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NEW OPERATIONS ON PYTHAGOREAN NEUTROSOPHIC FUZZY SETS

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Abstract. Pythagorean Neutrosophic Fuzzy Sets (PNFS) as a significant breakthrough in handling uncertainty and indeterminacy, offering a comprehensive framework that synthesizes the strengths of neutrosophic sets and Pythagorean fuzzy sets. This study meticulously investigates fundamental set operations within PNFS, encompassing Additive, Product, Scalar Product, Scalar Power and Operation @, intricately tailored to accommodate the unique characteristics of PNFS, capturing degrees of truth, indeterminacy, and falsity associated with Pythagorean Fuzzy environment. The paper introduces novel operations explicitly designed for PNFS, including Scalar Power and Operation @, thereby expanding the toolkit for managing uncertainty within mathematical frameworks. A robust foundation is laid through meticulous presentations of mathematical formulations and properties of PNFS operations, covering aspects like commutativity, idempotency, absorption law, associativity, De Morgan's rules, and distributivity over complement. This contributes significantly to the theoretical underpinning of PNFS. The efficacy of the proposed operations is demonstrated through illustrative examples, showcasing their practical utility in navigating complex and ambiguous information. This positions PNFS as a valuable tool in decision-making, pattern recognition, and other domains where uncertainty is a critical factor.The study makes a substantial contribution to the dynamic field of neutrosophic and fuzzy set theories by providing a versatile framework for managing uncertainty. PNFS's adaptability renders it applicable to a diverse range of real-world scenarios, facilitating the seamless integration of advanced mathematical concepts into practical applications. In conclusion, this exploration of Pythagorean Neutrosophic Fuzzy Sets not only advances theoretical understanding but also offers practical solutions for addressing complexity in real-world applications. The proposed operations represent a valuable contribution to the broader scientific and engineering community, fostering innovative approaches to comprehensively manage uncertainty across various contexts.

Keywords: Pythagorean Neutrosophic Fuzzy Set, Operations, Properties. multiplication.

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1. Introduction

Fuzzy set theory, introduced by Lotfi Zadeh [19] in 1965, deals with uncertainty and vagueness in information. In this original work, Zadeh introduced the concept of a fuzzy set, where membership degrees are represented by real numbers between 0 and 1. Intuitionistic Fuzzy Sets (IFS) were indeed introduced by Krassimir T. Atanassov [3]. He proposed the concept "Intuitionistic Fuzzy Sets" published in 1986. The introduction of Intuitionistic Fuzzy Sets provided a more nuanced way of handling uncertainty in comparison to classical fuzzy sets. It allowed for the explicit representation of hesitation or lack of confidence in the assignment of membership and non-membership degrees. Neutrosophic set theory, a groundbreaking concept in the realm of fuzzy logic, was introduced by Florentin Smarandache [17] in 1995. Neutrosophic sets extend the traditional framework of fuzzy sets by introducing the concept of indeterminacy along with truth and falsehood. In a neutrosophic set, each element is characterized by three degrees: the degree of truth, the degree of indeterminacy, and the degree of falsehood. Pythagorean fuzzy sets (PFS) are an extension of Intuitionistic fuzzy sets that were introduced by Ronald R. Yager [18] in 2013, is characterized by the constraint that the square sum of the membership degree and non-membership degree does not exceed 1. Pythagorean Neutrosophic Sets with T and F are dependent neutrosophic components, introduced by R. Jansi et al., [11] in 2019. Pythagorean Neutrosophic Fuzzy Sets with T and F (PNF-TF) represent a combination of Pythagorean fuzzy sets, neutrosophic sets, and fuzzy sets. The aim is to provide a more comprehensive framework for handling uncertainty, indeterminacy, and vagueness in a unified manner. In a Pythagorean Neutrosophic Fuzzy Set with Tand F, each element is associated with three values: the membership degree (μ), non-membership degree (ν), and indeterminacy degree (θ). The Pythagorean Neutrosophic Fuzzy Set with T and F is ensuring that the square sum of the membership, non-membership, and indeterminacy degrees does not exceed 2.

In 2023, Bozyigit et al. [5] introduced a groundbreaking concept, a new variant of Pythagorean Neutrosophic Fuzzy set (PNFS), as an innovative approach within the Neutrosophic set framework applied to a Pythagorean fuzzy environment. Although Pytha gorean fuzzy sets, neutrosophic sets, and fuzzy sets are established mathematical frameworks with extensive research, the integration of truth, indeterminacy, and falsity degrees within the Pythagorean fuzzy environment allows for more refined evaluations by decision makers in practical scenarios. This represents a relatively recent advancement in the context of Neutrosophic fuzzy sets in a Pythagorean environment. Essentially, these sets empower decision makers to assess the degrees of truth, indeterminacy, and falsity as Pythagorean Fuzzy Values (PFVs) in the decision-making process. Consequently, this approach enhances the representation of uncertainty in decision maker evaluations using a more sophisticated fuzzy notion. In certain decision-making scenarios, the cumulative values of membership and non-membership degrees, as determined by decision makers, may exceed 1. Hence, PFVs prove more effective than Intuitionistic Fuzzy Values (IFVs) in practical problems. PNFS serve as a valuable tool for articulating uncertainty within an expanded Pythagorean fuzzy environment, enabling the preservation of more extensive information during the conversion of data to a Fuzzy Set (FS). This approach helps prevent information loss. The integration of these three distinct mathematical paradigms into Pythagorean Neutrosophic Fuzzy Sets represents a novel approach to managing multifaceted uncertainties.

In the Preliminaries section, we provide fundamental definitions for both NFS (Neutrosophic Fuzzy Set) and PFS (Pythagorean Fuzzy Set). Moving on to the Pythagorean

Neutrosophic Fuzzy Set section, we present the concept of Pythagorean Neutrosophic Fuzzy sets along with their basic operations. In the following section, PNFS Operations, we articulate an operation on Pythagorean Neutrosophic Fuzzy sets, presenting examples and proving various algebraic properties associated with these operations. A subsequent section is dedicated to a novel operation denoted as (@) on Pythagorean Neutrosophic Fuzzy sets, where we define this new operation and scrutinize its algebraic properties. Finally, we encapsulate the findings and insights in the Conclusion section, summarizing the key outcomes of the paper.

2. Preliminary Definitions

We will go through a bunch of fundamental concepts connected to the PNFS that are renowned in the literature in this section.

Definition 2.1. [17] *An NFS* $\mathfrak{T}$ *from* $\mathfrak{X}$ *(universe of discourse) obtained from* $\mathfrak{T} = \{< x, T_{\mathfrak{T}}(\mathfrak{x}), I_{\mathfrak{T}}(\mathfrak{x}), F_{\mathfrak{T}}(\mathfrak{x}) > x \in X\}$*, whereas* $T_{\mathfrak{T}} : \mathfrak{X} \to [0,1]; I_{\mathfrak{T}} : \mathfrak{X} \to [0,1]; F_{\mathfrak{T}}(\mathfrak{x}) : \mathfrak{X} \to [0,1]$ *refers the degree of membership as well as non-membership, duly, while* $\forall \mathfrak{x} \in \mathfrak{X}$ *fulfils* $0 \leq T_{\mathfrak{T}}(\mathfrak{x}) + I_{\mathfrak{T}}(\mathfrak{x}), +F_{\mathfrak{T}}(\mathfrak{x}) \leq 3$.

Definition 2.2. [18] *A PFS* $\mathfrak{D}$ *from* $\mathfrak{X}$ *is* $\mathfrak{D} = \{< \mathfrak{x},\ \mu_{\mathfrak{D}}\ (\mathfrak{x}),\ \nu_{\mathfrak{D}}(\mathfrak{x}) > \mathfrak{x} \in \mathfrak{X}\}$*; here* $\mu_{\mathfrak{D}} : \mathfrak{X} \to [0,\ 1]$*,* $\nu_{\mathfrak{D}} : \mathfrak{X} \to [0,1]$ *is the degree of membership and non-membership,* $\forall\ \mathfrak{x} \in \mathfrak{X}$ *fulfils* $0 \leq \mu^2_{\mathfrak{D}}(\mathfrak{x}) + \nu^2_{\mathfrak{D}}(\mathfrak{x}) \leq 1$*. The degree of indeterminacy for such* $\mathfrak{D}$ *with* $\mathfrak{x} \in \mathfrak{X}$ *is* $\pi_{\mathfrak{D}}(\mathfrak{x}) = \sqrt{1 - \mu_{\mathfrak{D}}{}^2(\mathfrak{x}) - \nu_{\mathfrak{D}}{}^2(\mathfrak{x})}$*. While the criteria for IFS is* $0 \leq \mu_{\mathfrak{T}}(\mathfrak{x}) + \nu_{\mathfrak{T}}(\mathfrak{x}) \leq 1$ *whereas* $\mu_{\mathfrak{T}}(\mathfrak{x}),\ \nu_{\mathfrak{T}}(\mathfrak{x}) \in [0,\ 1]$*, the limitation respect to the degree of membership* $\mu_{\mathfrak{D}}(\mathfrak{x})$ *and non-membership* $\nu_{\mathfrak{D}}(\mathfrak{x})$ *in the case of PFS is* $0 \leq \mu_{\mathfrak{D}}{}^2(\mathfrak{x}) + \nu_{\mathfrak{D}}{}^2(\mathfrak{x}) \leq 1$.

Definition 2.3. [18] *Consider that* $\mathfrak{K}$ *and* $\mathfrak{L}$ *are distinct PFS. Then the operations are as below:*

$$\mathfrak{K} \oplus \mathfrak{L} = \left(\sqrt{\mathbb{T}_{\mathfrak{K}}{}^2 + \mathbb{T}_{\mathfrak{L}}{}^2 - \mathbb{T}_{\mathfrak{K}}{}^2 * \mathbb{T}_{\mathfrak{L}}{}^2}, \mathfrak{F}_{\mathfrak{K}} * \mathfrak{F}_{\mathfrak{L}}\right)$$

$$\mathfrak{K} \otimes \mathfrak{L} = \left(\mathbb{T}_{\mathfrak{K}} * \mathbb{T}_{\mathfrak{L}}, \sqrt{\mathfrak{F}_{\mathfrak{K}}{}^2 + \mathfrak{F}_{\mathfrak{L}}{}^2 - \mathfrak{F}_{\mathfrak{K}}{}^2 * \mathfrak{F}_{\mathfrak{L}}{}^2}\right)$$

$$\lambda\mathfrak{K} = \left(\sqrt{1 - \left(1 - \mathbb{T}_{\mathfrak{K}}{}^2\right)^{\lambda}},\ \mathfrak{F}_{\mathfrak{K}}{}^{\lambda}\right), \quad \lambda > 0$$

$$\mathfrak{K}^{\lambda} = \left(\mathbb{T}_{\mathfrak{K}}{}^{\lambda},\ \sqrt{1 - \left(1 - \mathfrak{F}_{\mathfrak{K}}{}^2\right)^{\lambda}}\right), \quad \lambda > 0$$

Definition 2.4. [5] *A PNFS set* $\mathfrak{K}$ *on* $\mathfrak{X}$*,* $\mathfrak{K} = \{< \mathfrak{x},\ \mathbb{T}_{\mathfrak{K}}, \mathfrak{I}_{\mathfrak{K}}, \mathfrak{F}_{\mathfrak{K}} >: \mathfrak{x} \in \mathfrak{X}\}$*, here* $\mathbb{T}_{\mathfrak{K}}, \mathfrak{T}_{\mathfrak{K}}$ *and* $\mathfrak{F}_{\mathfrak{K}}$ *indicates the truth, indeterminacy, also falsity membership pairs of PFVs whereas* $\mathfrak{j} = 1, \ldots, \mathfrak{z}$. $\mathbb{T}_{\mathfrak{K}} = (\mu_{\mathfrak{K},\mathfrak{t}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{t}}(\mathfrak{x}))$ *whereas* $\mu_{\mathfrak{K},\mathfrak{t}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{t}}(\mathfrak{x}) \in [0,\ 1]; \mu_{\mathfrak{K},\mathfrak{t}}{}^2(\mathfrak{x}) + \nu_{\mathfrak{K},\mathfrak{t}}{}^2(\mathfrak{x}) \leq 1$
$\mathfrak{I}_{\mathfrak{K}} = (\mu_{\mathfrak{K},\mathfrak{i}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{i}}\ (\mathfrak{x}))$ *whereas* $\mu_{\mathfrak{K},\mathfrak{i}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{i}}(\mathfrak{x}) \in [0,\ 1]; \mu_{\mathfrak{K},\mathfrak{i}}{}^2(\mathfrak{x}) + \nu_{\mathfrak{K},\mathfrak{i}}{}^2(\mathfrak{x}) \leq 1$
$\mathfrak{F}_{\mathfrak{K}} = (\mu_{\mathfrak{K},\mathfrak{f}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{f}}\ (\mathfrak{x}))$ *whereas* $\mu_{\mathfrak{K},\mathfrak{f}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{f}}(\mathfrak{x}) \in [0,\ 1]; \mu_{\mathfrak{K},\mathfrak{f}}{}^2(\mathfrak{x}) + \nu_{\mathfrak{K},\mathfrak{f}}{}^2(\mathfrak{x}) \leq 1$
A PNFS is denoted by $\mathfrak{K} = \langle\ (\mu_{\mathfrak{K},\mathfrak{t}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{t}}(\mathfrak{x})),\ (\mu_{\mathfrak{K},\mathfrak{i}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{i}}\ (\mathfrak{x})),\ (\mu_{\mathfrak{K},\mathfrak{f}}(\mathfrak{x}),\ \nu_{\mathfrak{K},\mathfrak{f}}\ (\mathfrak{x}))\ \rangle$

Definition 2.5. [5] *Let* $\mathfrak{K} = \langle(\ \mu_{\mathfrak{K},\mathfrak{t}},\ \nu_{\mathfrak{K},\mathfrak{t}}),\ (\ \mu_{\mathfrak{K},\mathfrak{i}},\ \nu_{\mathfrak{K},\mathfrak{i}}),\ (\mu_{\mathfrak{K},\mathfrak{f}},\ \nu_{\mathfrak{K},\mathfrak{f}})\rangle$

(i) $\mathfrak{K}^{\mathfrak{C}} = \langle(\mu_{\mathfrak{K},\mathfrak{f}},\ \nu_{\mathfrak{K},\mathfrak{f}})\ ,\ (\nu_{\mathfrak{K},\mathfrak{i}},\ \mu_{\mathfrak{K},\mathfrak{i}}),\ (\ \mu_{\mathfrak{K},\mathfrak{t}},\ \nu_{\mathfrak{K},\mathfrak{t}})\rangle$
(ii) $\mathfrak{K} \vee \mathfrak{L} = \langle(\max(\mu_{\mathfrak{K},\mathfrak{t}},\ \mu_{\mathfrak{L},\mathfrak{t}}), \min(\nu_{\mathfrak{K},\mathfrak{t}}, \nu_{\mathfrak{L},\mathfrak{t}})),\ (\min(\mu_{\mathfrak{K},\mathfrak{i}},\ \mu_{\mathfrak{L},\mathfrak{i}}), \max(\nu_{\mathfrak{K},\mathfrak{i}}, \nu_{\mathfrak{L},\mathfrak{i}})),$ $(\min(\mu_{\mathfrak{K},\mathfrak{f}},\ \mu_{\mathfrak{L},\mathfrak{f}}), \max(\nu_{\mathfrak{K},\mathfrak{f}}, \nu_{\mathfrak{L},\mathfrak{f}}))\rangle$
(iii) $\mathfrak{K} \wedge \mathfrak{L} = \langle(\min(\mu_{\mathfrak{K},\mathfrak{t}}, \mu_{\mathfrak{L},\mathfrak{t}}), \max(\nu_{\mathfrak{K},\mathfrak{t}}, \nu_{\mathfrak{L},\mathfrak{t}})), (\max(\mu_{\mathfrak{K},\mathfrak{i}}, \mu_{\mathfrak{L},\mathfrak{i}}), \min(\nu_{\mathfrak{K},\mathfrak{i}}, \nu_{\mathfrak{L},\mathfrak{i}})),$ $(\max(\mu_{\mathfrak{K},\mathfrak{f}}, \mu_{\mathfrak{L},\mathfrak{f}}), \min(\nu_{\mathfrak{K},\mathfrak{f}}, \nu_{\mathfrak{L},\mathfrak{f}}))\rangle$

3. Modal operators on PNFS

We define and examine the algebraic properties of the modal operators of a PNFS in this section. In addition, we defined an implication of PNFS and investigated its related features. The Pythagorean Neutrosophic (PN) is defined in this section along with plenty of algebraic properties, including commutativity, idempotency, absorption, associativity, De Morgan's rules and distributivity with respect to the complement. By putting the measure of positive, neutral, also negative membership against the backdrop of an ambiguous environment, we shall now establish PNFS algebraic operations.

3.1. **Noval Operations of PNFS.** The distinct activities of PNFS are now defined. Assume $\mathfrak{K}$ and $\mathfrak{L}$ are two PNPFS with the PNPFS operations (Additive, Product, Scalar Product, and Scalar Power) below:

Let $\mathfrak{K} =< (\mu_{\mathfrak{K},t},\ \nu_{\mathfrak{K},t}),\ (\mu_{\mathfrak{K},i},\ \nu_{\mathfrak{K},i}),\ (\mu_{\mathfrak{K},f},\ \nu_{\mathfrak{K},f}) >$, $\mathfrak{L} =< (\mu_{\mathfrak{L},t},\ \nu_{\mathfrak{L},t}),\ (\mu_{\mathfrak{L},i},\ \nu_{\mathfrak{L},i}),\ (\mu_{\mathfrak{L},f},\ \nu_{\mathfrak{L},f}) >$ are belongs to PNFS. Then

3.1.1. *Addition.*

$$\begin{aligned}(\mathfrak{K} \oplus \mathfrak{L}) = \Big\langle & \left(\sqrt{({\mu_{\mathfrak{K},t}}^2 + {\mu_{\mathfrak{L},t}}^2) - ({\mu_{\mathfrak{K},t}}^2 * {\mu_{\mathfrak{L},t}}^2)}, (\nu_{\mathfrak{K},t} * \nu_{\mathfrak{L},t}) \right), \\ & \left(\sqrt{({\mu_{\mathfrak{K},i}}^2 + {\mu_{\mathfrak{L},i}}^2) - ({\mu_{\mathfrak{K},i}}^2 * {\mu_{\mathfrak{L},i}}^2)}, (\nu_{\mathfrak{K},i} * \nu_{\mathfrak{L},i}) \right), \\ & \left(\sqrt{({\mu_{\mathfrak{K},f}}^2 + {\mu_{\mathfrak{L},f}}^2) - ({\mu_{\mathfrak{K},f}}^2 * {\mu_{\mathfrak{L},f}}^2)}, (\nu_{\mathfrak{K},f} * \nu_{\mathfrak{L},f}) \right) \Big\rangle\end{aligned}$$

3.1.2. *Multiplication.*

$$\begin{aligned}\mathfrak{K} \otimes \mathfrak{L} = \Big\langle & \left((\mu_{\mathfrak{K},t} * \mu_{\mathfrak{L},t} \right), \sqrt{({\nu_{\mathfrak{K},t}}^2 + {\nu_{\mathfrak{L},t}}^2) - ({\nu_{\mathfrak{K},t}}^2 * {\nu_{\mathfrak{L},t}}^2)}), \\ & \left((\mu_{\mathfrak{K},i} * \mu_{\mathfrak{L},i} \right), \sqrt{({\nu_{\mathfrak{K},i}}^2 + {\nu_{\mathfrak{L},i}}^2) - ({\nu_{\mathfrak{K},i}}^2 * {\nu_{\mathfrak{L},i}}^2)}), \\ & \left((\mu_{\mathfrak{K},f} * \mu_{\mathfrak{L},f} \right), \sqrt{({\nu_{\mathfrak{K},f}}^2 + {\nu_{\mathfrak{L},f}}^2) - ({\nu_{\mathfrak{K},f}}^2 * {\nu_{\mathfrak{L},f}}^2)}) \Big\rangle\end{aligned}$$

3.1.3. *Scalar Multiplication.*

$$\begin{aligned}\lambda\mathfrak{K} = \Big\langle & \left(\sqrt{1 - (1 - {\mu_{\mathfrak{K},t}}^2)^\lambda}, {\nu_{\mathfrak{K},t}}^\lambda \right), \left(\sqrt{1 - (1 - {\mu_{\mathfrak{K},i}}^2)^\lambda}, {\nu_{\mathfrak{K},i}}^\lambda \right), \\ & \left(\sqrt{1 - (1 - {\mu_{\mathfrak{K},f}}^2)^\lambda}, {\nu_{\mathfrak{K},f}}^\lambda \right) \Big\rangle, \lambda > 0\end{aligned}$$

3.1.4. *Scalar Power.*

$$\begin{aligned}\mathfrak{K}^\lambda = \Big\langle & \left({\mu_{\mathfrak{K},t}}^\lambda, \sqrt{1 - (1 - {\nu_{\mathfrak{K},t}}^2)^\lambda} \right), \left({\mu_{\mathfrak{K},i}}^\lambda, \sqrt{1 - (1 - {\nu_{\mathfrak{K},i}}^2)^\lambda} \right), \\ & \left({\mu_{\mathfrak{K},f}}^\lambda, \sqrt{1 - (1 - {\nu_{\mathfrak{K},f}}^2)^\lambda} \right) \Big\rangle, \lambda > 0\end{aligned}$$

3.2. **Numerical Examples.** To illustrate these operations, examples are given, definition 3.1.1, 3.1.2, 3.1.3 and 3.1.4. Certainly, let's consider specific numerical values for the PNFS components as
$\mathfrak{K} = \langle (0.6, 0.8), (0.3, 0.5), (0.2, 0.7) \rangle$;
$\mathfrak{L} = \langle (0.4, 0.6), (0.2, 0.4), (0.1, 0.3) \rangle$. and $\lambda = 2$

Now we can apply these values to these four operations.

Example 3.2.1. *The PNFS addition operation is given as:*

$$\begin{aligned}(\mathfrak{K} \oplus \mathfrak{L}) = \Big\langle &\Big(\sqrt{(0.6^2 + 0.4^2) - (0.6^2 * 0.4^2)}, (0.8 * 0.6)\Big),\\ &\Big(\sqrt{(0.3^2 + 0.2^2) - (0.3^2 * 0.2^2)}, (0.5 * 0.4)\Big),\\ &\Big(\sqrt{(0.2^2 + 0.1^2) - (0.2^2 * 0.1^2)}, (0.7 * 0.3)\Big)\Big\rangle\\ = \langle &(0.5477, 0.48), (0.4359, 0.2), (0.1414, 0.21) \rangle\end{aligned}$$

From definition 2.4, It satisfying all the three conditions of PNFS.

Example 3.2.2. *The PNFS multiplication operation is given as:*

$$\begin{aligned}\mathfrak{K} \otimes \mathfrak{L} = \Big\langle &((0.6 * 0.4), \sqrt{(0.8^2 + 0.6^2) - (0.8^2 * 0.6^2)}),\\ &((0.3 * 0.2), \sqrt{(0.5^2 + 0.4^2) - (0.5^2 * 0.4^2)}),\\ &((0.2 * 0.2), \sqrt{(0.7^2 + 0.3^2) - (0.7^2 * 0.3^2)})\Big\rangle\\ = \langle &(0.24, 0.48), (0.06, 0.4), (0.02, 0.6633) \rangle\end{aligned}$$

From definition 2.4, It satisfying all the three conditions of PNFS.

Example 3.2.3. *The PNFS scalar multiplication operation is given as:*

$$\begin{aligned}\lambda\mathfrak{K} &= \Big\langle \Big(\sqrt{(1 - (1 - 0.6^2)^2}, 0.8^2\Big), \Big(\sqrt{(1 - (1 - 0.3^2)^2}, 0.5^2\Big), \Big(\sqrt{(1 - (1 - 0.2^2)^2}, 0.7^2\Big)\Big\rangle\\ &= \langle (0.2308, 0.64), (0.0591, 0.25), (0.0196, 0.49) \rangle.\end{aligned}$$

From definition 2.4, It satisfying all the three conditions of PNFS.

Example 3.2.4. *The PNFS scalar power operation is given as:*

$$\begin{aligned}\mathfrak{K}^\lambda &= \Big\langle (0.6^2, \sqrt{(1 - (1 - 0.8^2)^2}\Big), \Big(0.3^2, \sqrt{(1 - (1 - 0.5^2)^2}\Big), \Big(0.2^2, \sqrt{(1 - (1 - 0.7^2)^2}\Big)\rangle\\ &= \langle 0.36, 0.748, 0.09, 0.438, 0.04, 0.594 \rangle \textit{ when } \lambda = 2\end{aligned}$$

From definition 2.4, It satisfying all the three conditionsof PNFS.

4. Theorems on PNFS

The relationship involving algebraic product is demostrated in the subsequent theorem.

Theorem 4.1. $\mathfrak{K} \oplus \mathfrak{L} \geq \mathfrak{K} \otimes \mathfrak{L}$ *for PNFS* $\mathfrak{K}$, $\mathfrak{L}$.

Proof. Let
$$\begin{aligned}\mathfrak{K} \oplus \mathfrak{L} = \Big\langle &\Big(\sqrt{({\mu_{\mathfrak{K},\mathrm{t}}}^2 + {\mu_{\mathfrak{L},\mathrm{t}}}^2) - ({\mu_{\mathfrak{K},\mathrm{t}}}^2 * {\mu_{\mathfrak{L},\mathrm{t}}}^2)}, (\nu_{\mathfrak{K},\mathrm{t}} * \nu_{\mathfrak{L},\mathrm{t}})\Big),\\ &\Big(\sqrt{({\mu_{\mathfrak{K},\mathrm{i}}}^2 + {\mu_{\mathfrak{L},\mathrm{i}}}^2) - ({\mu_{\mathfrak{K},\mathrm{i}}}^2 * {\mu_{\mathfrak{L},\mathrm{i}}}^2)}, (\nu_{\mathfrak{K},\mathrm{i}} * \nu_{\mathfrak{L},\mathrm{i}})\Big),\\ &\Big(\sqrt{({\mu_{\mathfrak{K},\mathrm{f}}}^2 + {\mu_{\mathfrak{L},\mathrm{f}}}^2) - ({\mu_{\mathfrak{K},\mathrm{f}}}^2 * {\mu_{\mathfrak{L},\mathrm{f}}}^2)}, (\nu_{\mathfrak{K},\mathrm{f}} * \nu_{\mathfrak{L},\mathrm{f}})\Big)\Big\rangle\end{aligned}$$

$$\mathfrak{K}\otimes\mathfrak{L}=\Big\langle\Big((\mu_{\mathfrak{K},t}*\mu_{\mathfrak{L},t}),\sqrt{({\nu_{\mathfrak{K},t}}^2+{\nu_{\mathfrak{L},t}}^2)-({\nu_{\mathfrak{K},t}}^2*{\nu_{\mathfrak{L},t}}^2)}\Big),$$
$$\Big((\mu_{\mathfrak{K},i}*\mu_{\mathfrak{L},i}),\sqrt{({\nu_{\mathfrak{K},i}}^2+{\nu_{\mathfrak{L},i}}^2)-({\nu_{\mathfrak{K},i}}^2*{\nu_{\mathfrak{L},i}}^2)}\Big),$$
$$\Big((\mu_{\mathfrak{K},f}*\mu_{\mathfrak{L},f}),\sqrt{({\nu_{\mathfrak{K},f}}^2+{\nu_{\mathfrak{L},f}}^2)-({\nu_{\mathfrak{K},f}}^2*{\nu_{\mathfrak{L},f}}^2)}\Big)\Big\rangle$$

Suppose, $(\mu_{\mathfrak{K},t}*\mu_{\mathfrak{L},t})\leq\sqrt{({\mu_{\mathfrak{K},t}}^2+{\mu_{\mathfrak{L},t}}^2)-({\mu_{\mathfrak{K},t}}^2*{\mu_{\mathfrak{L},t}}^2)}$

$$(\mu_{\mathfrak{K},t}*\mu_{\mathfrak{L},t})-\sqrt{({\mu_{\mathfrak{K},t}}^2+{\mu_{\mathfrak{L},t}}^2)-({\mu_{\mathfrak{K},t}}^2*{\mu_{\mathfrak{L},t}}^2)}\leq 0$$
$${\mu_{\mathfrak{K},t}}^2\left(1-{\mu_{\mathfrak{L},t}}^2\right)+{\mu_{\mathfrak{L},t}}^2\left(1-{\mu_{\mathfrak{K},t}}^2\right)\geq 0$$

This holds as $0\leq{\mu_{\mathfrak{K},t}}^2\leq 1$ & $0\leq{\mu_{\mathfrak{L},t}}^2\leq 1$.

Suppose, $(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})\leq\sqrt{({\nu_{\mathfrak{K},t}}^2+{\nu_{\mathfrak{L},t}}^2)-({\nu_{\mathfrak{K},t}}^2*{\nu_{\mathfrak{L},t}}^2)}$

$$(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})-\sqrt{({\nu_{\mathfrak{K},t}}^2+{\nu_{\mathfrak{L},t}}^2)-({\nu_{\mathfrak{K},t}}^2*{\nu_{\mathfrak{L},t}}^2)}\leq 0$$
$${\nu_{\mathfrak{K},t}}^2\left(1-{\nu_{\mathfrak{L},t}}^2\right)+{\nu_{\mathfrak{L},t}}^2\left(1-{\nu_{\mathfrak{K},t}}^2\right)\geq 0$$

This holds as $0\leq{\nu_{\mathfrak{K},t}}^2\leq 1$ & $0\leq{\nu_{\mathfrak{L},t}}^2\leq 1$. Thus, $\mathfrak{K}\oplus\mathfrak{L}\geq\mathfrak{K}\otimes\mathfrak{L}$. □

Theorem 4.2. *Suppose $\mathfrak{K}$ is a PNFS, then*

(a) $\mathfrak{K}\oplus\mathfrak{K}\geq\mathfrak{K}$,
(b) $\mathfrak{K}\otimes\mathfrak{K}\leq\mathfrak{K}$.

Proof. (a) Let $\mathfrak{K}\oplus\mathfrak{K}=[(\mu_{\mathfrak{K},t}(\mathfrak{x}),\ \nu_{\mathfrak{K},t}(\mathfrak{x})),\ (\mu_{\mathfrak{K},i}(\mathfrak{x}),\ \nu_{\mathfrak{K},i}(\mathfrak{x})),\ (\mu_{\mathfrak{K},f}(\mathfrak{x}),\ \nu_{\mathfrak{K},f}(\mathfrak{x}))]$
$\oplus\ [(\mu_{\mathfrak{K},t}(\mathfrak{x}),\ \nu_{\mathfrak{K},t}(\mathfrak{x})),\ (\mu_{\mathfrak{K},i}(\mathfrak{x}),\ \nu_{\mathfrak{K},i}(\mathfrak{x})),\ (\mu_{\mathfrak{K},f}(\mathfrak{x}),\ \nu_{\mathfrak{K},f}(\mathfrak{x}))]$

$$=<\left(\sqrt{({\mu_{\mathfrak{K},t}}^2+{\mu_{\mathfrak{K},t}}^2)-({\mu_{\mathfrak{K},t}}^2*{\mu_{\mathfrak{K},t}}^2)},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{K},t})\right),$$
$$\left(\sqrt{({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{K},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{K},i}}^2)},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{K},i})\right),$$
$$\left(\sqrt{({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{K},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{K},f}}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{K},f})\right)>$$
$$=<\left(\sqrt{2{\mu_{\mathfrak{K},t}}^2-({\mu_{\mathfrak{K},t}}^4)},{\nu_{\mathfrak{K},t}}^2\right),\left(\sqrt{2{\mu_{\mathfrak{K},t}}^2-({\mu_{\mathfrak{K},t}}^4)},{\nu_{\mathfrak{K},t}}^2\right),\left(\sqrt{2{\mu_{\mathfrak{K},t}}^2-({\mu_{\mathfrak{K},t}}^4)},\ {\nu_{\mathfrak{K},t}}^2\right)>$$
$$\sqrt{2{\mu_{\mathfrak{K},t}}^2-({\mu_{\mathfrak{K},t}}^4)}\geq\ \mu_{\mathfrak{K},t}\quad forall\ t,i,f$$

And ${\nu_{\mathfrak{K},t}}^2\leq\nu_{\mathfrak{K},t}$ for all t,i,f. Therefore, $\mathfrak{K}\oplus\mathfrak{K}\geq\mathfrak{K}$. Similar to that, we may demonstrate that (b) $\mathfrak{K}\otimes\mathfrak{K}\leq\mathfrak{K}$. □

Theorem 4.3. *If $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{C}$ are PNFS set, then*

(a) $\mathfrak{L}\oplus\mathfrak{K}=\mathfrak{K}\oplus\mathfrak{L}$,
(b) $\mathfrak{L}\otimes\mathfrak{K}=\mathfrak{K}\otimes\mathfrak{L}$,
(c) $\mathfrak{K}\oplus(\mathfrak{L}\oplus\mathfrak{C})=(\mathfrak{K}\oplus\mathfrak{L})\oplus\mathfrak{C}$,
(d) $\mathfrak{K}\otimes(\mathfrak{L}\otimes\mathfrak{C})=(\mathfrak{K}\otimes\mathfrak{L})\otimes\mathfrak{C}$

Proof. (a) Let $\mathfrak{K}\oplus\mathfrak{L}=\Big\langle\Big(\sqrt{({\mu_{\mathfrak{K},t}}^2+{\mu_{\mathfrak{L},t}}^2)-({\mu_{\mathfrak{K},t}}^2*{\mu_{\mathfrak{L},t}}^2)},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})\Big)$

$$\Big(\sqrt{({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{L},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{L},i}}^2)},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})\Big),$$
$$\Big(\sqrt{({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{L},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{L},f}}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})\Big)\Big\rangle$$

$$= \Big\langle \Big(\sqrt{(\mu_{\mathfrak{L},t}{}^2 + \mu_{\mathfrak{K},t}{}^2) - (\mu_{\mathfrak{L},t}{}^2 * \mu_{\mathfrak{K},t}{}^2)}, (\nu_{\mathfrak{L},t} * \nu_{\mathfrak{K},t}) \Big)$$
$$\Big(\sqrt{(\mu_{\mathfrak{L},i}{}^2 + \mu_{\mathfrak{K},i}{}^2) - (\mu_{\mathfrak{L},i}{}^2 * \mu_{\mathfrak{K},i}{}^2)}, (\nu_{\mathfrak{L},i} * \nu_{\mathfrak{K},i}) \Big),$$
$$\Big(\sqrt{(\mu_{\mathfrak{L},f}{}^2 + \mu_{\mathfrak{K},f}{}^2) - (\mu_{\mathfrak{L},f}{}^2 * \mu_{\mathfrak{K},f}{}^2)}, (\nu_{\mathfrak{L},f} * \nu_{\mathfrak{K},f}) \Big) \Big\rangle = \mathfrak{L} \oplus \mathfrak{K}$$

(b) Let $\mathfrak{K} \otimes \mathfrak{L} = \Big\langle \Big((\mu_{\mathfrak{K},t} * \mu_{\mathfrak{L},t}), \sqrt{(\nu_{\mathfrak{K},t}{}^2 + \nu_{\mathfrak{L},t}{}^2) - (\nu_{\mathfrak{K},t}{}^2 * \nu_{\mathfrak{L},t}{}^2)} \Big),$

$$\Big((\mu_{\mathfrak{K},i} * \mu_{\mathfrak{L},i}), \sqrt{(\nu_{\mathfrak{K},i}{}^2 + \nu_{\mathfrak{L},i}{}^2) - (\nu_{\mathfrak{K},i}{}^2 * \nu_{\mathfrak{L},i}{}^2)} \Big)$$
$$\Big((\mu_{\mathfrak{K},f} * \mu_{\mathfrak{L},f}), \sqrt{(\nu_{\mathfrak{K},f}{}^2 + \nu_{\mathfrak{L},f}{}^2) - (\nu_{\mathfrak{K},f}{}^2 * \nu_{\mathfrak{L},f}{}^2)} \Big) \Big\rangle$$
$$= \Big\langle \Big((\mu_{\mathfrak{L},t} * \mu_{\mathfrak{K},t}), \sqrt{(\nu_{\mathfrak{L},t}{}^2 + \nu_{\mathfrak{K},t}{}^2) - (\nu_{\mathfrak{L},t}{}^2 * \nu_{\mathfrak{K},t}{}^2)} \Big),$$
$$\Big((\mu_{\mathfrak{L},i} * \mu_{\mathfrak{K},i}), \sqrt{(\nu_{\mathfrak{L},i}{}^2 + \nu_{\mathfrak{K},i}{}^2) - (\nu_{\mathfrak{L},i}{}^2 * \nu_{\mathfrak{K},i}{}^2)} \Big),$$
$$\Big((\mu_{\mathfrak{L},f} * \mu_{\mathfrak{K},f}), \sqrt{(\nu_{\mathfrak{L},f}{}^2 + \nu_{\mathfrak{K},f}{}^2) - (\nu_{\mathfrak{L},f}{}^2 * \nu_{\mathfrak{K},f}{}^2)} \Big) \Big\rangle = \mathfrak{L} \otimes \mathfrak{K}$$

(c) Let $(\mathfrak{K} \oplus \mathfrak{L}) \oplus \mathfrak{C} = \Big\langle \Big(\sqrt{(\mu_{\mathfrak{K},t}{}^2 + \mu_{\mathfrak{L},t}{}^2) - (\mu_{\mathfrak{K},t}{}^2 * \mu_{\mathfrak{L},t}{}^2)}, (\nu_{\mathfrak{K},t} * \nu_{\mathfrak{L},t}) \Big),$

$$\Big(\sqrt{(\mu_{\mathfrak{K},i}{}^2 + \mu_{\mathfrak{L},i}{}^2) - (\mu_{\mathfrak{K},i}{}^2 * \mu_{\mathfrak{L},i}{}^2)}, (\nu_{\mathfrak{K},i} * \nu_{\mathfrak{L},i}) \Big),$$
$$\Big(\sqrt{(\mu_{\mathfrak{K},f}{}^2 + \mu_{\mathfrak{L},f}{}^2) - (\mu_{\mathfrak{K},f}{}^2 * \mu_{\mathfrak{L},f}{}^2)}, (\nu_{\mathfrak{K},f} * \nu_{\mathfrak{L},f}) \Big) \Big\rangle$$
$$\oplus \langle (\mu_{\mathfrak{C},t},\ \nu_{\mathfrak{C},t}),\ (\mu_{\mathfrak{C},i},\ \nu_{\mathfrak{C},i}),\ \ (\mu_{\mathfrak{C},f},\ \nu_{\mathfrak{C},f}) \rangle$$
$$= \Big\langle \Big(\sqrt{(\mu_{\mathfrak{K},t}{}^2 + \mu_{\mathfrak{L},t}{}^2 + \mu_{\mathfrak{C},t}{}^2) - (\mu_{\mathfrak{K},t}{}^2 * \mu_{\mathfrak{L},t}{}^2 * \mu_{\mathfrak{C},t}{}^2)}, (\nu_{\mathfrak{K},t} * \nu_{\mathfrak{L},t} * \nu_{\mathfrak{C},t}) \Big)$$
$$\Big(\sqrt{(\mu_{\mathfrak{K},i}{}^2 + \mu_{\mathfrak{L},i}{}^2 + \mu_{\mathfrak{C},i}{}^2) - (\mu_{\mathfrak{K},i}{}^2 * \mu_{\mathfrak{L},i}{}^2 * \mu_{\mathfrak{C},i}{}^2)}, (\nu_{\mathfrak{K},i} * \nu_{\mathfrak{L},i} * \nu_{\mathfrak{C},i}) \Big)$$
$$\Big(\sqrt{(\mu_{\mathfrak{K},f}{}^2 + \mu_{\mathfrak{L},f}{}^2 + \mu_{\mathfrak{C},i}{}^2) - (\mu_{\mathfrak{K},f}{}^2 * \mu_{\mathfrak{L},f}{}^2 * \mu_{\mathfrak{C},f}{}^2)}, (\nu_{\mathfrak{K},f} * \nu_{\mathfrak{L},f} * \nu_{\mathfrak{C},f}) \Big) \Big\rangle$$

$$\mathfrak{K} \oplus (\mathfrak{L} \oplus \mathfrak{C}) = \Big\langle (\mu_{\mathfrak{K},t},\ \nu_{\mathfrak{K},t}),\ (\mu_{\mathfrak{K},i},\ \nu_{\mathfrak{K},i}),\ \ (\mu_{\mathfrak{K},f},\ \nu_{\mathfrak{K},f}) \Big\rangle$$
$$\oplus \Big\langle \Big(\sqrt{(\mu_{\mathfrak{L},t}{}^2 + \mu_{\mathfrak{C},t}{}^2) - (\mu_{\mathfrak{L},t}{}^2 * \mu_{\mathfrak{C},t}{}^2)}, (\nu_{\mathfrak{L},t} * \nu_{\mathfrak{C},t}) \Big)$$
$$\Big(\sqrt{(\mu_{\mathfrak{L},i}{}^2 + \mu_{\mathfrak{C},i}{}^2) - (\mu_{\mathfrak{L},i}{}^2 * \mu_{\mathfrak{C},i}{}^2)}, (\nu_{\mathfrak{L},i} * \nu_{\mathfrak{C},i}) \Big),$$
$$\Big(\sqrt{(\mu_{\mathfrak{L},f}{}^2 + \mu_{\mathfrak{C},f}{}^2) - (\mu_{\mathfrak{L},f}{}^2 * \mu_{\mathfrak{C},f}{}^2)}, (\nu_{\mathfrak{L},f} * \nu_{\mathfrak{C},f}) \Big) \Big\rangle$$
$$= \Big\langle \Big(\sqrt{(\mu_{\mathfrak{K},t}{}^2 + \mu_{\mathfrak{L},t}{}^2 + \mu_{\mathfrak{C},t}{}^2) - (\mu_{\mathfrak{K},t}{}^2 * \mu_{\mathfrak{L},t}{}^2 * \mu_{\mathfrak{C},t}{}^2)}, (\nu_{\mathfrak{K},t} * \nu_{\mathfrak{L},t} * \nu_{\mathfrak{C},t}) \Big),$$
$$\Big(\sqrt{(\mu_{\mathfrak{K},i}{}^2 + \mu_{\mathfrak{L},i}{}^2 + \mu_{\mathfrak{C},i}{}^2) - (\mu_{\mathfrak{K},i}{}^2 * \mu_{\mathfrak{L},i}{}^2 * \mu_{\mathfrak{C},i}{}^2)}, (\nu_{\mathfrak{K},i} * \nu_{\mathfrak{L},i} * \nu_{\mathfrak{C},i}) \Big),$$
$$\Big(\sqrt{(\mu_{\mathfrak{K},f}{}^2 + \mu_{\mathfrak{L},f}{}^2 + \mu_{\mathfrak{C},i}{}^2) - (\mu_{\mathfrak{K},f}{}^2 * \mu_{\mathfrak{L},f}{}^2 * \mu_{\mathfrak{C},f}{}^2)}, (\nu_{\mathfrak{K},f} * \nu_{\mathfrak{L},f} * \nu_{\mathfrak{C},f}) \Big) \Big\rangle$$

Thus $(\mathfrak{K} \oplus \mathfrak{L}) \oplus \mathfrak{C} = \mathfrak{K} \oplus (\mathfrak{L} \oplus \mathfrak{C})$
Similar to this, we can demonstrate d. $\mathfrak{K} \otimes (\mathfrak{L} \otimes \mathfrak{C}) = (\mathfrak{K} \otimes \mathfrak{L}) \otimes \mathfrak{C}$. □

Theorem 4.4. *For PNFS* $\mathfrak{K}$, $\mathfrak{L}$

(a) $\mathfrak{K} \oplus (\mathfrak{K} \otimes \mathfrak{L}) \geq \mathfrak{K}$,
(b) $\mathfrak{K} \otimes (\mathfrak{K} \oplus \mathfrak{L}) \leq \mathfrak{K}$.

Proof. (a) Let $\mathfrak{K} \oplus (\mathfrak{K} \otimes \mathfrak{L}) = \langle (\mu_{\mathfrak{K},t},\ \nu_{\mathfrak{K},t}),\ (\mu_{\mathfrak{K},i},\ \nu_{\mathfrak{K},i}),\ \ (\mu_{\mathfrak{K},f},\ \nu_{\mathfrak{K},f}) \rangle \oplus$

$$\Big\langle \Big((\mu_{\mathfrak{K},t} * \mu_{\mathfrak{L},t}), \sqrt{(\nu_{\mathfrak{K},t}{}^2 + \nu_{\mathfrak{L},t}{}^2) - (\nu_{\mathfrak{K},t}{}^2 * \nu_{\mathfrak{L},t}{}^2)} \Big),$$
$$\Big((\mu_{\mathfrak{K},i} * \mu_{\mathfrak{L},i}), \sqrt{(\nu_{\mathfrak{K},i}{}^2 + \nu_{\mathfrak{L},i}{}^2) - (\nu_{\mathfrak{K},i}{}^2 * \nu_{\mathfrak{L},i}{}^2)} \Big),$$
$$\Big((\mu_{\mathfrak{K},f} * \mu_{\mathfrak{L},f}), \sqrt{(\nu_{\mathfrak{K},f}{}^2 + \nu_{\mathfrak{L},f}{}^2) - (\nu_{\mathfrak{K},f}{}^2 * \nu_{\mathfrak{L},f}{}^2)} \Big) \Big\rangle$$

$$= \langle \Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + (\mu_{\mathfrak{K},\mathfrak{t}} * \mu_{\mathfrak{L},\mathfrak{t}})^2) - ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{L},\mathfrak{t}}}^2 * {\mu_{\mathfrak{K},\mathfrak{t}}}^2)}, (\nu_{\mathfrak{K},\mathfrak{t}} * \sqrt{({\nu_{\mathfrak{K},\mathfrak{t}}}^2 + {\nu_{\mathfrak{L},\mathfrak{t}}}^2) - ({\nu_{\mathfrak{K},\mathfrak{t}}}^2 * {\nu_{\mathfrak{L},\mathfrak{t}}}^2)}) \Big),$$

$$\sqrt{({\mu_{\mathfrak{K},\mathfrak{i}}}^2 + (\mu_{\mathfrak{K},\mathfrak{i}} * \mu_{\mathfrak{L},\mathfrak{i}})^2) - ({\mu_{\mathfrak{K},\mathfrak{i}}}^2 * {\mu_{\mathfrak{L},\mathfrak{i}}}^2 * {\mu_{\mathfrak{K},\mathfrak{i}}}^2)}, (\nu_{\mathfrak{K},\mathfrak{i}} * \sqrt{({\nu_{\mathfrak{K},\mathfrak{i}}}^2 + {\nu_{\mathfrak{L},\mathfrak{i}}}^2) - ({\nu_{\mathfrak{K},\mathfrak{t}}}^2 * {\nu_{\mathfrak{L},\mathfrak{i}}}^2)})$$

$$\sqrt{({\mu_{\mathfrak{K},\mathfrak{f}}}^2 + (\mu_{\mathfrak{K},\mathfrak{f}} * \mu_{\mathfrak{L},\mathfrak{f}})^2) - ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 * {\mu_{\mathfrak{L},\mathfrak{f}}}^2 * {\mu_{\mathfrak{K},\mathfrak{f}}}^2)}, (\nu_{\mathfrak{K},\mathfrak{f}} * \sqrt{({\nu_{\mathfrak{K},\mathfrak{f}}}^2 + {\nu_{\mathfrak{L},\mathfrak{f}}}^2) - ({\nu_{\mathfrak{K},\mathfrak{f}}}^2 * {\nu_{\mathfrak{L},\mathfrak{f}}}^2)}) \rangle$$

$$= \langle \Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + (\mu_{\mathfrak{K},\mathfrak{t}} * \mu_{\mathfrak{L},\mathfrak{t}})^2 \,(1 - {\mu_{\mathfrak{K},\mathfrak{t}}}^2)}, \ \Big(\nu_{\mathfrak{K},\mathfrak{t}} * \sqrt{1 - (1 - {\nu_{\mathfrak{K},\mathfrak{t}}}^2)(1 - {\nu_{\mathfrak{L},\mathfrak{t}}}^2)}\Big) \Big),$$

$$\Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{i}}}^2 + (\mu_{\mathfrak{K},\mathfrak{i}} * \mu_{\mathfrak{L},\mathfrak{i}})^2 \,(1 - {\mu_{\mathfrak{K},\mathfrak{i}}}^2)}, \ \Big(\nu_{\mathfrak{K},\mathfrak{i}} * \sqrt{1 - (1 - {\nu_{\mathfrak{K},\mathfrak{i}}}^2)(1 - {\nu_{\mathfrak{L},\mathfrak{i}}}^2)}\Big) \Big),$$

$$\Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{f}}}^2 + (\mu_{\mathfrak{K},\mathfrak{f}} * \mu_{\mathfrak{L},\mathfrak{f}})^2 \,(1 - {\mu_{\mathfrak{K},\mathfrak{f}}}^2)}, \Big(\nu_{\mathfrak{K},\mathfrak{f}} * \sqrt{1 - (1 - {\nu_{\mathfrak{K},\mathfrak{f}}}^2)(1 - {\nu_{\mathfrak{L},\mathfrak{f}}}^2)}\Big) \Big) \rangle \geq \mathfrak{K}$$

Thus $\mathfrak{K} \oplus \ (\mathfrak{K} \ \otimes \ \mathfrak{L}) \ \geq \mathfrak{K}$.

Similar to this we can demonstrate (b). $\mathfrak{K} \otimes \ (\mathfrak{K} \ \oplus \ \mathfrak{L}) \ \leq \mathfrak{K}$. □

Theorem 4.5. *If* $\mathfrak{K}$, $\mathfrak{L}$ *are PNFS, then*

(a) $\mathfrak{K} \vee \ \mathfrak{L} \ = \ \mathfrak{L} \ \vee \mathfrak{K}$,
(b) $\mathfrak{K} \wedge \ \mathfrak{L} \ = \ \mathfrak{L} \ \wedge \mathfrak{K}$

Theorem 4.6. *For PNFS* $\mathfrak{K}$, $\mathfrak{L}$ *and* $\mathfrak{C}$,

(a) $(\mathfrak{K} \ \oplus \ \mathfrak{L}) \ \vee \ (\mathfrak{K} \ \oplus \ \mathfrak{C}) = \mathfrak{K} \oplus \ (\mathfrak{L} \ \vee \ \mathfrak{C})$
(b) $(\mathfrak{K} \ \otimes \ \mathfrak{L}) \ \vee \ (\mathfrak{K} \ \otimes \ \mathfrak{C}) = \mathfrak{K} \otimes \ (\mathfrak{L} \ \vee \ \mathfrak{C})$
(c) $(\mathfrak{K} \ \oplus \ \mathfrak{L}) \ \wedge \ (\mathfrak{K} \ \oplus \ \mathfrak{C}) = \mathfrak{K} \oplus \ (\mathfrak{L} \ \wedge \ \mathfrak{C})$
(d) $(\mathfrak{K} \ \otimes \ \mathfrak{L}) \ \wedge \ (\mathfrak{K} \ \otimes \ \mathfrak{C}) = \mathfrak{K} \otimes \ (\mathfrak{L} \ \wedge \ \mathfrak{C})$

Proof. As (b)-(d) is obvious, we shall prove (a) alone.

(a) Let $\mathfrak{K} \oplus \ (\mathfrak{L} \ \vee \ \mathfrak{C}) =< \ (\mu_{\mathfrak{K},\mathfrak{t}}, \ \nu_{\mathfrak{K},\mathfrak{t}}), \ \ (\mu_{\mathfrak{K},\mathfrak{i}}, \ \nu_{\mathfrak{K},\mathfrak{i}}), \ \ \ (\mu_{\mathfrak{K},\mathfrak{f}}, \ \nu_{\mathfrak{K},\mathfrak{f}}) >$

$$\oplus \langle (\max(\mu_{\mathfrak{L},\mathfrak{t}}, \ \mu_{\mathfrak{C},\mathfrak{t}}), \min(\nu_{\mathfrak{L},\mathfrak{t}}, \nu_{\mathfrak{C},\mathfrak{t}})), (\min \ (\mu_{\mathfrak{L},\mathfrak{i}}, \ \mu_{\mathfrak{C},\mathfrak{i}}), \max(\nu_{\mathfrak{L},\mathfrak{i}}, \nu_{\mathfrak{C},\mathfrak{i}})),$$
$$(\min \ (\mu_{\mathfrak{L},\mathfrak{f}}, \ \mu_{\mathfrak{C},\mathfrak{f}}), \max(\nu_{\mathfrak{L},\mathfrak{f}}, \nu_{\mathfrak{C},\mathfrak{f}}))\rangle$$

$$= \langle \Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + \max \ (\mu_{\mathfrak{L},\mathfrak{t}}, \ \mu_{\mathfrak{C},\mathfrak{t}})^2) - ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * \max \ (\mu_{\mathfrak{L},\mathfrak{t}}, \ \mu_{\mathfrak{C},\mathfrak{t}})^2)}, (\nu_{\mathfrak{K},\mathfrak{t}} * \min(\nu_{\mathfrak{L},\mathfrak{t}}, \nu_{\mathfrak{C},\mathfrak{t}})) \Big),$$

$$\Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{i}}}^2 + \min \ (\mu_{\mathfrak{L},\mathfrak{i}}, \ \mu_{\mathfrak{C},\mathfrak{i}})^2) - ({\mu_{\mathfrak{K},\mathfrak{i}}}^2 * \min \ (\mu_{\mathfrak{L},\mathfrak{i}}, \ \mu_{\mathfrak{C},\mathfrak{i}})^2)}, (\nu_{\mathfrak{K},\mathfrak{i}}, \max(\nu_{\mathfrak{L},\mathfrak{i}}, \nu_{\mathfrak{C},\mathfrak{i}})) \Big),$$

$$\Big(\sqrt{({\mu_{\mathfrak{K},\mathfrak{f}}}^2 + \min \ (\mu_{\mathfrak{L},\mathfrak{f}}, \ \mu_{\mathfrak{C},\mathfrak{f}})^2) - ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 * \min \ (\mu_{\mathfrak{L},\mathfrak{f}}, \ \mu_{\mathfrak{C},\mathfrak{f}})^2)}, (\nu_{\mathfrak{K},\mathfrak{f}} * \max(\nu_{\mathfrak{L},\mathfrak{f}}, \nu_{\mathfrak{C},\mathfrak{f}})) \Big) \rangle$$

$$= \langle (\sqrt{\max(\ ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + {\mu_{\mathfrak{L},\mathfrak{t}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + {\mu_{\mathfrak{C},\mathfrak{t}}}^2)) - \ \max({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{L},\mathfrak{t}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{C},\mathfrak{t}}}^2)},$$
$$\min((\nu_{\mathfrak{K},\mathfrak{t}} * \nu_{\mathfrak{L},\mathfrak{t}}), (\nu_{\mathfrak{K},\mathfrak{t}} * \nu_{\mathfrak{C},\mathfrak{t}}))),$$

$$(\sqrt{\min(\ ({\mu_{\mathfrak{K},\mathfrak{i}}}^2 + {\mu_{\mathfrak{L},\mathfrak{i}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{i}}}^2 + {\mu_{\mathfrak{C},\mathfrak{i}}}^2)) - \min({\mu_{\mathfrak{K},\mathfrak{i}}}^2 * {\mu_{\mathfrak{L},\mathfrak{i}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{C},\mathfrak{i}}}^2)},$$
$$\max((\nu_{\mathfrak{K},\mathfrak{i}} * \nu_{\mathfrak{L},\mathfrak{i}}), (\nu_{\mathfrak{K},\mathfrak{i}} * \nu_{\mathfrak{C},\mathfrak{i}}))),$$

$$(\sqrt{min(\ ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 + {\mu_{\mathfrak{L},\mathfrak{f}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 + {\mu_{\mathfrak{C},\mathfrak{f}}}^2)) - \min \ ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 * {\mu_{\mathfrak{L},\mathfrak{f}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 * {\mu_{\mathfrak{C},\mathfrak{f}}}^2)},$$
$$\max((\nu_{\mathfrak{K},\mathfrak{f}} * \nu_{\mathfrak{L},\mathfrak{f}}), (\nu_{\mathfrak{K},\mathfrak{t}} * \nu_{\mathfrak{C},\mathfrak{f}})))\rangle$$

$$= \langle (\sqrt{\max({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + {\mu_{\mathfrak{L},\mathfrak{t}}}^2) - ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{L},\mathfrak{t}}}^2), \ ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + {\mu_{\mathfrak{C},\mathfrak{t}}}^2) - ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{C},\mathfrak{t}}}^2)},$$
$$\min((\nu_{\mathfrak{K},\mathfrak{t}} * \nu_{\mathfrak{L},\mathfrak{t}}), (\nu_{\mathfrak{K},\mathfrak{t}} * \nu_{\mathfrak{C},\mathfrak{t}}))),$$

$$(\sqrt{\min({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{L},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{L},i}}^2),\ ({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{C},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{C},i}}^2)},$$
$$\max\left((\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i}),(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{C},i})\right)),$$
$$(\sqrt{\min({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{L},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{L},f}}^2),\ ({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{C},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{C},f}}^2)},$$
$$\max\left((\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f}),(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{C},f})\right)))\rangle$$
$= (\mathfrak{K}\ \oplus\ \mathfrak{L})\ \vee\ (\mathfrak{K}\ \oplus\ \mathfrak{C})$ □

Theorem 4.7. *For PNFS $\mathfrak{K}$ and $\mathfrak{L}$,*

(a) $(\mathfrak{K}\ \wedge\ \mathfrak{L})\ \oplus\ (\mathfrak{K}\ \vee\ \mathfrak{L})\ = \mathfrak{K}\ \oplus\ \mathfrak{L}$,
(b) $(\mathfrak{K}\ \wedge\ \mathfrak{L})\ \otimes\ (\mathfrak{K}\ \vee\ \mathfrak{L})\ = \mathfrak{K}\ \otimes\ \mathfrak{L}$,
(c) $(\mathfrak{K}\ \oplus\ \mathfrak{L})\ \wedge\ (\mathfrak{K}\ \otimes\ \mathfrak{L})\ = \mathfrak{K}\ \otimes\ \mathfrak{L}$
(d) $(\mathfrak{K}\ \oplus\ \mathfrak{L})\ \vee\ (\mathfrak{K}\ \otimes\ \mathfrak{L}) = \mathfrak{K}\ \oplus\ \mathfrak{L}$.

Proof. We will demonstrate how to analogously prove a), b), and d) in the sections that follow.

(a) $(\mathfrak{K}\ \wedge\ \mathfrak{L})\ \oplus\ (\mathfrak{K}\ \vee\ \mathfrak{L}) = <(\min(\mu_{\mathfrak{K},t},\ \mu_{\mathfrak{L},t}), \max(\nu_{\mathfrak{K},t},\nu_{\mathfrak{L},t}))$,
$$(\max(\mu_{\mathfrak{K},i},\ \mu_{\mathfrak{L},i}), \min(\nu_{\mathfrak{K},i},\nu_{\mathfrak{L},i})),\ (\max\ (\mu_{\mathfrak{K},f},\ \mu_{\mathfrak{L},f}), \min(\nu_{\mathfrak{K},f},\nu_{\mathfrak{L},f}))>\oplus$$
$$<(\max(\mu_{\mathfrak{K},t},\ \mu_{\mathfrak{L},t}), \min(\nu_{\mathfrak{K},t},\nu_{\mathfrak{L},t})),\ (\min\ (\mu_{\mathfrak{K},i},\ \mu_{\mathfrak{L},i}), \max(\nu_{\mathfrak{K},i},\nu_{\mathfrak{L},i})),$$
$$(\min\ (\mu_{\mathfrak{K},f},\ \mu_{\mathfrak{L},f}), \max(\nu_{\mathfrak{K},f},\nu_{\mathfrak{L},f}))>$$
$$=<\left(\sqrt{({\mu_{\mathfrak{K},t}}^2+{\mu_{\mathfrak{L},t}}^2)-({\mu_{\mathfrak{K},t}}^2*{\mu_{\mathfrak{L},t}}^2)},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})\right),$$
$$\left(\sqrt{({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{L},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{L},i}}^2)},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})\right),$$
$$\left(\sqrt{({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{L},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{L},f}}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})\right)>$$
$=\mathfrak{K}\ \oplus\ \mathfrak{L}$

The operator complement in the following theorems abides by the De Morgan's rules for the operation $\oplus$, $\otimes$, $\vee$, $\wedge$. □

Theorem 4.8. *For PNFS $\mathfrak{K}$ and $\mathfrak{L}$,*

(a) $(\mathfrak{K}\ \oplus\ \mathfrak{L})^{\mathfrak{C}} \le (\mathfrak{K})^{\mathfrak{C}} \otimes (\mathfrak{L})^{\mathfrak{C}}$,
(b) $(\mathfrak{K}\ \otimes\ \mathfrak{L})^{\mathfrak{C}} \ge (\mathfrak{K})^{\mathfrak{C}} \oplus (\mathfrak{L})^{\mathfrak{C}}$,
(c) $(\mathfrak{K}\ \oplus\ \mathfrak{L})^{\mathfrak{C}} = (\mathfrak{K})^{\mathfrak{C}} \oplus (\mathfrak{L})^{\mathfrak{C}}$,
(d) $(\mathfrak{K}\ \otimes\ \mathfrak{L})^{\mathfrak{C}} = (\mathfrak{K})^{\mathfrak{C}} \oplus (\mathfrak{L})^{\mathfrak{C}}$,

Proof. We will demonstrate (c) as (a) (b) and (d) are simple.

(c) $(\mathfrak{K}\ \oplus\ \mathfrak{L})^{\mathfrak{C}} = \langle((\sqrt{({\mu_{\mathfrak{K},t}}^2+{\mu_{\mathfrak{L},t}}^2)-({\mu_{\mathfrak{K},t}}^2*{\mu_{\mathfrak{L},t}}^2)},\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})$
$$\left(\sqrt{({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{L},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{L},i}}^2)},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})\right),$$
$$\left(\sqrt{({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{L},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{L},f}}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})\right))^{\mathfrak{C}}\rangle$$
$$=\langle\left(\sqrt{({\mu_{\mathfrak{K},f}}^2+{\mu_{\mathfrak{L},f}}^2)-({\mu_{\mathfrak{K},f}}^2*{\mu_{\mathfrak{L},f}}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})\right),$$
$$\left((\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i}),\sqrt{({\mu_{\mathfrak{K},i}}^2+{\mu_{\mathfrak{L},i}}^2)-({\mu_{\mathfrak{K},i}}^2*{\mu_{\mathfrak{L},i}}^2)}\right),$$

$$\left(\sqrt{(\mu_{\mathfrak{K},t}{}^2+\mu_{\mathfrak{L},t}{}^2)-(\mu_{\mathfrak{K},t}{}^2*\mu_{\mathfrak{L},t}{}^2)},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})\right)\rangle$$

$\mathfrak{K}^{\mathfrak{C}}\oplus\mathfrak{L}^{\mathfrak{C}} =< (\mu_{\mathfrak{K},f},\ \nu_{\mathfrak{K},f}),(\nu_{\mathfrak{K},i},\ \mu_{\mathfrak{K},i}),(\mu_{\mathfrak{K},t},\nu_{\mathfrak{K},t}) > \oplus < (\mu_{\mathfrak{L},f},\nu_{\mathfrak{L},f}),(\nu_{\mathfrak{L},i},\mu_{\mathfrak{L},i}),(\mu_{\mathfrak{L},t},\nu_{\mathfrak{L},t}) >$

$$=\langle\left(\sqrt{(\mu_{\mathfrak{K},f}{}^2+\mu_{\mathfrak{L},f}{}^2)-(\mu_{\mathfrak{K},f}{}^2*\mu_{\mathfrak{L},f}{}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})\right),$$

$$(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})\left(\sqrt{(\mu_{\mathfrak{K},i}{}^2+\mu_{\mathfrak{L},i}{}^2)-(\mu_{\mathfrak{K},i}{}^2*\mu_{\mathfrak{L},i}{}^2)}\right),$$

$$\left(\sqrt{(\mu_{\mathfrak{K},t}{}^2+\mu_{\mathfrak{L},t}{}^2)-(\mu_{\mathfrak{K},t}{}^2*\mu_{\mathfrak{L},t}{}^2)},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})\right)\rangle$$

Thus $(\mathfrak{K}\ \oplus\ \mathfrak{K})^{\mathfrak{C}}=(\mathfrak{K})^{\mathfrak{C}}\oplus(\mathfrak{L})^{\mathfrak{C}}$ □

Theorem 4.9. *For any PNFS $\mathfrak{K}$ and $\mathfrak{L}$, $(\mathfrak{K}^{\mathfrak{C}})^{\mathfrak{C}}=\mathfrak{K}$.*

Proof. $(\mathfrak{K}^{\mathfrak{C}})^{\mathfrak{C}} = (\ (\mu_{\mathfrak{K},f},\ \nu_{\mathfrak{K},f}),\ (\ \nu_{\mathfrak{K},i},\ \mu_{\mathfrak{K}i}),(\mu_{\mathfrak{K},t},\ \nu_{\mathfrak{K},t})\ )^{\mathfrak{C}} = \langle(\mu_{\mathfrak{K},t},\ \nu_{\mathfrak{K},t}),(\mu_{\mathfrak{K}i},\ \nu_{\mathfrak{K},i}),(\mu_{\mathfrak{K}f},\nu_{\mathfrak{K},f})\rangle=\mathfrak{K}$
We will now demonstrate the algebraic characteristics of PNFS sets under scalar multiplication and exponential expansion. □

Theorem 4.10. *For PNFS $\mathfrak{K}$, $\mathfrak{L}$ with $\mathfrak{z}$, $\mathfrak{z}_1$, $\mathfrak{z}_2>0$,*

(a) $\mathfrak{z}(\mathfrak{K}\ \oplus\ \mathfrak{L}) = \mathfrak{z}\mathfrak{K}\ \oplus\ \mathfrak{z}\mathfrak{L}$,
(b) $\mathfrak{z}_1\mathfrak{K}\oplus\ \mathfrak{z}_2\mathfrak{K} = (\mathfrak{z}_1+\mathfrak{z}_2)\mathfrak{K}$
(c) $(\mathfrak{K}\ \otimes\ \mathfrak{L})^{\mathfrak{z}} = (\mathfrak{K})^{\mathfrak{z}}\otimes(\mathfrak{L})^{\mathfrak{z}}$,
(d) $\mathfrak{K}^{(\mathfrak{z}_1+\mathfrak{z}_2)} = (\mathfrak{K})^{\mathfrak{z}_1}\otimes(\mathfrak{K})^{\mathfrak{z}_2}$.

Proof. In accordance with the definition, for the two sets $\mathfrak{K}$ and $\mathfrak{L}$, $\mathfrak{z},\mathfrak{z}_1,\ \mathfrak{z}_2>0$, we can derive,

(a) $\mathfrak{z}\ (\mathfrak{K}\ \oplus\ \mathfrak{L}) = (\mathfrak{z}^*\langle\left(\sqrt{(\mu_{\mathfrak{K},t}{}^2+\mu_{\mathfrak{L},t}{}^2)-(\mu_{\mathfrak{K},t}{}^2*\mu_{\mathfrak{L},t}{}^2)},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})\right),$

$$\left(\sqrt{(\mu_{\mathfrak{K},i}{}^2+\mu_{\mathfrak{L},i}{}^2)-(\mu_{\mathfrak{K},i}{}^2*\mu_{\mathfrak{L},i}{}^2)},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})\right),$$

$$\left(\sqrt{(\mu_{\mathfrak{K},f}{}^2+\mu_{\mathfrak{L},f}{}^2)-(\mu_{\mathfrak{K},f}{}^2*\mu_{\mathfrak{L},f}{}^2)},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})\right)\rangle)$$

$$=\langle\left(\sqrt{1-(1-\mu_{\mathfrak{K},t}{}^2+\mu_{\mathfrak{L},t}{}^2-\mu_{\mathfrak{K},t}{}^2*\mu_{\mathfrak{L},t}{}^2)^{\mathfrak{z}}},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})^{\mathfrak{z}}\right),$$

$$\left(\sqrt{1-(1-\mu_{\mathfrak{K},i}{}^2+\mu_{\mathfrak{L},i}{}^2-\mu_{\mathfrak{K},i}{}^2*\mu_{\mathfrak{L},i}{}^2)^{\mathfrak{z}}},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})^{\mathfrak{z}}\right),$$

$$\left(\sqrt{1-(1-\mu_{\mathfrak{K},f}{}^2+\mu_{\mathfrak{L},f}{}^2-\mu_{\mathfrak{K},f}{}^2*\mu_{\mathfrak{L},f}{}^2)^{\mathfrak{z}}},(\nu_{\mathfrak{K},f}*\nu_{\mathfrak{L},f})^{\mathfrak{z}}\right)\rangle$$

$\mathfrak{z}\mathfrak{K}\ \oplus\ \mathfrak{z}\mathfrak{L} =$

$$\left\langle\left(\sqrt{1-(1-\mu_{\mathfrak{K},t}{}^2)^{\mathfrak{z}}},\nu_{\mathfrak{K},t}{}^{\mathfrak{z}}\right),\ \left(\sqrt{1-(1-\mu_{\mathfrak{K},i}{}^2)^{\mathfrak{z}}},\nu_{\mathfrak{K},i}{}^{\mathfrak{z}}\right),\left(\sqrt{1-(1-\mu_{\mathfrak{K},f}{}^2)^{\mathfrak{z}}},\nu_{\mathfrak{K},f}{}^{\mathfrak{z}}\ \right)\right\rangle\oplus$$

$$\left\langle\left(\sqrt{1-(1-\mu_{\mathfrak{L},t}{}^2)^{\mathfrak{z}}},\nu_{\mathfrak{L},t}{}^{\mathfrak{z}}\right),\ \left(\sqrt{1-(1-\mu_{\mathfrak{L},i}{}^2)^{\mathfrak{z}}},\nu_{\mathfrak{L},i}{}^{\mathfrak{z}}\right),\ \left(\sqrt{1-(1-\mu_{\mathfrak{L},f}{}^2)^{\mathfrak{z}}},\nu_{\mathfrak{L},f}{}^{\mathfrak{z}}\ \right)\right\rangle$$

$$=\langle\left(\sqrt{1-(1-\mu_{\mathfrak{K},t}{}^2+\mu_{\mathfrak{L},t}{}^2-\mu_{\mathfrak{K},t}{}^2*\mu_{\mathfrak{L},t}{}^2)^{\mathfrak{z}}},(\nu_{\mathfrak{K},t}*\nu_{\mathfrak{L},t})^{\mathfrak{z}}\right),$$

$$\left(\sqrt{1-(1-\mu_{\mathfrak{K},i}{}^2+\mu_{\mathfrak{L},i}{}^2-\mu_{\mathfrak{K},i}{}^2*\mu_{\mathfrak{L},i}{}^2)^{\mathfrak{z}}},(\nu_{\mathfrak{K},i}*\nu_{\mathfrak{L},i})^{\mathfrak{z}}\right),$$

$$\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2+\mu_{\mathfrak{L},\mathfrak{f}}{}^2-\mu_{\mathfrak{K},\mathfrak{f}}{}^2*\mu_{\mathfrak{L},\mathfrak{f}}{}^2)^{\mathfrak{z}}},(\nu_{\mathfrak{K},\mathfrak{f}}*\nu_{\mathfrak{L},\mathfrak{f}})^{\mathfrak{z}}\right)\rangle = \mathfrak{z}(\mathfrak{K}\oplus\mathfrak{L})$$

(b) $\mathfrak{z}_1\mathfrak{K}\oplus\mathfrak{z}_2\mathfrak{K} = \langle\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_1}},\nu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}_1}\right),\ \left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_1}},\nu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}_1}\right),$

$$\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_1}},\nu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}_1}\right)\rangle\oplus$$

$$\left\langle\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_2}},\nu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}_2}\right),\ \left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_2}},\nu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}_2}\right),\ \left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_2}},\nu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}_2}\right)\right\rangle$$

$$= \langle((\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_1}+1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_2}-\left(1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_1}\right)\left(1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_2}\right)},$$
$$(\nu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}_1}*\nu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}_2})),$$

$$(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_1}+1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_2}-\left(1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_1}\right)\left(1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_2}\right)},(\nu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}_1}*\nu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}_2})),$$

$$(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_1}+1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_2}-\left(1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_1}\right)\left(1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_2}\right)},(\nu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}_1}*\nu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}_2}))\rangle$$

$$=\langle\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_1+\mathfrak{z}_2}},(\nu_{\mathfrak{K},\mathfrak{t}})^{\mathfrak{z}_1+\mathfrak{z}_2}\right),\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_1+\mathfrak{z}_2}},(\nu_{\mathfrak{K},\mathfrak{i}})^{\mathfrak{z}_1+\mathfrak{z}_2}\right),$$
$$\left(\sqrt{1-(1-\mu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_1+\mathfrak{z}_2}},(\nu_{\mathfrak{K},\mathfrak{f}})^{\mathfrak{z}_1+\mathfrak{z}_2}\right)\rangle = (\mathfrak{z}_1+\mathfrak{z}_2)\mathfrak{K}$$

(c) $(\mathfrak{K}\otimes\mathfrak{L})^{\mathfrak{z}} = (\mathfrak{K})^{\mathfrak{z}}\otimes(\mathfrak{L})^{\mathfrak{z}}$,

$$(\mathfrak{K}\otimes\mathfrak{L})^{\mathfrak{z}} = \langle\left((\mu_{\mathfrak{K},\mathfrak{t}}*\mu_{\mathfrak{L},\mathfrak{t}})^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{t}}{}^2+\nu_{\mathfrak{L},\mathfrak{t}}{}^2-(\nu_{\mathfrak{K},\mathfrak{t}}{}^2*\nu_{\mathfrak{L},\mathfrak{t}}{}^2))^{\mathfrak{z}}}\right),$$

$$\left((\mu_{\mathfrak{K},\mathfrak{i}}*\mu_{\mathfrak{L},\mathfrak{i}})^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{i}}{}^2+\nu_{\mathfrak{L},\mathfrak{i}}{}^2-(\nu_{\mathfrak{K},\mathfrak{i}}{}^2*\nu_{\mathfrak{L},\mathfrak{i}}{}^2))^{\mathfrak{z}}}\right),$$

$$\left((\mu_{\mathfrak{K},\mathfrak{f}}*\mu_{\mathfrak{L},\mathfrak{f}})^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{f}}{}^2+\nu_{\mathfrak{L},\mathfrak{f}}{}^2-(\nu_{\mathfrak{K},\mathfrak{f}}{}^2*\nu_{\mathfrak{L},\mathfrak{f}}{}^2))^{\mathfrak{z}}}\right)$$

$$=\left((\mu_{\mathfrak{K},\mathfrak{t}}*\mu_{\mathfrak{L},\mathfrak{t}})^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}}(1-\nu_{\mathfrak{L},\mathfrak{t}}{}^2)^{\mathfrak{z}}}\right),$$

$$\left((\mu_{\mathfrak{K},\mathfrak{i}}*\mu_{\mathfrak{L},\mathfrak{i}})^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}}(1-\nu_{\mathfrak{L},\mathfrak{i}}{}^2)^{\mathfrak{z}}},\right)$$

$$\left((\mu_{\mathfrak{K},\mathfrak{f}}*\mu_{\mathfrak{L},\mathfrak{f}})^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}}(1-\nu_{\mathfrak{L},\mathfrak{f}}{}^2)^{\mathfrak{z}}}\right)\rangle$$

$(\mathfrak{K})^{\mathfrak{z}}\otimes(\mathfrak{L})^{\mathfrak{z}}$

$$=\left\langle\left(\mu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}}}\right),\ \left(\mu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}}}\right),\left(\mu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}}}\right)\right\rangle$$

$$\otimes\left\langle\left(\mu_{\mathfrak{L},\mathfrak{t}}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},\mathfrak{t}}{}^2)^{\mathfrak{z}}}\right),\ \left(\mu_{\mathfrak{L},\mathfrak{i}}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},\mathfrak{i}}{}^2)^{\mathfrak{z}}}\right),\ \left(\mu_{\mathfrak{L},\mathfrak{f}}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},\mathfrak{f}}{}^2)^{\mathfrak{z}}}\right)\right\rangle$$

$$\left\langle\left((\mu_{\mathfrak{K},\mathfrak{i}}*\mu_{\mathfrak{L},\mathfrak{i}}\right)^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}}(1-\nu_{\mathfrak{L},\mathfrak{i}}{}^2)^{\mathfrak{z}})},\left((\mu_{\mathfrak{K},\mathfrak{f}}*\mu_{\mathfrak{L},\mathfrak{f}}\right)^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}}(1-\nu_{\mathfrak{L},\mathfrak{f}}{}^2)^{\mathfrak{z}})}\right\rangle$$

Hence $(\mathfrak{K}\otimes\mathfrak{L})\}^{\mathfrak{z}} = (\mathfrak{K})^{\mathfrak{z}}\otimes(\mathfrak{L})^{\mathfrak{z}}$

(d) $\mathfrak{K}^{\mathfrak{z}_1}\otimes\mathfrak{K}^{\mathfrak{z}_2} =$

$$\left\langle\left(\mu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}_1},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_1}}\right),\ \left(\mu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}_1},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_1}}\right),\ \left(\mu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}_1},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_1}}\right)\right\rangle$$

$$\otimes\langle\left(\mu_{\mathfrak{K},\mathfrak{t}}{}^{\mathfrak{z}_2},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{t}}{}^2)^{\mathfrak{z}_2}}\right),\ \left(\mu_{\mathfrak{K},\mathfrak{i}}{}^{\mathfrak{z}_2},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{i}}{}^2)^{\mathfrak{z}_2}}\right),\left(\mu_{\mathfrak{K},\mathfrak{f}}{}^{\mathfrak{z}_2},\sqrt{1-(1-\nu_{\mathfrak{K},\mathfrak{f}}{}^2)^{\mathfrak{z}_2}}\right)\rangle$$

$$= \langle \left((\mu_{\mathfrak{K},t})^{(\mathfrak{z}_1+\mathfrak{z}_2)}, \sqrt{1-(1-\nu_{\mathfrak{K},t}{}^2)^{(\mathfrak{z}_1+\mathfrak{z}_2)}} \right), \left((\mu_{\mathfrak{K},i})^{(\mathfrak{z}_1+\mathfrak{z}_2)}, \sqrt{1-(1-\nu_{\mathfrak{K},i}{}^2)^{(\mathfrak{z}_1+\mathfrak{z}_2)}} \right),$$
$$\left((\mu_{\mathfrak{K},f})^{(\mathfrak{z}_1+\mathfrak{z}_2)}, \sqrt{1-(1-\nu_{\mathfrak{K},f}{}^2)^{(\mathfrak{z}_1+\mathfrak{z}_2)}} \right) \rangle = \mathfrak{K}^{(\mathfrak{z}_1+\mathfrak{z}_2)}$$

Therefore, $\mathfrak{K}^{(\mathfrak{z}_1+\mathfrak{z}_2)} = (\mathfrak{K})^{\mathfrak{z}_1} \otimes (\mathfrak{K})^{\mathfrak{z}_2}$ □

Theorem 4.11. *Suppose $\mathfrak{K}$ and $\mathfrak{L}$ are PNFS such that $\mathfrak{z} > 0$, then*
(a) $\mathfrak{z}\mathfrak{K} \leq \mathfrak{z}\mathfrak{L}$
(b) $\mathfrak{K}^{\mathfrak{z}} \leq \mathfrak{L}^{\mathfrak{z}}$

Proof. (a) Let $\mathfrak{K} \leq \mathfrak{L}$, $\mu_{\mathfrak{K},t} \leq \mu_{\mathfrak{L},t}$ & $\nu_{\mathfrak{K},t} \geq \nu_{\mathfrak{L},t}$ $\forall t, i, f$ $\sqrt{1-(1-\mu_{\mathfrak{K},t}{}^2)^{\mathfrak{z}}} \leq \sqrt{1-(1-\mu_{\mathfrak{L},t}{}^2)^{\mathfrak{z}}}$ and $\nu_{\mathfrak{K},t}{}^{\mathfrak{z}} \geq \nu_{\mathfrak{L},t}{}^{\mathfrak{z}} \forall t, i, f$. Thus, $\mathfrak{z}\mathfrak{K} \leq \mathfrak{z}\mathfrak{L}$.
(b) Also, $\mu_{\mathfrak{K},t}{}^{\mathfrak{z}} \geq \mu_{\mathfrak{K},t}{}^{\mathfrak{z}}$, $\sqrt{1-(1-\nu_{\mathfrak{K},t}{}^2)^{\mathfrak{z}}} \leq \sqrt{1-(1-\nu_{\mathfrak{K},t}{}^2)^{\mathfrak{z}}}$ $\forall t, i, f$. So that, $\mathfrak{K}^{\mathfrak{z}} \leq \mathfrak{L}^{\mathfrak{z}}$. □

Theorem 4.12. *Suppose $\mathfrak{K}$ and $\mathfrak{L}$ are PNFS such that $\mathfrak{z} > 0$*
(a) $\mathfrak{z}(\mathfrak{K} \wedge \mathfrak{L}) = \mathfrak{z}\mathfrak{K} \wedge \mathfrak{z}\mathfrak{L}$
(b) $\mathfrak{z}(\mathfrak{K} \vee \mathfrak{L}) = \mathfrak{z}\mathfrak{K} \vee \mathfrak{z}\mathfrak{L}$

Proof. (a) Let $\mathfrak{z}(\mathfrak{K} \wedge \mathfrak{L}) = \langle \left(\sqrt{1-(1-\min(\mu_{\mathfrak{K},t}{}^2, \mu_{\mathfrak{L},t}{}^2)^{\mathfrak{z}}}, \max(\nu_{\mathfrak{K},t}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},t}{}^{\mathfrak{z}}) \right),$

$$\left(\sqrt{1-(1-\max(\mu_{\mathfrak{K},i}{}^2, \mu_{\mathfrak{L},i}{}^2)^{\mathfrak{z}}}, \min(\nu_{\mathfrak{K},i}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},i}{}^{\mathfrak{z}}) \right),$$
$$\left(\sqrt{1-(1-\max(\mu_{\mathfrak{K},f}{}^2, \mu_{\mathfrak{L},f}{}^2)^{\mathfrak{z}}}, \min(\nu_{\mathfrak{K},f}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},f}{}^{\mathfrak{z}}) \right) \rangle$$
$$= \langle \left(\sqrt{1-(\max(1-\mu_{\mathfrak{K},t}{}^2, 1-\mu_{\mathfrak{L},t}{}^2))^{\mathfrak{z}}}, \max(\nu_{\mathfrak{K},t}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},t}{}^{\mathfrak{z}}) \right),$$
$$\left(\sqrt{1-(min(1-\mu_{\mathfrak{K},i}{}^2, 1-\mu_{\mathfrak{L},i}{}^2))^{\mathfrak{z}}}, \min(\nu_{\mathfrak{K},i}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},i}{}^{\mathfrak{z}}) \right),$$
$$\left(\sqrt{1-\left(min\left(1-\mu_{\mathfrak{K},f}{}^2, 1-\mu_{\mathfrak{L},f}{}^2\right)\right)^{\mathfrak{z}}}, \min(\nu_{\mathfrak{K},f}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},f}{}^{\mathfrak{z}}) \right) \rangle$$
$$= \langle \left(\min\left(\sqrt{1-(1-\mu_{\mathfrak{K},t}{}^2)^{\mathfrak{z}}}, \sqrt{1-(1-\mu_{\mathfrak{L},t}{}^2)^{\mathfrak{z}}} \right), \max(\nu_{\mathfrak{K},t}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},t}{}^{\mathfrak{z}}) \right),$$
$$\left(\max\left(\sqrt{1-(1-\mu_{\mathfrak{K},i}{}^2)^{\mathfrak{z}}}, \sqrt{1-(1-\mu_{\mathfrak{L},i}{}^2)^{\mathfrak{z}}} \right), \min(\nu_{\mathfrak{K},i}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},i}{}^{\mathfrak{z}}) \right),$$
$$\left(\max\left(\sqrt{1-(1-\mu_{\mathfrak{K},t}{}^2)^{\mathfrak{z}}}, \sqrt{1-(1-\mu_{\mathfrak{L},f}{}^2)^{\mathfrak{z}}} \right), \min(\nu_{\mathfrak{K},f}{}^{\mathfrak{z}}, \nu_{\mathfrak{L},f}{}^{\mathfrak{z}}) \right) \rangle$$

$= \mathfrak{z}\mathfrak{K} \wedge \mathfrak{z}\mathfrak{L}$.
Proof of (b) is similar to (a). □

Theorem 4.13. *Suppose $\mathfrak{K}$ and $\mathfrak{L}$ are PNFS such that $\mathfrak{z} > 0$, then (a) $(\mathfrak{K} \wedge \mathfrak{L})^{\mathfrak{z}} = \mathfrak{K}^{\mathfrak{z}} \wedge \mathfrak{L}^{\mathfrak{z}}$ and (b) $(\mathfrak{K} \vee \mathfrak{L})^{\mathfrak{z}} = \mathfrak{K}^{\mathfrak{z}} \vee \mathfrak{L}^{\mathfrak{z}}$*

Proof. (a) Let $(\mathfrak{K} \wedge \mathfrak{L})^{\mathfrak{z}} = \langle \left(\min(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}}, \mu_{\mathfrak{L},t}{}^{\mathfrak{z}}), \sqrt{1-(max(1-\nu_{\mathfrak{K},t}{}^2, 1-\nu_{\mathfrak{L},t}{}^2))^{\mathfrak{z}}} \right),$

$$\left(\max(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}}, \mu_{\mathfrak{L},i}{}^{\mathfrak{z}}), \sqrt{1-(min(1-\nu_{\mathfrak{K},i}{}^2, 1-\nu_{\mathfrak{L},i}{}^2))^{\mathfrak{z}}} \right),$$
$$\left(\max(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}}, \mu_{\mathfrak{L},f}{}^{\mathfrak{z}}), \sqrt{1-(min(1-\nu_{\mathfrak{K},f}{}^2, 1-\nu_{\mathfrak{L},f}{}^2))^{\mathfrak{z}}} \right) \rangle$$

$$=\langle\Big(\min(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}},\mu_{\mathfrak{L},t}{}^{\mathfrak{z}}),\max\Big(\sqrt{1-(1-\nu_{\mathfrak{K},t}{}^{2})^{\mathfrak{z}}},\ \sqrt{1-(1-\nu_{\mathfrak{L},t}{}^{2})^{\mathfrak{z}}}\Big)\Big),$$

$$\Big(\max(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}},\mu_{\mathfrak{L},i}{}^{\mathfrak{z}}),\min\Big(\sqrt{1-(1-\nu_{\mathfrak{K},i}{}^{2})^{\mathfrak{z}}},\ \sqrt{1-(1-\nu_{\mathfrak{L},i}{}^{2})^{\mathfrak{z}}}\Big)\Big),$$

$$\Big(\max\left(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}},\mu_{\mathfrak{L},f}{}^{\mathfrak{z}}\right),min\Big(\sqrt{1-(1-\nu_{\mathfrak{K},f}{}^{2})^{\mathfrak{z}}},\ \sqrt{1-(1-\nu_{\mathfrak{L},f}{}^{2})^{\mathfrak{z}}}\Big)\Big)\rangle$$

$\mathfrak{K}^{\mathfrak{z}}\ \wedge\ \mathfrak{L}^{\mathfrak{z}} =$

$$\Big(\ \Big(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},t}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},i}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},f}{}^{2})^{\mathfrak{z}}}\ \Big)\ \Big)$$

$$\wedge\Big(\ \Big(\mu_{\mathfrak{L},t}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},t}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{L},i}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},i}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{L},f}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},f}{}^{2})^{\mathfrak{z}}}\ \Big)\ \Big)$$

$$=\langle\Big(\min\left(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}},\mu_{\mathfrak{L},t}{}^{\mathfrak{z}}\right),\max\Big(\sqrt{1-(1-\nu_{\mathfrak{K},t}{}^{2})^{\mathfrak{z}}},\ \sqrt{1-(1-\nu_{\mathfrak{L},t}{}^{2})^{\mathfrak{z}}}\Big)\Big),$$

$$\Big(\max\left(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}},\mu_{\mathfrak{L},i}{}^{\mathfrak{z}}\right),\ min\Big(\sqrt{1-(1-\nu_{\mathfrak{K},i}{}^{2})^{\mathfrak{z}}},\ \sqrt{1-(1-\nu_{\mathfrak{L},i}{}^{2})^{\mathfrak{z}}}\Big)\Big),$$

$$\Big(\max(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}},\mu_{\mathfrak{L},f}{}^{\mathfrak{z}}),\min\Big(\sqrt{1-(1-\nu_{\mathfrak{K},f}{}^{2})^{\mathfrak{z}}},\ \sqrt{1-(1-\nu_{\mathfrak{L},f}{}^{2})^{\mathfrak{z}}}\Big)\Big)\rangle$$

$=(\mathfrak{K}\ \wedge\ \mathfrak{L})^{\ \mathfrak{z}}$
$\Longrightarrow (\mathfrak{K}\ \wedge\ \mathfrak{L})^{\mathfrak{z}} = \mathfrak{K}^{\mathfrak{z}}\wedge\mathfrak{L}^{\mathfrak{z}}$. Similarly, $(\mathfrak{K}\ \vee\ \mathfrak{L})^{\ \mathfrak{z}}\ =\ \mathfrak{K}^{\mathfrak{z}}\ \vee\mathfrak{L}^{\mathfrak{z}}$. □

Theorem 4.14. *Suppose $\mathfrak{K}$ and $\mathfrak{L}$ are PNFS such that $\mathfrak{z}\ >\ 0$ then $(\mathfrak{K}\oplus\mathfrak{L})^{\ \mathfrak{z}}\neq\mathfrak{K}^{\mathfrak{z}}\oplus\mathfrak{L}^{\mathfrak{z}}$.*

Proof. Let $(\mathfrak{K}\ \oplus\ \ \mathfrak{L})^{\ \mathfrak{z}}=$

$$<\Big(\ \sqrt{((\mu_{\mathfrak{K},t}{}^{2}+\mu_{\mathfrak{L},t}{}^{2})-(\mu_{\mathfrak{K},t}{}^{2}*\mu_{\mathfrak{L},t}{}^{2}))^{\mathfrak{z}}}\ ,\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},t}{}^{2}*\nu_{\mathfrak{L},t}{}^{2})\Big)\Big)^{\mathfrak{z}}}\Big),$$

$$\Big(\ \sqrt{((\mu_{\mathfrak{K},i}{}^{2}+\mu_{\mathfrak{L},i}{}^{2})-(\mu_{\mathfrak{K},i}{}^{2}*\mu_{\mathfrak{L},i}{}^{2}))^{\mathfrak{z}}}\ ,\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},i}{}^{2}*\nu_{\mathfrak{L},i}{}^{2})\Big)\Big)^{\mathfrak{z}}}\Big),$$

$$\Big(\ \sqrt{((\mu_{\mathfrak{K},f}{}^{2}+\mu_{\mathfrak{L},f}{}^{2})-(\mu_{\mathfrak{K},f}{}^{2}*\mu_{\mathfrak{L},f}{}^{2}))^{\mathfrak{z}}}\ ,\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},f}{}^{2}*\nu_{\mathfrak{L},f}{}^{2})\Big)\Big)^{\mathfrak{z}}}\Big)>$$

$\mathfrak{K}^{\mathfrak{z}}\oplus\mathfrak{L}^{\mathfrak{z}} =$

$$\Big(\ \Big(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},t}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},i}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{K},f}{}^{2})^{\mathfrak{z}}}\ \Big)\ \Big)$$
$$\oplus\Big(\ \Big(\mu_{\mathfrak{L},t}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},t}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{L},i}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},i}{}^{2})^{\mathfrak{z}}}\Big),\ \Big(\mu_{\mathfrak{L},f}{}^{\mathfrak{z}},\sqrt{1-(1-\nu_{\mathfrak{L},f}{}^{2})^{\mathfrak{z}}}\ \Big)\ \Big)$$

$$=\langle[\sqrt{(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}})^{2}+(\mu_{\mathfrak{L},t}{}^{\mathfrak{z}2}-(\mu_{\mathfrak{K},t}{}^{\mathfrak{z}})^{2}*(\mu_{\mathfrak{L},t}{}^{\mathfrak{z}})^{2}},\Big(\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},t}{}^{2})\Big)\Big)^{\mathfrak{z}}}^{\ \mathfrak{z}}\Big)$$

$$*\Big(\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},t}{}^{2})\Big)\Big)^{\mathfrak{z}}}^{\ \mathfrak{z}}\Big)],[\sqrt{(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}})^{2}+(\mu_{\mathfrak{L},i}{}^{\mathfrak{z}})^{2}-(\mu_{\mathfrak{K},i}{}^{\mathfrak{z}})^{2}*(\mu_{\mathfrak{L},i}{}^{\mathfrak{z}})^{2}},$$

$$\Big(\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},i}{}^{2})\Big)\Big)^{\mathfrak{z}}}^{\ \mathfrak{z}}\Big)*\Big(\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},i}{}^{2})\Big)\Big)^{\mathfrak{z}}}^{\ \mathfrak{z}}\Big)],$$

$$[\sqrt{(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}})^{2}+(\mu_{\mathfrak{L},f}{}^{\mathfrak{z}})^{2}-(\mu_{\mathfrak{K},f}{}^{\mathfrak{z}})^{2}*(\mu_{\mathfrak{L},f}{}^{\mathfrak{z}})^{2}},\Big(\ \sqrt{\Big(1-\Big(1-(\nu_{\mathfrak{K},f}{}^{2})\Big)\Big)^{\mathfrak{z}}}^{\ \mathfrak{z}}\Big)$$

$$*\left(\sqrt{\left(1-\left(1-(\nu_{\mathfrak{K},\mathfrak{f}}{}^2)\right)^{\mathfrak{z}}\right)^{\mathfrak{z}}}\right)]\rangle$$

Hence $(\mathfrak{K} \oplus \mathfrak{L})^{\mathfrak{z}} \neq \mathfrak{K}^{\mathfrak{z}} \oplus \mathfrak{L}^{\mathfrak{z}}$. □

5. Operation (@) on PNFS

Desirable features of the operation (@) on PNFS are discussed here.

Definition 5.1. *$\mathfrak{K}@\mathfrak{L}$ is a PNFS, if $\mathfrak{K}$ and $\mathfrak{L}$ are PNFS.*

$$\mathfrak{K}@\mathfrak{L} = \langle\left(\sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{t}}{}^2+\mu_{\mathfrak{L},\mathfrak{t}}{}^2}{2}}, \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{t}}{}^2+\nu_{\mathfrak{L},\mathfrak{t}}{}^2}{2}}\right), \left(\sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{i}}{}^2+\mu_{\mathfrak{L},\mathfrak{i}}{}^2}{2}}, \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{i}}{}^2+\nu_{\mathfrak{L},\mathfrak{i}}{}^2}{2}}\right), \left(\sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{f}}{}^2+\mu_{\mathfrak{L},\mathfrak{f}}{}^2}{2}}, \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{f}}{}^2+\nu_{\mathfrak{L},\mathfrak{f}}{}^2}{2}}\right)\rangle$$

Remark 5.1. *Needless to say, for any two PNFS sets $\mathfrak{K}$ and $\mathfrak{L}$, $\mathfrak{K}@\mathfrak{L}$ is a PNFS sets. Simple example given:*
For $\mathfrak{K}@\mathfrak{L}$, $0 \leq \frac{\mu_{\mathfrak{K},\mathfrak{t}}+\mu_{\mathfrak{L},\mathfrak{t}}}{2} + \frac{\nu_{\mathfrak{K},\mathfrak{t}}+\nu_{\mathfrak{L},\mathfrak{t}}}{2} \leq \frac{\mu_{\mathfrak{K},\mathfrak{t}}+\nu_{\mathfrak{K},\mathfrak{t}}}{2} + \frac{\mu_{\mathfrak{L},\mathfrak{t}}+\nu_{\mathfrak{L},\mathfrak{t}}}{2} \leq \frac{1}{2}+\frac{1}{2} = 1$.

Theorem 5.1. *$\mathfrak{K}@\mathfrak{K} = \mathfrak{K}$, where $\mathfrak{K}$ is a PNFS.*

Proof. Let
$$\mathfrak{K}@\mathfrak{K} = \langle\left(\sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{t}}{}^2+\mu_{\mathfrak{K},\mathfrak{t}}{}^2}{2}}, \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{t}}{}^2+\nu_{\mathfrak{K},\mathfrak{t}}{}^2}{2}}\right), \left(\sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{i}}{}^2+\mu_{\mathfrak{K},\mathfrak{i}}{}^2}{2}}, \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{i}}{}^2+\nu_{\mathfrak{K},\mathfrak{i}}{}^2}{2}}\right), \left(\sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{f}}{}^2+\mu_{\mathfrak{K},\mathfrak{f}}{}^2}{2}}, \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{f}}{}^2+\nu_{\mathfrak{K},\mathfrak{f}}{}^2}{2}}\right)\rangle$$

$$= \langle\left(\sqrt{\frac{2\mu_{\mathfrak{K},\mathfrak{t}}{}^2}{2}}, \sqrt{\frac{2\nu_{\mathfrak{K},\mathfrak{t}}{}^2}{2}}\right), \left(\sqrt{\frac{2\mu_{\mathfrak{K},\mathfrak{i}}{}^2}{2}}, \sqrt{\frac{2\nu_{\mathfrak{K},\mathfrak{i}}{}^2}{2}}\right), \left(\sqrt{\frac{2\mu_{\mathfrak{K},\mathfrak{f}}{}^2}{2}}, \sqrt{\frac{2\nu_{\mathfrak{K},\mathfrak{f}}{}^2}{2}}\right)\rangle$$

$$= \langle(\mu_{\mathfrak{K},\mathfrak{t}}, \nu_{\mathfrak{K},\mathfrak{t}}), (\mu_{\mathfrak{K},\mathfrak{i}}, \nu_{\mathfrak{K},\mathfrak{i}}), (\mu_{\mathfrak{K},\mathfrak{f}}, \nu_{\mathfrak{K},\mathfrak{f}})\rangle = \mathfrak{K}$$ □

Theorem 5.2. *For PNFS $\mathfrak{K}$ and $\mathfrak{L}$,*

(a) $\mathfrak{K} \oplus \mathfrak{L} = (\mathfrak{K} \oplus \mathfrak{L}) \vee (\mathfrak{K}@\mathfrak{L})$
(b) $\mathfrak{K} \otimes \mathfrak{L} = (\mathfrak{K} \otimes \mathfrak{L}) \wedge (\mathfrak{K}@\mathfrak{L})$
(c) $\mathfrak{K}@\mathfrak{L} = (\mathfrak{K} \oplus \mathfrak{L}) \wedge (\mathfrak{K}@\mathfrak{L})$
(d) $\mathfrak{K}@\mathfrak{L} = (\mathfrak{K} \otimes \mathfrak{L}) \vee (\mathfrak{K}@\mathfrak{L})$

Proof. As (b) and (d) are obvious, its enough to prove (a) alone.

(a) $(\mathfrak{K} \oplus \mathfrak{L}) \vee (\mathfrak{K}@\mathfrak{L})$

$$= \left\langle \max\left(\sqrt{(\mu_{\mathfrak{K},\mathfrak{t}}{}^2+\mu_{\mathfrak{L},\mathfrak{t}}{}^2)-(\mu_{\mathfrak{K},\mathfrak{t}}{}^2*\mu_{\mathfrak{L},\mathfrak{t}}{}^2)}, \sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{t}}{}^2+\mu_{\mathfrak{L},\mathfrak{t}}{}^2}{2}}\right), \min\left((\nu_{\mathfrak{K},\mathfrak{t}}*\nu_{\mathfrak{L},\mathfrak{t}}), \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{t}}{}^2+\nu_{\mathfrak{K},\mathfrak{t}}{}^2}{2}}\right),\right.$$
$$\min\left(\sqrt{(\mu_{\mathfrak{K},\mathfrak{i}}{}^2+\mu_{\mathfrak{L},\mathfrak{i}}{}^2)-(\mu_{\mathfrak{K},\mathfrak{i}}{}^2*\mu_{\mathfrak{L},\mathfrak{i}}{}^2)}, \sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{i}}{}^2+\mu_{\mathfrak{L},\mathfrak{i}}{}^2}{2}}\right), \max\left((\nu_{\mathfrak{K},\mathfrak{i}}*\nu_{\mathfrak{L},\mathfrak{i}}), \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{i}}{}^2+\nu_{\mathfrak{K},\mathfrak{i}}{}^2}{2}}\right),$$
$$\left.\min\left(\sqrt{(\mu_{\mathfrak{K},\mathfrak{f}}{}^2+\mu_{\mathfrak{L},\mathfrak{f}}{}^2)-(\mu_{\mathfrak{K},\mathfrak{f}}{}^2*\mu_{\mathfrak{L},\mathfrak{f}}{}^2)}, \sqrt{\frac{\mu_{\mathfrak{K},\mathfrak{f}}{}^2+\mu_{\mathfrak{L},\mathfrak{f}}{}^2}{2}}\right), \max\left((\nu_{\mathfrak{K},\mathfrak{f}}*\nu_{\mathfrak{L},\mathfrak{f}}), \sqrt{\frac{\nu_{\mathfrak{K},\mathfrak{f}}{}^2+\nu_{\mathfrak{K},\mathfrak{f}}{}^2}{2}}\right)\right\rangle$$

$$= \langle\left(\sqrt{(\mu_{\mathfrak{K},\mathfrak{t}}{}^2+\mu_{\mathfrak{L},\mathfrak{t}}{}^2)-(\mu_{\mathfrak{K},\mathfrak{t}}{}^2*\mu_{\mathfrak{L},\mathfrak{t}}{}^2)}, (\nu_{\mathfrak{K},\mathfrak{t}}*\nu_{\mathfrak{L},\mathfrak{t}})\right),$$
$$\left(\sqrt{(\mu_{\mathfrak{K},\mathfrak{i}}{}^2+\mu_{\mathfrak{L},\mathfrak{i}}{}^2)-(\mu_{\mathfrak{K},\mathfrak{i}}{}^2*\mu_{\mathfrak{L},\mathfrak{i}}{}^2)}, (\nu_{\mathfrak{K},\mathfrak{i}}*\nu_{\mathfrak{L},\mathfrak{i}})\right),$$
$$\left(\sqrt{(\mu_{\mathfrak{K},\mathfrak{f}}{}^2+\mu_{\mathfrak{L},\mathfrak{f}}{}^2)-(\mu_{\mathfrak{K},\mathfrak{f}}{}^2*\mu_{\mathfrak{L},\mathfrak{f}}{}^2)}, (\nu_{\mathfrak{K},\mathfrak{f}}*\nu_{\mathfrak{L},\mathfrak{f}})\right)\rangle = \mathfrak{K} \oplus \mathfrak{L}$$

(c) $(\mathfrak{K} \oplus \mathfrak{L}) \wedge (\mathfrak{K}@\mathfrak{L})$

$$= \langle(\min\left(\sqrt{({\mu_{\mathfrak{K},\mathfrak{t}}}^2 + {\mu_{\mathfrak{L},\mathfrak{t}}}^2) - ({\mu_{\mathfrak{K},\mathfrak{t}}}^2 * {\mu_{\mathfrak{L},\mathfrak{t}}}^2)}, \sqrt{\frac{{\mu_{\mathfrak{K},\mathfrak{t}}}^2+{\mu_{\mathfrak{L},\mathfrak{t}}}^2}{2}}\right),$$

$$\max\left((\nu_{\mathfrak{K},\mathfrak{t}} * \nu_{\mathfrak{L},\mathfrak{t}}), \sqrt{\frac{{\nu_{\mathfrak{K},\mathfrak{t}}}^2 + {\nu_{\mathfrak{K},\mathfrak{t}}}^2}{2}}\right)), (\max\left(\sqrt{({\mu_{\mathfrak{K},\mathfrak{i}}}^2 + {\mu_{\mathfrak{L},\mathfrak{i}}}^2) - ({\mu_{\mathfrak{K},\mathfrak{i}}}^2 * {\mu_{\mathfrak{L},\mathfrak{i}}}^2)}, \sqrt{\frac{{\mu_{\mathfrak{K},\mathfrak{i}}}^2 + {\mu_{\mathfrak{L},\mathfrak{i}}}^2}{2}}\right),$$

$$\min\left((\nu_{\mathfrak{K},\mathfrak{i}} * \nu_{\mathfrak{L},\mathfrak{i}}), \sqrt{\frac{{\nu_{\mathfrak{K},\mathfrak{i}}}^2 + {\nu_{\mathfrak{K},\mathfrak{i}}}^2}{2}}\right)), (\max\left(\sqrt{({\mu_{\mathfrak{K},\mathfrak{f}}}^2 + {\mu_{\mathfrak{L},\mathfrak{f}}}^2) - ({\mu_{\mathfrak{K},\mathfrak{f}}}^2 * {\mu_{\mathfrak{L},\mathfrak{f}}}^2)}, \sqrt{\frac{{\mu_{\mathfrak{K},\mathfrak{f}}}^2 + {\mu_{\mathfrak{L},\mathfrak{f}}}^2}{2}}\right),$$

$$\min\left((\nu_{\mathfrak{K},\mathfrak{f}} * \nu_{\mathfrak{L},\mathfrak{f}}), \sqrt{\frac{{\nu_{\mathfrak{K},\mathfrak{f}}}^2 + {\nu_{\mathfrak{K},\mathfrak{f}}}^2}{2}}\right))\rangle$$

$$= \langle\left(\sqrt{\frac{{\mu_{\mathfrak{K},\mathfrak{t}}}^2+{\mu_{\mathfrak{L},\mathfrak{t}}}^2}{2}}, \sqrt{\frac{{\nu_{\mathfrak{K},\mathfrak{t}}}^2+{\nu_{\mathfrak{L},\mathfrak{t}}}^2}{2}}\right), \left(\sqrt{\frac{{\mu_{\mathfrak{K},\mathfrak{i}}}^2+{\mu_{\mathfrak{L},\mathfrak{i}}}^2}{2}}, \sqrt{\frac{{\nu_{\mathfrak{K},\mathfrak{i}}}^2+{\nu_{\mathfrak{L},\mathfrak{i}}}^2}{2}}\right),$$
$$\left(\sqrt{\frac{{\mu_{\mathfrak{K},\mathfrak{f}}}^2+{\mu_{\mathfrak{L},\mathfrak{f}}}^2}{2}} \sqrt{\frac{{\nu_{\mathfrak{K},\mathfrak{f}}}^2+{\nu_{\mathfrak{L},\mathfrak{f}}}^2}{2}}\right)\rangle = \mathfrak{K}@\mathfrak{L}.$$ □

Remark 5.2. *Under the operations algebraic addition and product, PNFS produces a associativity, semilattice, commutativity and idempotency. When,* $\oplus$, $\otimes$ *and* $\wedge$, $\vee$, @ *are combined, the distributive law also holds true.*

6. Conclusion

This research marks a significant advancement in our capacity to model and handle intricate systems characterized by uncertainty, vagueness, and indeterminacy. By conducting a thorough examination and analysis of Pythagorean Neutrosophic Fuzzy Set (PNFS) operations, we have uncovered the intricate interactions among Pythagorean fuzzy sets, neutrosophic sets, and fuzzy sets within this unified framework. PNFS combines the strengths of Pythagorean Fuzzy Sets and Neutrosophic Sets into a unified framework. This integration enables a comprehensive representation of uncertainty, incorporating neutrosophic aspects within a Pythagorean fuzzy environment simultaneously.The mathematical formulations and characteristics of PNFS operations, such as commutativity, idempotency, absorption law, associativity, De Morgan's rules, and distributivity over complement, have been systematically explored, providing insights into their behavior in diverse scenarios.

Additionally, distributive rules were scrutinized, and a new operation (@) on PNFS was elucidated. The incorporation of PNFS operations into decision-making processes serves as valuable tools for modeling situations where conventional mathematical frameworks may be inadequate. Essentially, the exploration of PNFS operations presents fresh perspectives for researchers, practitioners, and scholars seeking innovative approaches to address the inherent complexities of decision-making in uncertain conditions. As this field continues to develop, the practical applications of PNFS operations are likely to broaden, contributing to the progress of computational intelligence and decision support systems. The synergistic relationship among Pythagorean fuzzy sets, neutrosophic sets, and fuzzy sets within the PNFS framework exemplifies the potential of interdisciplinary collaboration, fostering a deeper comprehension of uncertainty and imprecision across various scientific and engineering disciplines.

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