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HYBRID RIGHT AND LEFT BI-QUASI IDEALS OF SEMIRINGS

S. MEENAKSHI[1], G. MUHIUDDIN[2], K. PORSELVI[3], B. ELAVARASAN[4*], §

Abstract. The fuzzy set is an excellent tool for dealing with indeterminacy that can be clearly and effectively analysed from the decision-maker's viewpoint, and it is extremely helpful for showing people's hesitations in their everyday interactions. In order to deal with practical problems, soft set theory has recently been created. By combining the fuzzy and soft sets, Jun et al. created hybrid structures. Hybrid structures are soft-set and fuzzy-set speculations. The concepts of hybrid bi-quasi ideal, hybrid right bi-quasi ideal, and hybrid left bi-quasi ideal of semirings are addressed in this paper. We also explore the equivalent conditions for a subset of a semiring to be a right bi-quasi ideal and for the semiring to be regular. Hybrid right (left) bi-quasi ideals in semirings are used to characterise the regular semiring.

Keywords: Semiring, ideal, right (left) bi-quasi ideal, hybrid structure, hybrid ideal, hybrid right (left) bi-quasi ideal.

AMS Subject Classification: 16Y60, 08A72

1. Introduction

In 1934, Vandiver [26] developed the concept of a semiring, which is now widely recognised as a common algebra and one of the fundamental structures of mathematics. Semirings are used to study determinants, matrices, coding theory, automata, graph theory, functional analysis, optimisation theory, and many other areas. Applications of semiring theory include information sciences, theoretical computer science, idempotent analysis, etc. Semirings were already addressed in the initial work on the basics of ring ideals. Semirings diverge from ring ideals, even though they are generalisations of rings. Ideals have a big impact on progressed research and the utilisation of algebraic structures. Many

[1,3,4] Department of Mathematics - Karunya Institute of Technology and Sciences - Coimbatore - 641114 - Tamilnadu - India.
e-mail: meenu2696@gmail.com; ORCID no. 0000-0003-0720-3647.
e-mail: porselvi94@yahoo.co.in; ORCID no. 0000-0003-4097-3467.
e-mail: belavarasan@gmail.com; ORCID no. 0000-0002-1414-2814 .
[2] Department of Mathematics - University of Tabuk - P.O. Box-741 - Tabuk - 71491 - Saudi Arabia.
e-mail: chishtygym@gmail.com; ORCID no. 0000-0002-5596-5841.
[*] Corresponding author.
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mathematical researchers have shown significant outcomes and analyses of algebraic structures through the idea of ideals.

In [9], Henriksen focused on a restricted class of semiring ideals termed k-ideals and obtained their various properties. Iseki [10] established the concept of quasi-ideals in semirings. Lajos and Szasz [14] pioneered the idea of bi-ideals in rings. In [20], the concept of the left bi-quasi ideal of semiring was established by Murali Krishna Rao.

In 1965, Fuzzy Set was initially presented by Zadeh [27] and might be effectively utilized in a variety of fields, involving engineering, robotics, image processing, industrial automation, and control systems. Graph theory, statistics, ring theory, decision-making, topological spaces, group theory, and other engineering applications are just a few of the many areas in which fuzzy set theory is used. Many issues, such as pattern recognition, intelligent data analysis, and information processing, are being tackled by the utilisation of fuzzy sets. Fuzzy semirings were briefly discussed by Ahsan [1]. In semirings, Rao [25] established the idea of fuzzy right (left) bi-quasi ideals and also addressed the regular semirings using fuzzy right and left bi-quasi ideals.

In 1999, Soft sets were first developed by Molodtsov [17], who also created the fundamental findings of the novel notions. He effectively utilized the soft sets in many areas, including Riemann integration, operation research, smoothness of function, game theory, the theory of probability, etc. Maji et al. [15] pinpointed a soft set parameter reduction and focused on a soft set application in decision-making issues.

In a gathering of attributes over a base universe set, Jun et al. [11] developed and looked into the concept of hybrid structure, which comprises the theories of soft sets and fuzzy sets. Using this theory, they claimed some of the notions of a hybrid linear space and a hybrid subalgebra. Anis et al. looked into the definitions of hybrid ideals and hybrid subsemigroups in semigroups [2]. They also looked at how these ideas have been linked to the idea of hybrid products.

Elavarasan et al. [3] discussed the notion of generalization of hybrid bi-ideals in semigroups. Elavarasan and Jun [4, 5, 6] examined the ideas of hybrid ideals in semirings and put forth certain circumstances that are equivalent for a semigroup to be intra-regular and regular. They also obtained some results related to hybrid ideals as well as hybrid bi-ideals in semigroups. Elavarasan et al. [7, 8] looked into the idea of hybrid products and hybrid intersections of hybrid left ideals in near rings that are zero-symmetric. They also looked into hybrid k-ideals in semirings.

In [12], Keerthika et al. discussed the notion of hybrid ordered ideals in ordered semirings and also characterised the regularity of ordered semirings in terms of hybrid structures. In [13], Keerthika et al. explored the idea of hybrid congruence and hybrid strong h-ideals in hemirings. The hybrid hemiring homomorphisms and quotient rings through the hybrid strong h-ideals were also discussed. In [16], Meenakshi et al. examined the concepts of hybrid ideals in near-subtraction semigroups, and their associated outcomes were discussed. They also established the notion of homomorphism of a hybrid structure in a near-subtraction semigroups.

The ideas of hybrid structures applied to modules over semirings [18] and hybrid ideals as well as k-hybrid ideals in ternary semirings [19] were discussed. Porselvi and Elavarasan [21] established the idea of hybrid interior ideals in semigroups and demonstrated how these ideals coincide for both intra-regular and regular semigroups. Hybrid structures have been utilised to solve a number of algebraic systems with varying results [22, 23, 24].

In this work, the ideas of hybrid left bi-quasi ideal, hybrid right bi-quasi ideal, and hybrid bi-quasi ideal of semirings are examined. Using hybrid right (left) bi-quasi ideals in semirings, we characterise the regular semiring. We also examine the equivalent conditions for a subset of a semiring to be a right bi-quasi ideal and for the semiring to be regular.

2. Preliminaries

The primary ideas and definitions required for this paper will be briefly reviewed in this section. Hereafter, $\mathfrak{D}$ represents a semiring and the power set of a set L can be represented by $\mathcal{P}(L)$.

Definition 2.1. *A semiring $\mathfrak{D}$ with two binary operations " $+$ " and " $\cdot$ " that fulfils the below axioms:*
(i) $(\mathfrak{D}, +)$ *is a semigroup.*
(ii) $(\mathfrak{D}, \cdot)$ *is a semigroup.*
(iii) $(x+f)\cdot a = x\cdot a + f\cdot a$ *and* $x\cdot(f+a) = x\cdot f + x\cdot a \ \forall\ x, f, a \in \mathfrak{D}$.
(iv) $\exists\ 0 \in \mathfrak{D}\ :\ \varpi_1 + 0 = \varpi_1 = 0 + \varpi_1$ *and* $\varpi_1 \cdot 0 = 0 \cdot \varpi_1 = 0$, *for all* $\varpi_1 \in \mathfrak{D}$.

Definition 2.2. *A subset $\mathfrak{I}(\neq \emptyset)$ of $\mathfrak{D}$ is known as*
(i) a subsemiring of $\mathfrak{D}$ if $(\mathfrak{I}, +)$ is a subsemigroup of $(\mathfrak{D}, +)$.
(ii) a left (right) bi-quasi ideal of $\mathfrak{D}$ if $(\mathfrak{I}, +)$ is a subsemigroup of $(\mathfrak{D}, +)$ and $\mathfrak{D}\mathfrak{I} \cap \mathfrak{I}\mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I}$ $(\mathfrak{I}\mathfrak{D} \cap \mathfrak{I}\mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I})$.
(iii) a bi-quasi ideal of $\mathfrak{D}$ if it is both a left and right bi-quasi ideal of $\mathfrak{D}$.

Definition 2.3. *A subsemiring $\mathfrak{I}(\neq \emptyset)$ of $\mathfrak{D}$ is known as*
(i) an interior ideal of $\mathfrak{D}$ if $\mathfrak{D}\mathfrak{I}\mathfrak{D} \subseteq \mathfrak{I}$.
(ii) a bi-ideal of $\mathfrak{D}$ if $\mathfrak{I}\mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I}$.
(iii) a left (right) ideal of $\mathfrak{D}$ if $\mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I}$ $(\mathfrak{I}\mathfrak{D} \subseteq \mathfrak{I})$.
(iv) a $k-$ideal if $\mathfrak{I}\mathfrak{D} \subseteq \mathfrak{I}$ and $\mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I}$, and if $j_0 \in \mathfrak{D}, j_0 + s_0 \in \mathfrak{I}, s_0 \in \mathfrak{I}$, then $j_0 \in \mathfrak{I}$.
(v) an ideal if $\mathfrak{I}\mathfrak{D} \subseteq \mathfrak{I}$ and $\mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I}$.
(vi) a quasi ideal of $\mathfrak{D}$ if $\mathfrak{I}\mathfrak{D} \cap \mathfrak{D}\mathfrak{I} \subseteq \mathfrak{I}$.

Definition 2.4. *An element b of a semiring $\mathfrak{D}$ is called a regular element if there exists an element k of $\mathfrak{D}$ such that $b = bkb$. If every element of $\mathfrak{D}$ is a regular element, then $\mathfrak{D}$ is called a regular semiring.*

Definition 2.5. [11] *A universal set can be represented by $\mathfrak{Q}$ and a hybrid structure in $\mathfrak{D}$ over $\mathfrak{Q}$ is a mapping $\tilde{d}_\varpi := (\tilde{d}, \varpi) : \mathfrak{D} \to \mathcal{P}(\mathfrak{Q}) \times [0,1],\ u \mapsto (\tilde{d}(u), \varpi(u))$, where $\tilde{d} : \mathfrak{D} \to \mathcal{P}(\mathfrak{Q})$ and $\varpi : \mathfrak{D} \to [0,1]$ are mappings.*

Define a relation $\ll$ on the family of all hybrid structures, represented by $\mathscr{H}(\mathfrak{D})$, in $\mathfrak{D}$ over $\mathfrak{Q}$ as below:

$$\left(\forall \tilde{d}_\varpi, \tilde{y}_\omega \in \mathscr{H}(\mathfrak{D})\right)\left(\tilde{d}_\varpi \ll \tilde{y}_\omega \iff \tilde{d}\ \tilde{\subseteq}\ \tilde{y}, \varpi \succeq \omega\right)$$

where $\tilde{d}\ \tilde{\subseteq}\ \tilde{y}$ means that $\tilde{d}(u) \subseteq \tilde{y}(u)$ and $\varpi \succeq \omega$ means that $\varpi(u) \geq \omega(u)\ \forall u \in \mathfrak{D}$. Then the set $(\mathscr{H}(\mathfrak{D}), \ll)$ is partially ordered.

Definition 2.6. [11] *Let $\tilde{u}_\zeta \in \mathscr{H}(\mathfrak{D})$. For any $(\Theta, \psi) \in \mathcal{P}(\mathfrak{Q}) \times [0,1]$, we define $\mathfrak{D}^{\Theta}_{\tilde{u}} := \{z \in \mathfrak{D}\ :\ \tilde{u}(z) \supseteq \Theta\}$ and $\mathfrak{D}^{\psi}_{\zeta} := \{z \in \mathfrak{D}\ :\ \zeta(z) \leq \psi\}$.*

Definition 2.7. [11] *For $\tilde{v}_\varrho \in \mathscr{H}(\mathfrak{D})$, the set $\tilde{v}_\varrho[\Gamma, \beta] := \{j_1 \in \mathfrak{D}\ :\ \tilde{v}(j_1) \supseteq \Gamma$ and $\varrho(j_1) \leq \beta\}$ is known as $[\Gamma, \beta]-$hybrid cut of $\tilde{v}_\varrho$, where $\Gamma \in \mathcal{P}(\mathfrak{Q})$ and $\beta \in [0,1]$. Note that $\mathfrak{D}^{\Gamma}_{\tilde{v}} \cap \mathfrak{D}^{\beta}_{\varrho} = \tilde{v}_\varrho[\Gamma, \beta]$.*

Definition 2.8. [11] *For $\tilde{c}_\eta \in \mathscr{H}(\mathfrak{D})$ and $\mathcal{L} \in \mathcal{P}(\mathfrak{D})\backslash\{\emptyset\}$, the characteristic hybrid structure in $\mathfrak{D}$ over $\mathfrak{Q}$ is represented by $\chi_{\mathcal{L}}(\tilde{c}_\eta)$ and it is described as,*

$$\begin{aligned}\chi_{\mathcal{L}}(\tilde{c}_\eta) = (\chi_{\mathcal{L}}(\tilde{c}), \chi_{\mathcal{L}}(\eta)) : \mathfrak{D} \longrightarrow \mathcal{P}(\mathfrak{Q}) \times [0,1],\\ h_1 \mapsto (\chi_{\mathcal{L}}(\tilde{c})(h_1), \chi_{\mathcal{L}}(\eta)(h_1)),\end{aligned}$$

where

$$\chi_{\mathcal{L}}(\tilde{c}) : \mathfrak{D} \to \mathcal{P}(\mathfrak{Q}), h_1 \mapsto \begin{cases} \mathfrak{Q} & if\ h_1 \in \mathcal{L} \\ \emptyset & otherwise, \end{cases}$$

$$\chi_{\mathcal{L}}(\eta) : \mathfrak{D} \to [0,1], h_1 \mapsto \begin{cases} 0 & if\ h_1 \in \mathcal{L} \\ 1 & otherwise \end{cases}$$

for any $h_1 \in \mathfrak{D}$.

Definition 2.9. [11] *For $\tilde{t}_\varrho, \tilde{q}_\varpi \in \mathscr{H}(\mathfrak{D})$,*

(i) the hybrid product $\tilde{t}_\varrho \odot \tilde{q}_\varpi$ is described as $\tilde{t}_\varrho \odot \tilde{q}_\varpi := (\tilde{t} \circ \tilde{q}, \varrho \circ \varpi)$, where

$$(\tilde{t} \circ \tilde{q})(w) = \begin{cases} \bigcup_{w=sl} \{\tilde{t}(s) \cap \tilde{q}(l)\} & if\ w = sl \\ \emptyset & otherwise, \end{cases}$$

$$(\varrho \circ \varpi)(w) = \begin{cases} \bigwedge_{w=sl} \{\varrho(s) \vee \varpi(l)\} & if\ w = sl \\ 1 & otherwise \end{cases}$$

for $s, l, w \in \mathfrak{D}$.

(ii) the hybrid intersection $\tilde{t}_\varrho \Cap \tilde{q}_\varpi$ is described as $\tilde{t}_\varrho \Cap \tilde{q}_\varpi := (\tilde{t}\tilde{\cap}\tilde{q}, \varrho \vee \varpi)$, where

$$\begin{aligned}\tilde{t}\tilde{\cap}\tilde{q} : \mathfrak{D} \to \mathcal{P}(\mathfrak{Q}), s \mapsto \tilde{t}(s) \cap \tilde{q}(s),\\ \varrho \vee \varpi : \mathfrak{D} \to [0,1], s \mapsto \varrho(s) \vee \varpi(s)\ for\ s \in \mathfrak{D}.\end{aligned}$$

Definition 2.10. *Let $\tilde{w}_\kappa \in \mathscr{H}(\mathfrak{D})$. Then $\tilde{w}_\kappa$ is known as a hybrid subsemiring of $\mathfrak{D}$ if it fulfils the following criteria:*

(i) $(\forall g_0, i_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{w}(g_0 + i_0) \supseteq \tilde{w}(g_0) \cap \tilde{w}(i_0) \\ \kappa(g_0 + i_0) \leq \kappa(g_0) \vee \kappa(i_0) \end{pmatrix}$.

(ii) $(\forall g_0, i_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{w}(g_0 i_0) \supseteq \tilde{w}(g_0) \cap \tilde{w}(i_0) \\ \kappa(g_0 i_0) \leq \kappa(g_0) \vee \kappa(i_0) \end{pmatrix}$.

Definition 2.11. *Let $\tilde{n}_\rho \in \mathscr{H}(\mathfrak{D})$. Then $\tilde{n}_\rho$ is termed as*

(i) a hybrid right (left) ideal of $\mathfrak{D}$ if it fulfils the below axioms:

(a) $(\forall w_0, b_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{n}(w_0 + b_0) \supseteq \tilde{n}(w_0) \cap \tilde{n}(b_0) \\ \rho(w_0 + b_0) \leq \rho(w_0) \vee \rho(b_0) \end{pmatrix}$.

(b) $(\forall w_0, b_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{n}(w_0 b_0) \supseteq \tilde{n}(w_0) \\ \rho(w_0 b_0) \leq \rho(w_0) \end{pmatrix} \left(\begin{pmatrix} \tilde{n}(w_0 b_0) \supseteq \tilde{n}(b_0) \\ \rho(w_0 b_0) \leq \rho(b_0) \end{pmatrix} \right)$.

(ii) a hybrid ideal of $\mathfrak{D}$ if it fulfils the below axioms:

(a) $(\forall w_0, b_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{n}(w_0 + b_0) \supseteq \tilde{n}(w_0) \cap \tilde{n}(b_0) \\ \rho(w_0 + b_0) \leq \rho(w_0) \vee \rho(b_0) \end{pmatrix}$.

(b) $(\forall w_0, b_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{n}(w_0 b_0) \supseteq \tilde{n}(w_0) \cup \tilde{n}(b_0) \\ \rho(w_0 b_0) \leq \rho(w_0) \wedge \rho(b_0) \end{pmatrix}$.

(iii) a hybrid bi-ideal of $\mathfrak{D}$ if it fulfils the below axioms:

(a) $(\forall w_0, b_0 \in \mathfrak{D}) \begin{pmatrix} \tilde{n}(w_0 + b_0) \supseteq \tilde{n}(w_0) \cap \tilde{n}(b_0) \\ \rho(w_0 + b_0) \leq \rho(w_0) \vee \rho(b_0) \end{pmatrix}$.

(b) $(\forall \tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D}))\,(\tilde{n}_\rho \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{n}_\rho \ll \tilde{n}_\rho)$.

(iv) a hybrid quasi ideal of $\mathfrak{D}$ *if it fulfils the below axioms:*

(a) $(\forall w_0, b_0 \in \mathfrak{D}) \left(\begin{array}{c} \tilde{n}(w_0 + b_0) \supseteq \tilde{n}(w_0) \cap \tilde{n}(b_0) \\ \rho(w_0 + b_0) \leq \rho(w_0) \vee \rho(b_0) \end{array} \right)$.

(b) $(\forall \tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D}))\,((\tilde{n}_\rho \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{n}_\rho) \ll \tilde{n}_\rho)$.

3. Hybrid right and left bi-quasi ideals

As a generalisation of the hybrid bi-ideal of $\mathfrak{D}$, we look into the concepts of hybrid left (right) bi-quasi ideal and also obtain some associated properties in this portion.

Definition 3.1. *Let* $\tilde{b}_\varsigma \in \mathscr{H}(\mathfrak{D})$. *Then* $\tilde{b}_\varsigma$ *is termed as a hybrid left (right) bi-quasi ideal if*

(i) $(\forall w_0, a_0 \in \mathfrak{D}) \left(\begin{array}{c} \tilde{b}(w_0 + a_0) \supseteq \tilde{b}(w_0) \cap \tilde{b}(a_0) \\ \varsigma(w_0 + a_0) \leq \varsigma(w_0) \vee \varsigma(a_0) \end{array} \right)$.

(ii) $(\forall \tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D}))\ (\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{b}_\varsigma) \Cap (\tilde{b}_\varsigma \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{b}_\varsigma) \ll \tilde{b}_\varsigma\ ((\tilde{b}_\varsigma \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\tilde{b}_\varsigma \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{b}_\varsigma) \ll \tilde{b}_\varsigma)$.

If $\tilde{b}_\varsigma$ *is both a hybrid right and a hybrid left bi-quasi ideal of* $\mathfrak{D}$, *then it is a hybrid bi-quasi ideal of* $\mathfrak{D}$.

Below are the examples for the hybrid bi-quasi ideal of $\mathfrak{D}$.

Example 3.1. *Let* $\mathcal{M}$ *be the set of all rational numbers. Then, with respect to usual matrix addition and usual matrix multiplication,* $\mathfrak{D} = \left\{ \begin{pmatrix} y_0 & c_0 \\ 0 & r_0 \end{pmatrix} \mid y_0, c_0, r_0 \in \mathcal{M} \right\}$ *is a semiring. Let* $\mathcal{I} = \left\{ \begin{pmatrix} y_0 & 0 \\ 0 & c_0 \end{pmatrix} \mid y_0, 0 \neq c_0 \in \mathcal{M} \right\}$. *Then* $\mathcal{I}$ *is a right bi-quasi ideal, but it is not a bi-ideal of* $\mathfrak{D}$ *(See* [25]*).*

Construct $\tilde{l}_\varsigma \in \mathscr{H}(\mathfrak{D})$ *such that* $\tilde{l}(f_1) = \begin{cases} \mathfrak{Q} & if\ f_1 \in \mathcal{I}, \\ \emptyset & otherwise \end{cases}$ *and* $\varsigma(f_1) = \begin{cases} 0 & if\ f_1 \in \mathcal{I}, \\ 1 & otherwise \end{cases}$. *Then* $\tilde{l}_\varsigma$ *is a hybrid right bi-quasi ideal of* $\mathfrak{D}$.

Example 3.2. *Let* $\mathcal{M}$ *be the set of all rational numbers. Then, with respect to usual matrix addition and usual matrix multiplication,* $\mathfrak{D} = \left\{ \begin{pmatrix} y_0 & c_0 \\ u_0 & r_0 \end{pmatrix} \mid y_0, c_0, u_0, r_0 \in \mathcal{M} \right\}$ *is a semiring. Let* $\mathcal{I} = \left\{ \begin{pmatrix} y_0 & c_0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq y_0, 0 \neq c_0 \in \mathcal{M} \right\}$. *Then* $\mathcal{I}$ *is a bi-quasi ideal, but it is not a left ideal of* $\mathfrak{D}$.

Construct $\tilde{l}_\varsigma \in \mathscr{H}(\mathfrak{D})$ *such that* $\tilde{l}(f_1) = \begin{cases} \mathfrak{Q} & if\ f_1 \in \mathcal{I}, \\ \emptyset & otherwise \end{cases}$ *and* $\varsigma(f_1) = \begin{cases} 0 & if\ f_1 \in \mathcal{I}, \\ 1 & otherwise \end{cases}$. *Then* $\tilde{l}_\varsigma$ *is a hybrid bi-quasi ideal of* $\mathfrak{D}$, *but it is not a hybrid left ideal of* $\mathfrak{D}$ *as* $\tilde{l}(ab) \not\supseteq \tilde{l}(b)$ *and* $\varsigma(ab) \not\leq \varsigma(b)$, *where* $a = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $b = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$.

Example 3.3. *Let* $\mathfrak{D}$ *be the set of all non-negative integers. Then, with respect to usual addition and multiplication,* $\mathfrak{D}$ *is a semiring. Construct* $\tilde{l}_\varsigma \in \mathscr{H}(\mathfrak{D})$ *such that* $\tilde{l}(f_1) = \begin{cases} \mathfrak{Q} & if\ f_1\ is\ even, \\ \emptyset & otherwise \end{cases}$ *and* $\varsigma(f_1) = \begin{cases} 0 & if\ f_1\ is\ even, \\ 1 & otherwise \end{cases}$. *Then* $\tilde{l}_\varsigma$ *is a hybrid bi-quasi ideal of* $\mathfrak{D}$.

Notation 3.1. *Let $\mathfrak{D}$ be a semiring. Then we use the following notations.*
(i) $\mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$ is the collection of all hybrid left ideals of $\mathfrak{D}$.
(ii) $\mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$ is the set of all hybrid right ideals of $\mathfrak{D}$.
(iii) $\mathscr{H}_{\mathfrak{I}}(\mathfrak{D})$ is the set of all hybrid ideals of $\mathfrak{D}$.
(iv) $\mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$ is the gathering of all hybrid left bi-quasi ideals of $\mathfrak{D}$.
(v) $\mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$ is the family of all hybrid right bi-quasi ideals of $\mathfrak{D}$.

Theorem 3.1. *If $\tilde{z}_\tau \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$, then $\tilde{z}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$.*

Proof. Let $\tilde{z}_\tau \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$ and $\tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D})$. Then, $\forall c_1 \in \mathfrak{D}$,

$$\begin{aligned}(\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}))(c_1) &= \bigcup_{c_1=n_1v_1} \{\tilde{z}(n_1) \cap \chi_{\mathfrak{D}}(\tilde{p})(v_1)\} \\ &= \bigcup_{c_1=n_1v_1} \tilde{z}(n_1) \subseteq \bigcup_{c_1=n_1v_1} \tilde{z}(n_1v_1) = \tilde{z}(c_1),\end{aligned}$$

$$\begin{aligned}(\tau \circ \chi_{\mathfrak{D}}(\gamma))(c_1) &= \bigwedge_{c_1=n_1v_1} \{\tau(n_1) \vee \chi_{\mathfrak{D}}(\gamma)(v_1)\} \\ &= \bigwedge_{c_1=n_1v_1} \tau(n_1) \geq \bigwedge_{c_1=n_1v_1} \tau(n_1v_1) = \tau(c_1).\end{aligned}$$

And

$$\begin{aligned}(\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(c_1) &= \bigcup_{c_1=j_1a_1y_1} \{(\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p})(j_1a_1)) \cap \tilde{z}(y_1)\} \\ &\subseteq \bigcup_{c_1=j_1a_1y_1} \{\tilde{z}(j_1a_1) \cap \tilde{z}(y_1)\} \subseteq \bigcup_{c_1=j_1a_1y_1} \tilde{z}(j_1a_1y_1) = \tilde{z}(c_1),\end{aligned}$$

$$\begin{aligned}(\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) &= \bigwedge_{c_1=j_1a_1y_1} \{(\tau \circ \chi_{\mathfrak{D}}(\gamma)(j_1a_1)) \vee \tau(y_1)\} \\ &\geq \bigwedge_{c_1=j_1a_1y_1} \{\tau(j_1a_1) \vee \tau(y_1)\} \geq \bigwedge_{c_1=j_1a_1y_1} \tau(j_1a_1y_1) = \tau(c_1).\end{aligned}$$

So,

$$((\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p})) \cap (\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z}))(c_1) = (\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}))(c_1) \cap (\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(c_1) \subseteq \tilde{z}(c_1),$$
$$((\tau \circ \chi_{\mathfrak{D}}(\gamma)) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau))(c_1) = (\tau \circ \chi_{\mathfrak{D}}(\gamma))(c_1) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \geq \tau(c_1).$$

Hence, $\tilde{z}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$. □

Theorem 3.2. *If $\tilde{z}_\tau \in \mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$, then $\tilde{z}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$.*

Proof. For $\tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D})$ and $\forall c_1 \in \mathfrak{D}$, we have

$$\begin{aligned}(\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(c_1) &= \bigcup_{c_1=n_1v_1} \{\chi_{\mathfrak{D}}(\tilde{p})(n_1) \cap \tilde{z}(v_1)\} \\ &= \bigcup_{c_1=n_1v_1} \{\mathfrak{Q} \cap \tilde{z}(v_1)\} = \bigcup_{c_1=n_1v_1} \tilde{z}(v_1) \subseteq \bigcup_{c_1=n_1v_1} \tilde{z}(n_1v_1) = \tilde{z}(c_1),\end{aligned}$$

$$\begin{aligned}(\chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) &= \bigwedge_{c_1=n_1v_1} \{\chi_{\mathfrak{D}}(\gamma)(n_1) \vee \tau(v_1)\} \\ &= \bigwedge_{c_1=n_1v_1} \{0 \vee \tau(v_1)\} = \bigwedge_{c_1=n_1v_1} \tau(v_1) \geq \bigwedge_{c_1=n_1v_1} \tau(n_1v_1) = \tau(c_1).\end{aligned}$$

And

$$(\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(c_1) = \bigcup_{c_1=j_1a_1y_1} \{\tilde{z}(j_1) \cap (\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(a_1y_1)\}$$
$$\subseteq \bigcup_{c_1=j_1a_1y_1} \{\tilde{z}(j_1) \cap \tilde{z}(a_1y_1)\} = \tilde{z}(c_1),$$
$$(\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) = \bigwedge_{c_1=j_1a_1y_1} \{\tau(j_1) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \tau)(a_1y_1)\}$$
$$\geq \bigwedge_{c_1=j_1a_1y_1} \{\tau(j_1) \vee \tau(a_1y_1)\} = \tau(c_1).$$

So,

$$((\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z}) \cap (\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z}))(c_1) = (\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(c_1) \cap (\tilde{z} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{z})(c_1) \subseteq \tilde{z}(c_1),$$
$$((\chi_{\mathfrak{D}}(\gamma) \circ \tau) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau))(c_1) = (\chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \geq \tau(c_1).$$

Hence, $\tilde{z}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$. □

Theorem 3.3. *If $\tilde{n}_\tau \in \mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$, then $\tilde{n}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$.*

Proof. Let $\tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D})$. Then, $\forall c_1 \in \mathfrak{D}$, we get, $(\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(c_1) \subseteq \tilde{n}(c_1)$ and $(\chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \geq \tau(c_1)$. And

$$(\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(c_1) = \bigcup_{c_1=j_1a_1y_1} \{\tilde{n}(j_1) \cap (\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(a_1y_1)\}$$
$$\subseteq \bigcup_{c_1=j_1a_1y_1} \{\tilde{n}(j_1) \cap \tilde{n}(a_1y_1)\} = \tilde{n}(c_1),$$
$$(\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) = \bigwedge_{c_1=j_1a_1y_1} \{\tau(j_1) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \tau)(a_1y_1)\}$$
$$\geq \bigwedge_{c_1=j_1a_1y_1} \{\tau(j_1) \vee \tau(a_1y_1)\} = \tau(c_1).$$

So,

$$((\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p})) \cap (\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n}))(c_1) = (\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}))(c_1) \cap (\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(c_1) \subseteq \tilde{n}(c_1),$$
$$((\tau \circ \chi_{\mathfrak{D}}(\gamma)) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau))(c_1) = (\tau \circ \chi_{\mathfrak{D}}(\gamma))(c_1) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \geq \tau(c_1).$$

Hence $\tilde{n}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$. □

Theorem 3.4. *Let $\tilde{n}_\tau \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$. Then $\tilde{n}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$.*

Proof. Let $\tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D})$. Then, $\forall c_1 \in \mathfrak{D}$, we get, $(\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}))(c_1) \subseteq \tilde{n}(c_1)$ and $(\tau \circ \chi_{\mathfrak{D}}(\gamma))(c_1) \geq \tau(c_1)$. And

$$(\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(c_1) = \bigcup_{c_1=j_1a_1y_1} \{(\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}))(j_1a_1) \cap \tilde{n}(y_1)\}$$
$$\subseteq \bigcup_{c_1=j_1a_1y_1} \{\tilde{n}(j_1a_1) \cap \tilde{n}(y_1)\} = \tilde{n}(c_1),$$
$$(\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) = \bigwedge_{c_1=j_1a_1y_1} \{(\tau \circ \chi_{\mathfrak{D}}(\gamma))(j_1a_1) \vee \tau(y_1)\}$$
$$\geq \bigwedge_{c_1=j_1a_1y_1} \{\tau(j_1a_1) \vee \tau(y_1)\} = \tau(c_1).$$

So,

$$(\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n}) \cap (\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n}))(c_1) = (\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(c_1) \cap (\tilde{n} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{n})(c_1) \subseteq \tilde{n}(c_1),$$
$$(\chi_{\mathfrak{D}}(\gamma) \circ \tau) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau))(c_1) = (\chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \vee (\tau \circ \chi_{\mathfrak{D}}(\gamma) \circ \tau)(c_1) \geq \tau(c_1).$$

Hence $\tilde{n}_\tau \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$. □

The following corollary follows from Theorem 3.3 and Theorem 3.4.

Corollary 3.1. *Let $\tilde{n}_\tau \in \mathscr{H}_{\mathfrak{J}}(\mathfrak{D})$. Then $\tilde{n}_\tau$ of $\mathfrak{D}$ is a hybrid bi-quasi ideal.*

Theorem 3.5. *Let $\tilde{v}_\varrho \in \mathscr{H}(\mathfrak{D})$. If $\tilde{v}_\varrho \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$, then the hybrid cut $\tilde{v}_\varrho[\Gamma, \beta]$ is a left bi-quasi ideal of $\mathfrak{D}$, $\forall \Gamma \in \mathcal{P}(\mathfrak{Q})$, $\beta \in [0, 1]$.*

Proof. For $\Gamma \in \mathcal{P}(\mathfrak{Q})$, $\beta \in [0, 1]$. Let $e_1, w_1 \in \tilde{v}_\varrho[\Gamma, \beta]$. Then $\tilde{v}(e_1+w_1) \supseteq \tilde{v}(e_1) \cap \tilde{v}(w_1) \supseteq \Gamma$ and $\varrho(e_1 + w_1) \leq \varrho(e_1) \vee \varrho(w_1) \leq \beta$. So $e_1 + w_1 \in \tilde{v}_\varrho[\Gamma, \beta]$.

Let $z \in \mathfrak{D}$. If $\exists\ f, l \in \mathfrak{D}$ and $u, y, w \in \tilde{v}_\varrho[\Gamma, \beta]$ such that $z = fu = ylw$. Then $(\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{v})(z) \supseteq \Gamma; (\chi_{\mathfrak{D}}(\gamma) \circ \varrho)(z) \leq \beta$ and $(\tilde{v} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{v})(z) \supseteq \Gamma; (\varrho \circ \chi_{\mathfrak{D}}(\gamma) \circ \varrho)(z) \leq \beta$ and $\tilde{v}(z) \supseteq ((\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{v}) \cap (\tilde{v} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{v}))(z) \supseteq \Gamma; \varrho(z) \leq ((\chi_{\mathfrak{D}}(\gamma) \circ \varrho) \vee (\varrho \circ \chi_{\mathfrak{D}}(\gamma) \circ \varrho))(z) \leq \beta$. Thus $z \in \tilde{v}_\varrho[\Gamma, \beta]$ and hence $\tilde{v}_\varrho[\Gamma, \beta]$ is a left bi-quasi-ideal of $\mathfrak{D}$. □

Theorem 3.6. *Let $\tilde{v}_\varrho \in \mathscr{H}(\mathfrak{D})$. If $\tilde{v}_\varrho \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$, then the hybrid cut $\tilde{v}_\varrho[\Gamma, \beta]$ is a right bi-quasi-ideal of $\mathfrak{D}, \forall \Gamma \in \mathcal{P}(\mathfrak{Q})$, $\beta \in [0, 1]$.*

Proof. It is similar to the proof of Theorem 3.5. □

Theorem 3.7. *Let $\tilde{v}_\varrho \in \mathscr{H}(\mathfrak{D})$ and $\mathscr{K} \in \mathcal{P}(\mathfrak{D}) \backslash \{\emptyset\}$. Then $\mathscr{K}$ of $\mathfrak{D}$ is a right bi-quasi ideal if and only if $\chi_{\mathscr{K}}(\tilde{v}_\varrho) \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$.*

Proof. Assume $\mathscr{K}$ of $\mathfrak{D}$ is a right bi-quasi ideal and for any $\tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D})$, we have $\chi_{\mathscr{K}}(\tilde{v}_\varrho)$ of $\mathfrak{D}$ is a hybrid subsemiring and $\mathscr{K}\mathfrak{D} \cap \mathscr{K}\mathfrak{D}\mathscr{K} \subseteq \mathscr{K}$. Thus $(\chi_{\mathscr{K}}(\tilde{v}_\varrho) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\chi_{\mathscr{K}}(\tilde{v}_\varrho) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \chi_{\mathscr{K}}(\tilde{v}_\varrho)) = \chi_{\mathscr{K}\mathfrak{D}}(\tilde{v}_\varrho) \Cap \chi_{\mathscr{K}\mathfrak{D}\mathscr{K}}(\tilde{v}_\varrho) = \chi_{\mathscr{K}\mathfrak{D} \cap \mathscr{K}\mathfrak{D}\mathscr{K}}(\tilde{v}_\varrho) \ll \chi_{\mathscr{K}}(\tilde{v}_\varrho)$. Therefore $\chi_{\mathscr{K}}(\tilde{v}_\varrho) \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$.

Conversely, let $\chi_{\mathscr{K}}(\tilde{v}_\varrho) \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$. Then $\mathscr{K}$ of $\mathfrak{D}$ is a subsemiring. As $(\chi_{\mathscr{K}}(\tilde{v}_\varrho) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\chi_{\mathscr{K}}(\tilde{v}_\varrho) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \chi_{\mathscr{K}}(\tilde{v}_\varrho)) \ll \chi_{\mathscr{K}}(\tilde{v}_\varrho)$, we get $\chi_{\mathscr{K}\mathfrak{D}}(\tilde{v}_\varrho) \Cap \chi_{\mathscr{K}\mathfrak{D}\mathscr{K}}(\tilde{v}_\varrho) \ll \chi_{\mathscr{K}}(\tilde{v}_\varrho)$ and $\chi_{\mathscr{K}\mathfrak{D} \cap \mathscr{K}\mathfrak{D}\mathscr{K}}(\tilde{v}_\varrho) \ll \chi_{\mathscr{K}}(\tilde{v}_\varrho)$. Thus $\mathscr{K}\mathfrak{D} \cap \mathscr{K}\mathfrak{D}\mathscr{K} \subseteq \mathscr{K}$. So $\mathscr{K}$ of $\mathfrak{D}$ is a right bi-quasi ideal. □

Theorem 3.8. *Let $\tilde{d}_\vartheta, \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$. Then $\tilde{d}_\vartheta \Cap \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$.*

Proof. For $a_0, n_0 \in \mathfrak{D}$ and $\tilde{p}_\gamma \in \mathscr{H}(\mathfrak{D})$, we get

$$\begin{aligned}
(\tilde{d} \cap \tilde{w})(a_0 + n_0) &= \tilde{d}(a_0 + n_0) \cap \tilde{w}(a_0 + n_0) \\
&\supseteq (\tilde{d}(a_0) \cap \tilde{d}(n_0)) \cap (\tilde{w}(a_0) \cap \tilde{w}(n_0)) \\
&= (\tilde{d}(a_0) \cap \tilde{w}(a_0)) \cap (\tilde{d}(n_0) \cap \tilde{w}(n_0)) = (\tilde{d} \cap \tilde{w})(a_0) \cap (\tilde{d} \cap \tilde{w})(n_0), \\
(\vartheta \cap \iota)(a_0 + n_0) &= \vartheta(a_0 + n_0) \vee \iota(a_0 + n_0) \\
&\leq (\vartheta(a_0) \vee \vartheta(n_0)) \vee (\iota(a_0) \vee \iota(n_0)) \\
&= (\vartheta(a_0) \vee \iota(a_0)) \vee (\vartheta(n_0) \vee \iota(n_0)) = (\vartheta \vee \iota)(a_0) \vee (\vartheta \vee \iota)(n_0).
\end{aligned}$$

and

$$
\begin{aligned}
(\chi_{\mathfrak{D}}(\tilde{p})\circ(\tilde{d}\cap\tilde{w}))(a_0) &= \bigcup_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\tilde{p})(r_0)\cap(\tilde{d}\cap\tilde{w})(e_0)\}\\
&= \bigcup_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\tilde{p})(r_0)\cap(\tilde{d}(e_0)\cap\tilde{w}(e_0))\}\\
&= \bigcup_{a_0=r_0e_0}\{(\chi_{\mathfrak{D}}(\tilde{p})(r_0)\cap\tilde{d}(e_0))\cap(\chi_{\mathfrak{D}}(\tilde{p})(r_0)\cap\tilde{w}(e_0))\}\\
&= \left\{\bigcup_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\tilde{p})(r_0)\cap\tilde{d}(e_0)\}\right\}\cap\left\{\bigcup_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\tilde{p})(r_0)\cap\tilde{w}(e_0)\}\right\}\\
&= (\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})(a_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w})(a_0)\\
&= ((\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w}))(a_0),\\
(\chi_{\mathfrak{D}}(\gamma)\circ(\vartheta\vee\iota))(a_0) &= \bigwedge_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\gamma)(r_0)\vee(\vartheta\vee\iota)(e_0)\}\\
&= \bigwedge_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\gamma)(r_0)\vee(\vartheta(e_0)\vee\iota(e_0))\}\\
&= \bigwedge_{a_0=r_0e_0}\{(\chi_{\mathfrak{D}}(\gamma)(r_0)\vee\vartheta(e_0))\vee(\chi_{\mathfrak{D}}(\gamma)(r_0)\vee\iota(e_0))\}\\
&= \left\{\bigwedge_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\gamma)(r_0)\vee\vartheta(e_0)\}\right\}\vee\left\{\bigwedge_{a_0=r_0e_0}\{\chi_{\mathfrak{D}}(\gamma)(r_0)\vee\iota(e_0)\}\right\}\\
&= (\chi_{\mathfrak{D}}(\gamma)\circ\vartheta)(a_0)\vee(\chi_{\mathfrak{D}}(\gamma)\circ\iota)(a_0)\\
&= ((\chi_{\mathfrak{D}}(\gamma)\circ\vartheta)\vee(\chi_{\mathfrak{D}}(\gamma)\circ\iota))(a_0).
\end{aligned}
$$

Thus $(\chi_{\mathfrak{D}}(\tilde{p}_\gamma)\odot(\tilde{d}_\vartheta\Cap\tilde{w}_\iota))=((\chi_{\mathfrak{D}}(\tilde{p}_\gamma)\odot\tilde{d}_\vartheta)\Cap(\chi_{\mathfrak{D}}(\tilde{p}_\gamma)\odot\tilde{w}_\iota))$. Now,

$$
\begin{aligned}
&((\tilde{d}\cap\tilde{w})\circ\chi_{\mathfrak{D}}(\tilde{p})\circ(\tilde{d}\cap\tilde{w}))(a_0) = \bigcup_{a_0=r_0e_0m_0}\{(\tilde{d}\cap\tilde{w})(r_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ(\tilde{d}\cap\tilde{w}))(e_0m_0)\}\\
&= \bigcup_{a_0=r_0e_0m_0}\{(\tilde{d}\cap\tilde{w})(r_0)\cap((\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w}))(e_0m_0)\}\\
&= \bigcup_{a_0=r_0e_0m_0}\{\tilde{d}(r_0)\cap\tilde{w}(r_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})(e_0m_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w})(e_0m_0)\}\\
&= \bigcup_{a_0=r_0e_0m_0}\{\tilde{d}(r_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})(e_0m_0)\cap\tilde{w}(r_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w})(e_0m_0)\}\\
&= \left\{\bigcup_{a_0=r_0e_0m_0}\{\tilde{d}(r_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})(e_0m_0)\}\right\}\cap\left\{\bigcup_{a_0=r_0e_0m_0}\{\tilde{w}(r_0)\cap(\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w})(e_0m_0)\}\right\}\\
&= (\tilde{d}\circ\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})(a_0)\cap(\tilde{w}\circ\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w})(a_0)\\
&= ((\tilde{d}\circ\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{d})\cap(\tilde{w}\circ\chi_{\mathfrak{D}}(\tilde{p})\circ\tilde{w}))(a_0),
\end{aligned}
$$

$$((\vartheta \vee \iota) \circ \chi_{\mathfrak{D}}(\gamma) \circ (\vartheta \vee \iota))(a_0) = \bigwedge_{a_0=r_0e_0m_0} \{(\vartheta \vee \iota)(r_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ (\vartheta \vee \iota))(e_0m_0)\}$$
$$= \bigwedge_{a_0=r_0e_0m_0} \{(\vartheta \vee \iota)(r_0) \vee ((\chi_{\mathfrak{D}}(\gamma) \circ \vartheta) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \iota))(e_0m_0)\}$$
$$= \bigwedge_{a_0=r_0e_0m_0} \{\vartheta(r_0) \vee \iota(r_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \vartheta)(e_0m_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \iota)(e_0m_0)\}$$
$$= \bigwedge_{a_0=r_0e_0m_0} \{\vartheta(r_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \vartheta)(e_0m_0) \vee \iota(r_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \iota)(e_0m_0)\}$$
$$= \left\{ \bigwedge_{a_0=r_0e_0m_0} \{\vartheta(r_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \vartheta)(e_0m_0)\} \right\} \vee \left\{ \bigwedge_{a_0=r_0e_0m_0} \{\iota(r_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \iota)(e_0m_0)\} \right\}$$
$$= (\vartheta \circ \chi_{\mathfrak{D}}(\gamma) \circ \vartheta)(a_0) \vee (\iota \circ \chi_{\mathfrak{D}}(\gamma) \circ \iota)(a_0)$$
$$= ((\vartheta \circ \chi_{\mathfrak{D}}(\gamma) \circ \vartheta) \vee (\iota \circ \chi_{\mathfrak{D}}(\gamma) \circ \iota))(a_0).$$

Thus $((\tilde{d}_\vartheta \Cap \tilde{w}_\iota) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) = ((\tilde{d}_\vartheta \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta) \Cap (\tilde{w}_\iota \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota))$. Hence $(\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) \Cap ((\tilde{d}_\vartheta \Cap \tilde{w}_\iota) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) = ((\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta) \Cap (\tilde{d}_\vartheta \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta)) \Cap ((\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota) \Cap (\tilde{w}_\iota \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota)) \ll (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)$. So $\tilde{d}_\vartheta \Cap \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$. □

Theorem 3.9. *If $\tilde{d}_\vartheta, \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$, then $\tilde{d}_\vartheta \Cap \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.8. □

Theorem 3.10. *Let $\tilde{d}_\vartheta \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$ and $\tilde{w}_\iota \in \mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$. Then $\tilde{d}_\vartheta \Cap \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$.*

Proof. For $a_0, n_0 \in \mathfrak{D}$, we get

$$\begin{aligned}
(\tilde{d} \cap \tilde{w})(a_0 + n_0) &= \tilde{d}(a_0 + n_0) \cap \tilde{w}(a_0 + n_0) \\
&\supseteq (\tilde{d}(a_0) \cap \tilde{d}(n_0)) \cap (\tilde{w}(a_0) \cap \tilde{w}(n_0)) \\
&= (\tilde{d}(a_0) \cap \tilde{w}(a_0)) \cap (\tilde{d}(n_0) \cap \tilde{w}(n_0)) = (\tilde{d} \cap \tilde{w})(a_0) \cap (\tilde{d} \cap \tilde{w})(n_0), \\
(\vartheta \vee \iota)(a_0 + n_0) &= \vartheta(a_0 + n_0) \vee \iota(a_0 + n_0) \\
&\leq (\vartheta(a_0) \vee \vartheta(n_0)) \vee (\iota(a_0) \vee \iota(n_0)) \\
&= (\vartheta(a_0) \vee \iota(a_0)) \vee (\vartheta(n_0) \vee \iota(n_0)) = (\vartheta \vee \iota)(a_0) \vee (\vartheta \vee \iota)(n_0).
\end{aligned}$$

By the proof of Theorem 3.8, we have, $(\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) = ((\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta) \Cap (\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota))$ and $((\tilde{d}_\vartheta \Cap \tilde{w}_\iota) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) = ((\tilde{d}_\vartheta \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta) \Cap (\tilde{w}_\iota \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota))$.

Hence $(\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) \Cap ((\tilde{d}_\vartheta \Cap \tilde{w}_\iota) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)) = ((\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta) \Cap (\tilde{d}_\vartheta \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{d}_\vartheta)) \Cap ((\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota) \Cap (\tilde{w}_\iota \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{w}_\iota)) \ll (\tilde{d}_\vartheta \Cap \tilde{w}_\iota)$. So, $\tilde{d}_\vartheta \Cap \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$. □

Theorem 3.11. *If $\tilde{d}_\vartheta \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$ and $\tilde{w}_\iota \in \mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$, then $\tilde{d}_\vartheta \Cap \tilde{w}_\iota \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$.*

Proof. It is similar to the proof of Theorem 3.10. □

Theorem 3.12. [20] *For any left ideal B and right ideal W of $\mathfrak{D}$, $WB = W \cap B$ if and only if $\mathfrak{D}$ is a regular semiring.*

Theorem 3.13. *If $\tilde{z}_\varsigma \in \mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$ and $\tilde{m}_\varphi \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$, then $\tilde{m}_\varphi \odot \tilde{z}_\varsigma = \tilde{m}_\varphi \Cap \tilde{z}_\varsigma$ if and only if $\mathfrak{D}$ is regular.*

Proof. Let $\tilde{z}_\varsigma \in \mathscr{H}_{\mathfrak{L}}(\mathfrak{D})$ and $\tilde{m}_\varphi \in \mathscr{H}_{\mathfrak{R}}(\mathfrak{D})$. Then $\tilde{m}_\varphi \odot \tilde{z}_\varsigma \ll \tilde{m}_\varphi \Cap \tilde{z}_\varsigma$. If $\mathfrak{D}$ is a regular, then for $q \in \mathfrak{D}$, $\exists\ w \in \mathfrak{D}$ such that $q = qwq$. Now

$$(\tilde{m} \circ \tilde{z})(q) = \bigcup_{q=bu} \{\tilde{m}(b) \cap \tilde{z}(u)\} \supseteq \tilde{m}(qw) \cap \tilde{z}(q) \supseteq \tilde{m}(q) \cap \tilde{z}(q) = (\tilde{m} \cap \tilde{z})(q),$$

$$(\varphi \circ \varsigma)(q) = \bigwedge_{q=bu} \{\varphi(b) \vee \varsigma(u)\} \leq \varphi(qw) \vee \varsigma(q) \leq \varphi(q) \vee \varsigma(q) = (\varphi \vee \varsigma)(q).$$

Thus $\tilde{m}_\varphi \Cap \tilde{z}_\varsigma \ll \tilde{m}_\varphi \odot \tilde{z}_\varsigma$ and hence $\tilde{m}_\varphi \odot \tilde{z}_\varsigma = \tilde{m}_\varphi \Cap \tilde{z}_\varsigma$.

Conversely, let W and B be a right and a left ideal of $\mathfrak{D}$ respectively. Then $\chi_W(\tilde{m}_\varphi)$ and $\chi_B(\tilde{m}_\varphi)$ are hybrid right ideal and hybrid left ideal of $\mathfrak{D}$ respectively. Moreover, $WB \subseteq W \cap B$. Let $w \in W \cap B$. Then $(\chi_W \tilde{m})(w) = \mathfrak{Q} = (\chi_B \tilde{m})(w)$ and $(\chi_W \varphi)(w) = 0 = (\chi_B \varphi)(w)$. Thus $(\chi_W \circ \chi_B)\tilde{m}(w) = (\chi_W \cap \chi_B)\tilde{m}(w) = \chi_W \tilde{m}(w) \cap \chi_B \tilde{m}(w) = \mathfrak{Q}$ and $(\chi_W \circ \chi_B)\varphi(w) = (\chi_W \vee \chi_B)\varphi(w) = \chi_W \varphi(w) \vee \chi_B \varphi(w) = 0$. So, $\chi_W \tilde{m}(w_1) \cap \chi_B \tilde{m}(w_2) = \mathfrak{Q}$ and $\chi_W \varphi(w_1) \vee \chi_B \varphi(w_2) = 0$ for some $w_1, w_2 \in \mathfrak{D}$ satisfying $w = w_1 w_2$ i.e., $w \in WB$. Hence $W \cap B = WB$ and so, by Theorem 3.12, $\mathfrak{D}$ is regular. □

Theorem 3.14. [20] *Let $\mathfrak{D}$ be a semiring. Then, for every left bi-quasi ideal P of $\mathfrak{D}$, $P = \mathfrak{D}P \cap P\mathfrak{D}P$ if and only if $\mathfrak{D}$ is a regular.*

Theorem 3.15. *Let $\mathfrak{D}$ be a semiring. Then, for $\tilde{y}_\varpi \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D}), \tilde{y}_\varpi = (\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi)$ if and only if $\mathfrak{D}$ is a regular.*

Proof. For a right bi-quasi ideal P of $\mathfrak{D}$, by Theorem 3.7, we have $\chi_P(\tilde{y}_\varpi) \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$. Hence $\chi_P(\tilde{y}_\varpi) = (\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \chi_P(\tilde{y}_\varpi)) \Cap (\chi_P(\tilde{y}_\varpi) \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \chi_P(\tilde{y}_\varpi)) = \chi_{\mathfrak{D}P}(\tilde{y}_\varpi) \Cap \chi_{P\mathfrak{D}P}(\tilde{y}_\varpi)$ and $P = \mathfrak{D}P \cap P\mathfrak{D}P$. By Theorem 3.14, $\mathfrak{D}$ is regular semiring.

Conversely, let $\tilde{y}_\varpi \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$. Then $(\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi) \ll \tilde{y}_\varpi$.

Let $q_0 \in \mathfrak{D}$. Then $\exists\ c_0 \in \mathfrak{D} : q_0 = q_0 c_0 q_0$. Now

$$(\tilde{y} \circ \chi_{\mathfrak{D}}(\tilde{p}))(q_0) \supseteq \bigcup_{q_0 = q_0 c_0 q_0} \{\tilde{y}(q_0) \cap \chi_{\mathfrak{D}}(\tilde{p})(c_0 q_0)\} = \tilde{y}(q_0),$$

$$(\varpi \circ \chi_{\mathfrak{D}}(\gamma))(q_0) \leq \bigwedge_{q_0 = q_0 c_0 q_0} \{\varpi(q_0) \vee \chi_{\mathfrak{D}}(\gamma)(c_0 q_0)\} = \varpi(q_0).$$

And

$$\begin{aligned}
(\tilde{y} \circ \chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{y})(q_0) &\supseteq \bigcup_{q_0 = q_0 c_0 q_0} \{\tilde{y}(q_0) \cap (\chi_{\mathfrak{D}}(\tilde{p}) \circ \tilde{y})(c_0 q_0)\} \\
&= \bigcup_{q_0 = q_0 c_0 q_0} \tilde{y}(q_0) \cap \bigcup_{c_0 q_0 = s_0 m_0} \{(\chi_{\mathfrak{D}}(\tilde{p})(m_0) \cap \tilde{y}(s_0))\} \\
&\supseteq \bigcup_{q_0 = q_0 q_0} \{\tilde{y}(q_0) \cap \tilde{y}(q_0)\} = \tilde{y}(q_0), \\
(\varpi \circ \chi_{\mathfrak{D}}(\gamma) \circ \varpi)(q_0) &\leq \bigwedge_{q_0 = q_0 c_0 q_0} \{\varpi(q_0) \vee (\chi_{\mathfrak{D}}(\gamma) \circ \varpi)(c_0 q_0)\} \\
&= \bigwedge_{q_0 = q_0 c_0 q_0} \varpi(q_0) \vee \bigwedge_{c_0 q_0 = s_0 m_0} \{(\chi_{\mathfrak{D}}(\gamma)(m_0) \vee \varpi(s_0))\} \\
&\leq \bigwedge_{q_0 = q_0 q_0} \{\varpi(q_0) \vee \varpi(q_0)\} = \varpi(q_0).
\end{aligned}$$

So, $(\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \Cap (\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi) = \tilde{y}_\varpi$. □

Theorem 3.16. *Let $\mathfrak{D}$ be a semiring. Then $\mathfrak{D}$ is a regular if and only if $(\tilde{y}_\varpi \Cap \tilde{f}_\xi) \ll (\tilde{f}_\xi \odot \tilde{y}_\varpi) \Cap (\tilde{y}_\varpi \odot \tilde{f}_\xi \odot \tilde{y}_\varpi)$, for $\tilde{y}_\varpi \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$ and $\tilde{f}_\xi \in \mathscr{H}_{\mathfrak{I}}(\mathfrak{D})$.*

Proof. Suppose $\mathfrak{D}$ is a regular and $v_1 \in \mathfrak{D}$, $\exists\ b_1 \in \mathfrak{D} : v_1b_1 = v_1b_1v_1b_1$. Now,

$$\begin{aligned}
(\tilde{y} \circ \tilde{f} \circ \tilde{y})(v_1) &\supseteq \bigcup_{v_1 = v_1b_1v_1} \{(\tilde{y} \circ \tilde{f})(v_1b_1) \cap \tilde{y}(v_1)\} \\
&\supseteq \bigcup_{v_1 = v_1b_1v_1} \{(\tilde{y}(v_1) \cap \tilde{f}(b_1)) \cap \tilde{y}(v_1)\} \\
&\supseteq \bigcup_{v_1b_1 = v_1b_1v_1b_1} \{(\tilde{y}(v_1) \cap \tilde{f}(b_1v_1b_1))\} \cap \tilde{y}(v_1) = \tilde{y}(v_1) \cap \tilde{f}(v_1), \\
(\varpi \circ \xi \circ \varpi)(v_1) &= \bigwedge_{v_1 = j_1j_2} \{(\varpi \circ \xi)(j_1) \vee \varpi(j_2)\} \\
&\leq \bigwedge_{v_1 = v_1b_1v_1} \{(\varpi \circ \xi)(v_1b_1) \vee \varpi(v_1)\} \\
&\leq \bigwedge_{v_1 = v_1b_1v_1} \{(\varpi(v_1) \vee \xi(b_1)) \vee \varpi(v_1)\} \\
&\leq \bigwedge_{v_1b_1 = v_1b_1v_1b_1} \{(\varpi(v_1) \vee \xi(b_1v_1b_1))\} \vee \varpi(v_1) = \varpi\,(v_1) \vee \xi\,(v_1)\,. \\
(\tilde{f} \circ \tilde{y})(v_1) &= \bigcup_{v_1 = e_1k_1} \{\tilde{f}(e_1) \cap \tilde{y}(k_1)\} \\
&\supseteq \bigcup_{v_1 = v_1b_1v_1} \{\tilde{f}(b_1) \cap \tilde{y}(v_1)\} \supseteq \tilde{f}(v_1) \cap \tilde{y}(v_1) = (\tilde{f} \cap \tilde{y})(v_1), \\
(\xi \circ \varpi)(v_1) &= \bigwedge_{v_1 = e_1k_1} \{\xi(e_1) \vee \varpi(k_1)\} \\
&\leq \bigwedge_{v_1 = v_1b_1v_1} \{\xi(b_1) \vee \varpi(v_1)\} \leq \xi(v_1) \vee \varpi(v_1) = (\xi \vee \varpi)(v_1).
\end{aligned}$$

Hence $(\tilde{y}_\varpi \Cap \tilde{f}_\xi) \ll (\tilde{y}_\varpi \odot \tilde{f}_\xi \odot \tilde{y}_\varpi)$. So $(\tilde{y}_\varpi \Cap \tilde{f}_\xi) \ll (\tilde{f}_\xi \odot \tilde{y}_\varpi) \Cap (\tilde{y}_\varpi \odot \tilde{f}_\xi \odot \tilde{y}_\varpi)$.

Conversely, if $\tilde{y}_\varpi \in \mathscr{H}\mathfrak{B}_{\mathfrak{L}}(\mathfrak{D})$, then $(\tilde{y}_\varpi \Cap \chi_{\mathfrak{D}}(\tilde{p}_\gamma)) \ll (\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi) \Cap (\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi)$ and $\tilde{y}_\varpi \ll (\chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi) \Cap (\tilde{y}_\varpi \odot \chi_{\mathfrak{D}}(\tilde{p}_\gamma) \odot \tilde{y}_\varpi)$. By Theorem 3.14, $\mathfrak{D}$ is a regular. □

Corollary 3.2. *Let $\mathfrak{D}$ be a semiring. Then for $\tilde{y}_\varpi \in \mathscr{H}\mathfrak{B}_{\mathfrak{R}}(\mathfrak{D})$ and $\tilde{f}_\xi \in \mathscr{H}_{\mathcal{I}}(\mathfrak{D})$, we have $(\tilde{y}_\varpi \Cap \tilde{f}_\xi) \ll (\tilde{y}_\varpi \odot \tilde{f}_\xi) \Cap (\tilde{y}_\varpi \odot \tilde{f}_\xi \odot \tilde{y}_\varpi)$ if and only if $\mathfrak{D}$ is a regular.*

4. Conclusion

In this work, we established the idea of hybrid left (right) bi-quasi ideals of semirings and obtained some results related to regular semirings using hybrid left (right) bi-quasi ideals of semirings. We also obtained equivalent conditions for a subset of a semiring to be a right bi-quasi ideal and for the semiring to be regular. Using the ideas and results presented in this paper, it is intended to demonstrate the concept of a hybrid prime (resp., semi) bi-quasi ideal and its related properties for a hybrid left (right) bi-quasi ideal to be a hybrid prime (resp., semi) left (right) bi-quasi ideal of semiring, as well as an intuitionistic hybrid left (right) bi-quasi ideal of semiring.

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Ms. S. Meenakshi received her master's degree at Alagappa University in 2018 and her master of philosophy at Seethalakshmi Achi College for Women in 2019. She works as a full-time research scholar in the Department of Mathematics at Karunya Institute of Technology and Sciences in Coimbatore. She has published a good number of research papers in mathematics journals and presented her findings at numerous international conferences. Her research interests include near-subtraction semigroups, semirings, and lattices in the context of hybrid ideals and hybrid structures.

Dr. G. Muhiuddin is a Professor of Mathematics at the University of Tabuk in Saudi Arabia. His research interests include logic-related algebras, fuzzy mathematics, hyper-algebraic structures, graph theory, decision analysis, and category theory. He has served as a referee for over 100 SCI journals and a reviewer for the American Math Society. He has published over 200 research papers in prestigious mathematical journals and is the editor-in-chief of the Annals of Communication in Mathematics. He has visited numerous universities around the world, giving invited talks. In 2021, 2022, and 2023, he was also named to Stanford University's list of the top 2.

Dr. K. Porselvi received her master's degree from the Department of Mathematics at Manonmaniam Sundaranar University, Tamil Nadu, and her doctorate in the Department of Mathematics at Karunya University in Tamil Nadu, India. She has been working in the Department of Mathematics at Karunya Institute of Technology and Sciences, Coimbatore, as an assistant professor. She has published a good number of research papers in mathematics journals and her areas of interest include algebra, partially ordered sets, and graph theory.

Dr. B. Elavarasan is a Professor (Assistant) of Mathematics at the Karunya Institute of Technology and Sciences, Coimbatore, India. He received his master's degree at Periyar University and his master of philosophy and doctorate in the Department of Mathematics at Annamalai University, Chidambaram. His research interests include algebra, discrete mathematics and various algebraic structures in the context of hybrid structures. He has published a good number of research papers in mathematical journals and acts as a reviewer for the various mathematics journals.