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FUZZY ASTERISK SUBALGEBRAS AND FUZZY ASTERISK IDEALS OF GK-ALGEBRA

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Abstract. In this paper, the notation of fuzzy asterisk subalgebras and fuzzy asterisk ideals of GK-algebra are introduced and investigated some of their properties. The homomorphic inverse image of fuzzy asterisk subalgebras and fuzzy asterisk ideals are studied. Also introduced the notation of fuzzy relations on the family of fuzzy asterisk subalgebras and fuzzy asterisk ideals of GK-algebra and investigated some related properties.

Keywords: GK-algebra, fuzzy asterisk subalgebra, fuzzy asterisk ideal, fuzzy $\rho-$product relation.

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1. Introduction

The notations of BCK/BCI - algebras [1]were initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Borumand Saeid[2] introduced fuzzy dot BCK/BCI-algebras. Neggers and Kim[3] introduced a new notation, called a B-algebras which is related to several classes of algebras of interest such as BCK/BCI-algebras. Kim and Kim[6] introduced the notation of BG-algebras, which is a generalization of B-algebras. J. Kavitha and R. Gowri[7] introduced the notation of GK-algebra in 2018 and investigate its properties.

For the general development of GK-algebras, the subalgebras and ideal theory play important role. Fuzzy asterisk subalgebras are very interesting algebraic structure and follow many interesting results. Also, fuzzy ideals are reach topics in any algebraic structure. Fuzzy asterisk ideals are also useful mathematical structure. To the best of our knowledge, no works are available on fuzzy asterisk subalgebras and fuzzy asterisk ideals of GK-algebras.

In this paper, fuzzy asterisk subalgebras of GK-algebras are defined and lot of properties are investigated. The notation of $\rho-$relations on the family of fuzzy asterisk subalgebras of GK-algebras are introduced with some related properties investigated. The remainder of this article is structured as follows: Section 2 proceeds with a recapitulation of

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all required definitions and properties. In Section 3, the concepts and operation of fuzzy asterisk subalgebras are introduced with their properties discussed in detail. In Section 4, notation of fuzzy asterisk ideals of GK-algebra are considered and ρ-product relations are given on fuzzy asterisk ideals. Finally in Section 5, conclusion is given.

2. Preliminaries

In this section some elementary aspects that are necessary for this paper are included. A GK-algebra is defined as follows.

Definition 2.1. [7] A non-empty set X with fixed constant 1 and a binary operation $\otimes$ is called a ***GK-algebra*** if it satisfies the following axioms

(1) $i \otimes i = 1$
(2) $i \otimes 1 = i$
(3) $i \otimes j = 1$ and $j \otimes i = 1$ implies $i = j$
(4) $(i \otimes j) \otimes (1 \otimes j) = i \;\; \forall\, i, j, k \;\in X$

Example 2.1. [8] *Consider the set X={ 1,l,m,n }. The binary operation $\otimes$ is defined as follows:*

$\otimes$	1	l	m	n
1	1	l	m	n
l	l	1	n	m
m	m	n	1	l
n	n	m	l	1

Hence $(X, \otimes, 1)$ is a GK-algebra.

Definition 2.2. [7] Let $(X, \otimes, 1)$ be a GK-algebra. A non-empty subset Y of X is called a sub algebra of X if $i \otimes j \;\in Y$ for any $i, j \;\in Y$. A mapping $f : X \to Y$ of GK-algebra is called a homomorphism if $f(x \otimes y) = f(x) \otimes f(y)$ for all $x, y \;\in X$.
We now review some fuzzy concepts as follows: Let X be a non empty set. A function *map* $\mu : X \to [0, 1]$ is called a **fuzzy set** on X. For any x $\in$ X, the number $\mu(x)$ is called the membership grade of x. The **complement** of μ, denoted by $\bar{\mu}$, is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x) \;\forall\; x \;\in X$.For any two fuzzy sets $A = \{ < x, \mu_A(x) >: x \;\in X \}$ and $B = \{ < x, \mu_B(x) >: x \;\in X \}$ in X, the following operations are defined,

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \;\forall\; x \;\in X,\; A \cap B = \min\{ \mu_A(x), \mu_B(x) \;\forall\; x \;\in X. \}$$

Let f be a mapping from the set X into the set Y. Let B be a fuzzy set in Y.Then the inverse image of B, denoted by $f^{\,-1}(B)$ in X is given by $f^{\,-1}(\mu_B(x)) = \mu_B(f(x))$.

Definition 2.3. [8] A fuzzy set A in X is called a fuzzy subalgebra if it satisfies the inequality $\mu_A(x \otimes y) \geq \min\{ \mu_A(x), \mu_A(y) \} \;\forall\; x, y \;\in X$.

Example 2.2. [8] *Consider the set X={ 1,2,3,4 } is a GK-algebra*

$\otimes$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Define a mapping $\mu_A : X \to [0, 1]$ by

$$\mu_A(x) = \begin{cases} 0.9 & if\;\, x = 1, 2 \\ 0.5 & if\;\, otherwise \end{cases}$$

Then μ_A is a fuzzy GK-subalgebra of X.

Definition 2.4. [8] A fuzzy set A in X is called a fuzzy ideal of X if it satisfies the inequality (i) $\mu_A(1) \geq \mu_A(x)$ and (ii) $\mu_A(x \otimes z) \geq \min\{ \mu_A(y \otimes z), \mu_A(y \otimes x) \} \; \forall \; x, y, z \in X$.

Example 2.3. *Consider the above example*(2.2)*. This is an example of fuzzy GK-ideal.*

3. Fuzzy asterisk subalgebra of GK-algebra

In this section, fuzzy asterisk subalgebra of GK-algebra are defined and some important properties are presented.In what follows,let $(X, \otimes, 1)$ or simply X denote a GK-algebra unless otherwise specified.

Definition 3.1. For all $x, y \in X$, Let A be a fuzzy set in a GK-algebra X. Then A is called a fuzzy asterisk subalgebra of X if $\mu_A(x \otimes y) \geq \mu_A(x) * \mu_A(y)$, where $*$ denotes ordinary multiplication.

Example 3.1. *Consider the set X={ 1,2,3,4,5 } with the following Cayley table:*

$\otimes$	1	2	3	4	5
1	1	5	4	3	2
2	2	1	5	4	3
3	3	2	1	5	4
4	4	3	2	1	5
5	5	4	3	2	1

Hence $(X, \otimes, 1)$ is a GK-algebra. Define a fuzzy set A in X by $\mu_A(1) = 0.65, \mu_A(2) = 0.53, \mu_A(3) = 0.7, \mu_A(4) = 0.57, \mu_A(5) = 0.8$. Then A is a fuzzy asterisk subalgebra of X. Note that every fuzzy subalgebra of X is a fuzzy asterisk subalgebra of X, but the converse is not true.

Remark 3.1. The fuzzy asterisk subalgebra in above example is not a fuzzy subalgebra, since $\mu_A(3 \otimes 3) = \mu_A(1) = 0.65 < 0.7 = \mu_A(3) = \min\{ \mu_A(3), \mu_A(3) \}$.

Theorem 3.1. *Every fuzzy asterisk subalgebras A of X satisfies the inequality $\mu_A(1) \geq (\mu_A(x))^2 \; \forall \; x \in X$.*

Proof. For all $x \in X$, we have $x \otimes x = 1$. Then $\mu_A(1) = \mu_A(x \otimes x) \geq \mu_A(x) * \mu_A(x) = (\mu_A(x))^2$. □

Theorem 3.2. *Let A be a fuzzy asterisk subalgebra of X .If there exists a sequence $\{ x_n \}$ in X such that $\lim_{n \to \infty}((\mu_A(x_n))^2)$=1, then $\mu_A(1) = 1$.*

Proof. By theorem (3.1), $\mu_A(1) \geq (\mu_A(x))^2$ for all $x \in X$. Therefore, $\mu_A(1) \geq (\mu_A(x_n))^2$ for every positive integer n. Consider, $1 \geq \mu_A(1) \geq \lim_{n \to \infty}((\mu_A(x_n))^2)$=1. Hence, $\mu_A(1) = 1$. □

Theorem 3.3. *Let A_1 and A_2 be two fuzzy asterisk subalgebras of X. Then $A_1 \cap A_2$ is a fuzzy asterisk subalgebra of X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2.

$$\begin{aligned} \text{Now, } \mu_{A_1 \cap A_2}(x \otimes y) &= \min\{ \mu_{A_1}(x \otimes y), \mu_{A_2}(x \otimes y) \} \\ &\geq \min\{ \mu_{A_1}(x) * \mu_{A_1}(y), \mu_{A_2}(x) * \mu_{A_2}(y) \} \\ &= (\min\{ \mu_{A_1}(x), \mu_{A_2}(x) \}) * (\min\{ \mu_{A_1}(y), \mu_{A_2}(y) \}) \\ &= \mu_{A_1 \cap A_2}(x) * \mu_{A_1 \cap A_2}(y) \end{aligned}$$

$$\text{Therefore, } \mu_{A_1 \cap A_2}(x \otimes y) \geq \mu_{A_1 \cap A_2}(x) * \mu_{A_1 \cap A_2}(y).$$

Hence, $A_1 \cap A_2$ is a fuzzy asterisk subalgebra of X.
The above can be generalised as follows. □

Theorem 3.4. *Let $\{ A_i \mid i = 1, 2, 3, \dots \}$ be a family of fuzzy asterisk subalgebras of X. Then $\bigcap A_i$ is also a fuzzy asterisk subalgebra of X, where $\bigcap A_i = \min\{ \mu_{A_i}(x) \}$.*

Theorem 3.5. *Let $f : X \to Y$ be a homomorphism of GK-algebra. If $B = \{ < x, \mu_B(x) >: x \in Y \}$ is a fuzzy asterisk subalgebra of Y, then the pre-image $f^{-1}(B) = \{ < x, f^{-1}(\mu_B(x)) >: x \in X \}$ of B under f is a fuzzy asterisk subalgebra of X.*

Proof. Assume that B is a fuzzy asterisk subalgebra of Y and let $x, y \in X$.Then
$f^{-1}(\mu_B(x \otimes y)) = \mu_B(f(x \otimes y)) = \mu_B(f(x) \otimes f(y)) \geq \mu_B(f(x)) * \mu_B(f(y)) = f^{-1}(\mu_B(x)) * f^{-1}(\mu_B(y))$.
Therefore,$f^{-1}(B) = \{ < x, f^{-1}(\mu_B(x)) >: x \in X \}$ is a fuzzy asterisk subalgebra of X. □

Theorem 3.6. *A fuzzy set A of a GK-algebra X is a fuzzy subalgebra of X if and only if for every $t \in [0, 1]$,the level subset $U(\mu_A(x) : t) = \{ x \in X \mid \mu_A(x) \geq t \}$ is either empty or GK- subalgebra of X.*

Remark 3.2. If A is a fuzzy asterisk subalgebra of X,then $U(\mu_A : t)$ need not be a subalgebra of X.In example (3.1),A is fuzzy asterisk subalgebra of X but $U(\mu_A : 0.65) = \{ x \in X \mid \mu_A(x) \geq 0.65 \} = \{ 1, 3, 5 \}$ is not a subalgebra of X since $1 \otimes 3 = 4 \notin U(\mu_A : 0.65)$.

Theorem 3.7. *Let A be a fuzzy asterisk subalgebra of X. Then $U(\mu_A : 1) = \{ x \in X \mid \mu_A(x) = 1 \}$ is either empty or is a subalgebra of X.*

Proof. Assume that $U(\mu_A : 1) \neq \emptyset$. Obviously $1 \in U(\mu_A : 1)$. If $x, y \in U(\mu_A : 1)$, then $\mu_A(x \otimes y) \geq \mu_A(x) * \mu_A(y) = 1$.Hence $\mu_A(x \otimes y) = 1$, which implies that $x \otimes y \in U(\mu_A : 1)$. Consequently,$U(\mu_A : 1)$ is a subalgebra of X. □

Definition 3.2. Let $A = \{ < x, \mu_A(x) >: x \in X \}$ and $B = \{ < x, \mu_B(x) >: x \in X \}$ be two fuzzy sets in X.The Cartesian product $A \times B : X \times X \to [0, 1]$ is defined by $(\mu_A \times \mu_B)(x, y) = \mu_A(x) * \mu_B(y)$ for all $x, y \in X$.

Theorem 3.8. *Let A and B be two fuzzy asterisk subalgebras of X, then $A \times B$ is a fuzzy asterisk subalgebra of $X \times X$.*

Proof. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$.

$$
\begin{aligned}
\text{Then, } (\mu_A \times \mu_B)((x_1, y_1) \otimes (x_2, y_2)) &= (\mu_A \times \mu_B)((x_1 \otimes x_2, y_1 \otimes y_2)) \\
&= \mu_A(x_1 \otimes x_2) * \mu_B(y_1 \otimes y_2) \\
&\geq ((\mu_A(x_1) * (\mu_A(x_2)) * ((\mu_B(y_1) * (\mu_B(y_2)) \\
&= ((\mu_A(x_1) * (\mu_B(y_1)) * ((\mu_A(x_2) * (\mu_B(y_2)) \\
&= (\mu_A \times \mu_B)((x_1, y_1) * (\mu_A \times \mu_B)((x_2, y_2)
\end{aligned}
$$

Therefore, $(\mu_A \times \mu_B)((x_1, y_1) \otimes (x_2, y_2)) \geq (\mu_A \times \mu_B)((x_1, y_1) * (\mu_A \times \mu_B)((x_2, y_2)$.

Hence, $A \times B$ is a fuzzy asterisk subalgebra of $X \times X$. □

4. Fuzzy asterisk ideal of GK-algebra.

In this section, fuzzy asterisk ideals of GK-algebra are defined and studied some of its result.

Definition 4.1. A fuzzy set A in X is called a fuzzy asterisk ideal of X if it satisfies the inequality (i) $\mu_A(1) \geq \mu_A(x)$ and (ii) $\mu_A(x \otimes z) \geq \mu_A(y \otimes z) * \mu_A(y \otimes x)$ $\forall\ x, y, z\ \in X$.

Example 4.1. *Consider the set X={1,2,3,4} be a set with the following Cayley table:*

$\otimes$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Hence $(X, \otimes, 1)$ is a GK-algebra.Define a fuzzy set A in X by $\mu_A(1) = 0.6, \mu_A(2) = 0.4, \mu_A(3) = 0.5, \mu_A(4) = 0.52$.Then A is a fuzzy asterisk ideal of X. Note that every fuzzy ideal of X is a fuzzy asterisk ideal of X, but the converse is not true.

Remark 4.1. The fuzzy asterisk ideal in above example is not a fuzzy ideal, since $\mu_A(2 \otimes 1) = \mu_A(2) = 0.4 < 0.5 = \min\{\ \mu_A(3), \mu_A(4)\ \} = \min\{\ \mu_A(4 \otimes 2), \mu_A(4 \otimes 1)\ \}$.

Theorem 4.1. *Let μ_A be fuzzy asterisk ideal of X.if $x \leq y$ holds in X.Then $\mu_A(x) \geq (\mu_A(y))^2$*

Proof. Let μ_A be fuzzy asterisk ideal of X.Let $x, y\ \in X$ and $x \leq y$ then $x \otimes y = y \otimes x = 1$.We know that $x \otimes 1 = x$.

$$\begin{aligned} \text{Now, } \mu_A((x) &= \mu_A(x \otimes 1)) \\ &\geq \mu_A(y \otimes 1) * \mu_A(y \otimes x) \\ &\geq (\mu_A(y) * \mu_A(1) \\ &\geq (\mu_A(y) * \mu_A(y) \\ &= (\mu_A(y))^2 \\ \text{Hence, } \mu_A(x) &\geq (\mu_A(y))^2. \end{aligned}$$

□

Theorem 4.2. *Let μ_A be fuzzy asterisk ideal of X.If the inequality $y \otimes x \leq z$ holds in X. Then $\mu_A(x) \geq \mu_A(y) * (\mu_A(z))^2$ $\forall\ x, y, z\ \in X$.*

Proof. Assume $y \otimes x \leq z$ holds in X.Then by theorem (4.1) $\mu_A(y \otimes x) \geq (\mu_A(z))^2$. By the definition of fuzzy asterisk ideal of GK-algebra,$\mu_A(x \otimes z) \geq \mu_A(y \otimes z) * \mu_A(y \otimes x)$.Put z=1 then $\mu_A(x \otimes 1) \geq \mu_A(y \otimes 1) * \mu_A(y \otimes x)$ implies $\mu_A(x) \geq \mu_A(y) * (\mu_A(z))^2$. □

Theorem 4.3. *If A is fuzzy asterisk ideal of X and $\mu_{A_\tau}(x) = \tau * \mu_A(x)$ $\forall\ x\ \in X$ and $\tau\ \in [0, 1]$, then $\mu_{A_\tau}(x)$ is fuzzy asterisk ideal of X.*

Proof. Let μ_A be fuzzy asterisk ideal of GK-algebra and $\tau\ \in [0, 1]$,hence $\mu_A(1) \geq \mu_A(x)$ $\forall\ x\ \in X$.Now $\mu_{A_\tau}(1) = \tau * \mu_A(1) \geq \tau * \mu_A(x) = \mu_{A_\tau}(x)$ $\forall\ x\ \in X$ and By the definition of fuzzy asterisk ideal of GK-algebra, $\mu_A(x \otimes z) \geq \mu_A(y \otimes z) * \mu_A(y \otimes x)$.

$$\begin{aligned} \text{Now, } \mu_{A_\tau}(x \otimes z) &= \tau * \mu_A(x \otimes z) \\ &\geq \tau * [\mu_A(y \otimes z) * \mu_A(y \otimes x)) \\ &= [\tau * \mu_A(y \otimes z)] * [\tau * \mu_A(y \otimes x)] \\ &= \mu_{A_\tau}(y \otimes z) * \mu_{A_\tau}(y \otimes x) \\ \text{Therefore, } \mu_{A_\tau}(x \otimes z) &\geq \mu_{A_\tau}(y \otimes z) * \mu_{A_\tau}(y \otimes x). \end{aligned}$$

Hence $\mu_{A_\tau}(x)$ is fuzzy asterisk ideal of X. □

Theorem 4.4. *If A and A^c are both fuzzy asterisk ideal of X.Then A is constant function.*

Proof. Let A and A^c are both fuzzy asterisk ideal of X. Then $\mu_A(1) \geq \mu_A(x)$ and $\mu_{A^c}(1) \geq \mu_{A^c}(x) \Rightarrow 1 - \mu_A(1) \geq 1 - \mu_A(x) \Rightarrow \mu_A(1) = \mu_A(x)$ for all $x \in X$. Therefore, $\mu_A(1) = \mu_A(x)$.Hence, A is a constant function. □

Theorem 4.5. *If A_1 and A_2 be two fuzzy asterisk ideals of X, then $A_1 \cap A_2$ is also a fuzzy asterisk ideal of X.*

Proof. Let A_1 and A_2 be two fuzzy asterisk ideals of X. Then for any $x \in X, \mu_{A_1}(1) \geq \mu_{A_1}(x)$, $\mu_{A_2}(1) \geq \mu_{A_2}(x)$. Now $\mu_{A_1 \cap A_2}(1)$ =min$\{ \mu_{A_1}(1), \mu_{A_2}(1) \} \geq \min\{ \mu_{A_1}(x), \mu_{A_2}(x) \}$ $= \mu_{A_1 \cap A_2}(x)$. Again for any $x, y \in X$,we have

$$\begin{aligned}
\mu_{A_1 \cap A_2}(x \otimes z) &= \min\{ \mu_{A_1}(x \otimes z), \mu_{A_2}(x \otimes z) \} \\
&\geq \min\{ \mu_{A_1}(y \otimes x) * \mu_{A_1}(y \otimes z), \mu_{A_2}(y \otimes x) * \mu_{A_2}(y \otimes z) \} \\
&= (\min\{ \mu_{A_1}(y \otimes x), \mu_{A_2}(y \otimes x) \})* \\
&\quad\ (\min\{ \mu_{A_1}(y \otimes z), \mu_{A_2}(y \otimes z) \}) \\
&= \mu_{A_1 \cap A_2}(y \otimes x) * \mu_{A_1 \cap A_2}(y \otimes z)
\end{aligned}$$

Therefore,$\mu_{A_1 \cap A_2}(x \otimes z) \geq \mu_{A_1 \cap A_2}(y \otimes x) * \mu_{A_1 \cap A_2}(y \otimes z)$.

Hence, $A_1 \cap A_2$ is a fuzzy asterisk ideal of X.
The above can be generalised as follows. □

Theorem 4.6. *Let $\{ A_i \mid i = 1, 2, 3, \dots \}$ be a family of fuzzy asterisk ideals of X. Then $\bigcap A_i$ is also a fuzzy asterisk ideal of X, where $\bigcap A_i = \min\{\mu_{A_i}(x)\}$.*

Theorem 4.7. *Let A and B be two fuzzy asterisk ideals of X, then $A \times B$ is a fuzzy asterisk ideal of $X \times X$.*

Proof. Let (x_1, y_1) , (x_2, y_2) and $(x_3, y_3) \in X \times X$.Then

$$\begin{aligned}
(\mu_A \times \mu_B)((x_1 \otimes x_3, y_1 \otimes y_3)) &= \mu_A(x_1 \otimes x_3) * \mu_B(y_1 \otimes y_3) \\
&\geq (\mu_A(x_2 \otimes x_3) * \mu_A(x_2 \otimes x_1))* \\
&\quad\ (\mu_B(y_2 \otimes y_3) * \mu_B(y_2 \otimes y_1)) \\
&= (\mu_A(x_2 \otimes x_3) * \mu_B(y_2 \otimes y_3))* \\
&\quad\ (\mu_A(x_2 \otimes x_1) * \mu_B(y_2 \otimes y_1)) \\
&= ((\mu_A \times \mu_B)(x_2 \otimes x_3, y_2 \otimes y_3))* \\
&\quad\ ((\mu_A \times \mu_B)(x_2 \otimes x_1, y_2 \otimes y_1))
\end{aligned}$$

Therefore,$(\mu_A \times \mu_B)((x_1 \otimes x_3, y_1 \otimes y_3)) \geq (\mu_A \times \mu_B)((x_2 \otimes x_3, y_2 \otimes y_3) *$
$(\mu_A \times \mu_B)((x_2 \otimes x_1, y_2 \otimes y_1)$. Hence, $A \times B$ is a fuzzy asterisk ideal of $X \times X$ □

Theorem 4.8. *A fuzzy set A of a GK-algebra X is a fuzzy ideal of X if and only if for every $t \in [0, 1]$, a non empty level subset $U(\mu_A : t) = \{x \in X : \mu_A(x) \geq t\}$ is an ideal of X.*

Proof. For any $t \in [0, 1]$,assume that $U(\mu_A : t)$ is non-empty.Let $x, y, z \in U(\mu_A : t)$. Now, let $(y \otimes x), (y \otimes z) \in U(\mu_A : t)$ .Hence, $\mu_A(x \otimes z) \geq \min\{ \mu_A(y \otimes x), \mu_A(y \otimes z) \} = \min\{ t, t \} = t$. Also $\mu_A(1) \geq \mu_A(x) \geq t$. Hence $1 \in U(\mu_A : t)$.Therefore $U(\mu_A : t)$ is an ideal of X.
Conversly, suppose $U(\mu_A : t)$ is a GK-ideal of X. Let $x, y, z \in X$. Take $t = \min\{ \mu_A(y \otimes x), \mu_A(y \otimes z) \}$. Then by assumption $\mu_A(x \otimes z) \geq t = \min\{ \mu_A(y \otimes x), \mu_A(y \otimes z) \}$. Also take $\mu_A(x) = t$,then $\mu_A(1) \geq t = \mu_A(x)$. Hence μ_A is fuzzy GK-ideal of X. □

Remark 4.2. If A is a fuzzy asterisk ideal of X,then $U(\ \mu_A : t\ )$ need not be an ideal of X.In example (3.1), we take the values $\mu_A(1) = \mu_A(2) = \mu_A(3) = \mu_A(4) = 0.6, \mu_A(5) = 0.4$ then A is fuzzy asterisk ideal of X but $U(\ \mu_A : 0.5) = \{\ x\ \in X \mid \mu_A(x) \geq 0.5\ \} = \{\ 1, 2, 3, 4\ \}$ is not an ideal of X since $(1 \otimes 1), (2 \otimes 1) \in U(\ \mu_A : 0.5)$ but $(1 \otimes 2) = 5 \notin U(\ \mu_A : 0.5)$.

Theorem 4.9. *Let $f : x \to y$ be a homomorphism of GK-algebra. If $B = \{\ < x, \mu_B(x) >: x\ \in Y\ \}$ is a fuzzy asterisk ideal of Y, then the pre-image $f^{\ -1}(B) = \{\ < x, f^{\ -1}(\mu_B(x)) >: x\ \in X\ \}$ of B under f is a fuzzy asterisk ideal of X.*

Proof. Assume that B is a fuzzy asterisk ideal of Y and let $x, y\ \in X$.Then $f^{\ -1}(\mu_B(x \otimes z)) = \mu_B(f(x \otimes z)) \geq \mu_B(f(y \otimes x)) * \mu_B(f(y \otimes z)) = f^{\ -1}(\mu_B(y \otimes x)) * f^{\ -1}(\mu_B(y \otimes z))$. Also, $\forall\ x\ \in X, f^{\ -1}(\mu_B(x)) = \mu_B(f(x)) \leq \mu_B(1) = \mu_B(f(1)) = f^{\ -1}(\mu_B)(1) \Rightarrow f^{\ -1}(\mu_B(x)) \leq f^{\ -1}(\mu_B)(1)$.
Therefore,$f^{\ -1}(B) = \{\ < x, f^{\ -1}(\mu_B(x)) >: x\ \in X\ \}$ is a fuzzy asterisk ideal of X. □

Theorem 4.10. *Let $f : x \to y$ be a epimorphism of GK-algebra. If $B = \{\ < x, \mu_B(x) >: x\ \in Y\ \}$ is a fuzzy asterisk ideal of Y, then the pre-image $f^{\ -1}(B) = \{\ < x, f^{\ -1}(\mu_B(x)) >: x\ \in X\ \}$ of B under f is a fuzzy asterisk ideal of X.*

Proof. For any $x\ \in Y$ thereexists $a\ \in X$ such that $f(a) = x$. Hence $\mu_B(x) = \mu_B(f(a)) = f^{\ -1}(\mu_B(a)) \leq f^{\ -1}(\mu_B(1)) = \mu_B(f(1)) = \mu_B(1)$.
Let $x, y, z\ \in Y$, $f(a) = x, f(b) = y, f(c) = z$ for some $a, b, c\ \in X$ $\mu_B(x \otimes z) = \mu_B(f(a) \otimes f(c)) = \mu_B(f(a \otimes c)) = f^{\ -1}(\mu_B)(a \otimes c) \geq f^{\ -1}(\mu_B)(b \otimes a) * f^{\ -1}(\mu_B)(b \otimes c) = \mu_B(f(b \otimes a)) * \mu_B(f(b \otimes c)) = \mu_B(f(b) \otimes f(a)) * \mu_B(f(b) \otimes f(c)) = \mu_B(y \otimes x) * \mu_B(y \otimes z)$.
Hence B is fuzzy asterisk ideal of Y. □

Definition 4.2. Let ρ be a fuzzy subset of X. A fuzzy relation μ on X is called a fuzzy $\rho-$product relation if $\mu(x, y) \geq \rho(x) * \rho(y)$

Theorem 4.11. *Let μ_ρ be the strongest fuzzy $\rho-$relation on X, where ρ is a subset of X. Then ρ is a fuzzy ideal of X if and only if μ_ρ is a fuzzy asterisk ideal of $X \times X$.*

Proof. Suppose that ρ is a fuzzy asterisk ideal of X. For any $x, y\ \in X$, we have $\mu_\rho(1, 1) = \rho(1) * \rho(1) \geq \rho(x) * \rho(y) = \mu_\rho(x, y)$.
Let (x_1, x_2),(y_1, y_2),$(z_1, z_2)\ \in X \times X$. Then

$$\begin{aligned}\mu_\rho(x_1 \otimes z_1, x_2 \otimes z_2) &= \rho(x_1 \otimes z_1) * \rho(x_2 \otimes z_2)\\ &\geq \rho(y_1 \otimes x_1) * \rho(y_1 \otimes z_1) * \rho(y_2 \otimes x_2) * \rho(y_2 \otimes z_2)\\ &= \mu_\rho(y_1 \otimes x_1, y_1 \otimes z_1) * \mu_\rho(y_2 \otimes x_2, y_2 \otimes z_2)\end{aligned}$$

Hence , μ_ρ is a fuzzy asterisk ideal of $X \times X$. Conversly, assume that μ_ρ is a fuzzy asterisk ideal of $X \times X, (\rho(1))^2 = \mu_\rho(1, 1) \geq \mu_\rho(x, x) = (\rho(x))^2\ \forall\ x\ \in X$.Also,$(\rho(x \otimes z))^2 = \mu_\rho(x \otimes z, x \otimes z) = \mu_\rho((x, x) \otimes (z, z)) \geq \mu_\rho((y, y) \otimes (x, x)) * \mu_\rho((y, y) \otimes (z, z)) = \mu_\rho(y \otimes x, y \otimes x) * \mu_\rho(y \otimes z, y \otimes z) = (\rho(y \otimes x))^2 * (\rho(y \otimes z))^2$, which implies that $\rho(x \otimes z) \geq \rho(y \otimes x) * \rho(y \otimes z)$ for all $x, y\ \in X$. Therefore , ρ is a fuzzy asterisk ideal of X. □

5. Conclusions

In the present paper,the notations of fuzzy asterisk subalgebras and fuzzy asterisk ideals of GK-algebras are introduced with some of its important properties. The relationship are discussed between fuzzy asterisk subalgebras and fuzzy asterisk ideals of GK-algebras.

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