condition for Theorem 3.1, since the result fails to hold in general for an incomplete partial metric space. In this regard, the investigation of the existence of common fixed point for our derived type of mappings considering incomplete partial metric space is a scope for further study. In [4], Amiri et al. established the existence of solutions for the Reimann-Liouville and Atangana-Baleanu fractional integral inclusion by using common fixed point results for α_c-admissible multi-valued mappings in the framework of complex-valued double controlled metric space. Similar study can be carried out for the mappings discussed in this paper in case of double controlled partial metric space.

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