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A DISCUSSION ON CONTROLLABILITY OF SEMILINEAR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. The study of controllability results ensure the essential conditions required for a solution. Keeping the importance of the study, we discuss the controllability of semilinear impulsive functional-differential equations (SIFDE) of second order. For this purpose we use the idea of strongly continuous cosine families (SCCF) of linear operators and Banach Contraction principle. Lastly, an example is set to explain the abstract theory. The achieved results reveal that the proposed method is systematic and appropriate for dealing with the semilinear impulsive functional-differential problem that arises in engineering and physics.

Keywords: Functional-differential equation; Second order; Impulsive condition; Banach contraction principle; Cosine family.

AMS Subject Classification: 34K30, 47H10, 34A37, 93B05.

1. Introduction

The idea of controllability is pivotal notions in mathematical control theory. Various significant problems of control theory like optimal control, stabilization of unstable system by feedback, structural decomposition, observer design and structural engineering, can be explained by assuming that the system is controllable. Roughly, controllability means by using the set of admissible control move a system to any initial state to final desired state.

The concept of controllability is also applied in many real world phenomena, see for instance, controlling sugar level in the blood, missile and anti-missile problems, rocket launching problems for satellite, reactor control, quantum system control etc. Due to the wide applications, the problem of controllability of different system such as integrodifferential system, impulsive system, fractional system and stochastic system is studied by

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many authors ([4], [14], [26], [27]). Ma et.al. [17] studied the sufficient condition for approximate controllability of Hilfer fractional inclusion system in Hilbert space by applying Bohnenblust- Karlin's fixed point theorem. By employing fixed point theorem, multivalued analysis, fractional calculus, and semi-group theories the approximate controllability is studied for Atangana-Baleanu nonlocal fractional neutral integrodifferential stochastic systems with infinite delay in [18]. Klamka ([12], [13]) investigated the controllability of linear system with delays in control.

To express instant changes in the functioning of the system, the impulses in differential equations are implemented. It is known fact in literature that a lot of real word occurrences like population dynamics [23], control theory [11], drug resistance model in biology and medicine ([6], [20], [29]) etc. demonstrate the impulse effect and modeled in the form of impulsive differential equation. The importance of nonlocal conditions is exhibited in diffusion process by the author [7]. So, the use of nonlocal condition in impulsive differential system is very interesting field of research. For example, incorporating the nonlocal and impulsive condition both in the phenomena of traveling of sound wave through a non-uniform rod is very useful to model such type of system.

On the other hand, a variety of physical phenomena modeled in second order equations and it can be seen in ([2], [9], [19]). Therefore, it is quite relevant to analyze the controllability for such kinds of system in Banach space. Several times it is very useful to explore the abstract differential system of second order without transforming it into first order system see ([2], [9]). Idea of SCCF of operators is very helpful apparatus for the analysis of various kinds of abstract second order system, see ([30], [31]).

Many researchers ([3], [21], [33]) have paid their attention for the examination of controllability of second order differential systems with or without impulse in recent years. Arthi et al. [1] analyzed the controllability of damped integrodifferential system and neutral system with impulsive condition in Banach space by employing the Banach fixed point thorem. The approximate controllability of various kinds of differential and integrodifferential systems is derived in the works ([16], [22], [28]).

Author develop appropriate criteria for the controllability of ISFDE of second order with nonlocal condition evolved from the above mentioned literature survey and inspired due to the fact that no work has been carried out to the best of our understating. We shall apply the fixed point technique and SCCF of linear operators to find the required outcome.

2. Preliminaries

Consider the semilinear impulsive functional-differential control system as follows:

$$\begin{cases} v''(\xi) = Av(\xi) + K\Big(\xi, v(\xi), v(\gamma_1(\xi)), ..., v(\gamma_q(\xi)), v'(\xi)\Big) + Gy(\xi),\ \xi \in [0,b] = H \\ v(0) = v_0, \\ v'(0) + h(v) = v_1, \\ \Delta v(\varsigma_i) = \ell_i\big(v(\varsigma_i)\big), \Delta v'(\varsigma_i) = \bar{\ell}_i\big(v'(\varsigma_i^+)\big), \xi \neq \varsigma_i, i = 1, 2, .., m, \end{cases} \tag{1}$$

where the functions $\gamma_i : H \to H (i = 1, 2, ..., q)$, and $q \in \mathbb{N}$. The functions $K : H \times W^{q+2} \to W, \ell_i, \bar{\ell}_i : W \to W$ and $h : PC(H, W) \to PC(H, W)$ are continuous functions to be specified later. Furthermore, $\Delta v(\varsigma_i) = v(\varsigma_i + 0) - v(\varsigma_i - 0), \Delta v'(\varsigma_i) = v'(\varsigma_i + 0) - v'(\varsigma_i - 0)$, where the character $v(\varsigma_i + 0)$ and $v(\varsigma_i - 0)$ describe the right and left limits of v at ς_i respectively for $0 = \varsigma_0 < \varsigma_1 < \varsigma_2 < ... < \varsigma_m < \varsigma_{m+1} = b, m \in \mathbb{N}$. Let W be the Banach space with the supremum norm $\|.\|$. Considering the symbol in ([10], [15]), $PC(H, W) =$

$\big\{v : v$ is a function from H to W in such a manner that $v(\xi)$ is continuous at $\xi \neq \varsigma_i$, left continuous at $\xi = \varsigma_i$, and the right limit $v(\varsigma_i + 0)$ exists for $i = 1, 2, ..., m\big\}$. It is noted that $PC^1(H, W)$ is a Banach space with the supremum norm $\|v\|_1 = sup\big\{\|v(\xi)\| + \|v'(\xi)\| : \xi \in H\big\}$, see [5]. Here A be the infinitesimal generator of a SCCF of linear operators $\{S_1(\xi) : \xi \in \mathbb{R}\}$ on W. Moreover, the bounded linear operator $G : U \to W$ and the control function $y(.)$ is specified in $L^2(H, U)$, i.e., a Banach space of admissible control functions. Here the interval $H = [0, b]$ and U be a Banach space.

Moreover, we state some results and notations which concerned with the theory of cosine functions of operators. This theory is required to establish our outcome.

Definition 2.1. The family $\{S_1(\xi) : \xi \in \mathbb{R}\}$ is termed as SCCF if it holds the following

- (i) $S_1(0) = I$.
- (ii) $S_1(\xi + \zeta) + S_1(\xi - \zeta) = 2S_1(\xi)S_1(\zeta), \forall\ \xi, \zeta \in \mathbb{R}$.
- (iii) for all $v \in W$, the map $\xi \to S_1(\xi)v$ is strongly continuous.

The infinitesimal generator $A : W \to W$ of the cosine family $\{S_1(\xi) : \xi \in \mathbb{R}\}$ is given by $Av = \frac{d^2}{d\xi^2} S_1(\xi)v\big|_{\xi=0}, v \in dom(A)$, where $dom(A) = \{v \in W : S_1(.)v$ is of class C^2 with respect to $\xi\}$. Define $V = \{v \in W : S_1(\xi)v$ is of class C^1 with respect to $\xi\}$.

Moreover, the sine family $\{S_2(\xi) : \xi \in \mathbb{R}\}$ is expressed by $S_2(\xi)v = \int_0^{\xi} S_1(\Phi)v d\Phi$, for $\xi \in \mathbb{R}, v \in W$, which is related to the $\{S_1(\xi) : \xi \in \mathbb{R}\}$. For detailed information, see for instance Travis and Webb [30] and Fattorini [8].

Proposition 2.1. [32] *Let $\{S_1(\xi) : \xi \in \mathbb{R}\}$ be SCCF with infinitesimal generator A and associated sine family $S_2(\xi), \xi \in \mathbb{R}$. The following are true.*

- (i) *If $v \in W, S_2(\xi)v \in V$.*
- (ii) *If $v \in V, S_1(\xi)v \in V$.*
- (iii) *If $v \in V, S_2(\xi)v \in dom(A)$ and $\frac{d}{d\xi} S_1(\xi)v = AS_2(\xi)v$.*
- (iv) *If $v \in V$, $S_2(\xi)v \in dom(A)$, then* $\frac{d}{d\xi} S_1(\xi)v = AS_2(\xi)v$ *and* $\frac{d^2}{d\xi^2} S_2(\xi)v = AS_2(\xi)v$.
- (v) *If $v \in dom(A)$, $S_1(\xi)v \in dom(A)$, then* $\frac{d^2}{d\xi^2} S_1(\xi)v = AS_1(\xi)v = S_1(\xi)Av$.
- (vi) *If $v \in V, S_2(\xi)v \in dom(A)$ and $\frac{d^2}{d\xi^2} S_2(\xi)v = AS_2(\xi)v$.*

Definition 2.2. [10] A function $v(\xi) \in PC^1(H, W)$ is stated as a mild solution of equation (1) if it satisfies

$$v(\xi) = S_1(\xi)v_0 + S_2(\xi)v_1 - S_2(\xi)h(v) + \int_0^{\xi} S_2(\xi - \Phi)\times$$

$$K\bigg(\Phi, v(\Phi), v\big(\gamma_1(\Phi)\big), ..., v\big(\gamma_q(\Phi)\big), v'(\Phi)\bigg)d\Phi + \int_0^{\xi} S_2(\xi - \Phi)Gy(\Phi)d\Phi$$

$$+ \sum_{0<\varsigma_i<\xi} S_1(\xi - \varsigma_i)\ell_i\big(v(\varsigma_i)\big) + \sum_{0<\varsigma_i<\xi} S_2(\xi - \varsigma_i)\bar{\ell}_i\big(v'(\varsigma_i^+)\big), \forall\ \xi \in [0, b] = H. \quad (2)$$

Now, we state the following assumptions:

(A_1) The function $\gamma_i : H \to H$ is continuous on H. Let $K : H \times W^{q+2} \to W$ be continuous function in such a way that there exists $F_0 > 0$ such that

$$\|K(\xi, x_1, x_2, ..., x_{q+2}) - K(\xi, \bar{x}_1, \bar{x}_2, ..., \bar{x}_{q+2}) \le F_0 \sum_{i=1}^{q+2} \|x_i - \bar{x}_i\|$$

for $\xi \in H, x_i, \bar{x}_i \in W, i = 1, 2, ..., q+2$.

(A_2) Let ℓ_i and $\bar{\ell}_i$ be continuous function in such a manner that there exist $F_i > 0$ and $\bar{F}_i > 0$, so that

$$\|\ell_i(x) - \ell_i(u)\| \le F_i\|x - u\|,\ \|\bar{\ell}_i(x) - \bar{\ell}_i(u)\| \le \bar{F}_i\|x - u\|,$$

for all $x, u \in W$.

(A_3) Let $h : PC(H, W) \to PC(H, W)$ be a continuous function in such a way that there exists $L_0 > 0$, so that

$$\|h(x) - h(u)\| \le L_0\|x - u\|; \text{ for all } x, u \in W.$$

Also,

$$F_i^* = max\{F_i, \bar{F}_i\}.$$

(A_4) $G : U \to W$ is a continuous operator. Also, the linear operator $Q : L^2(H, U) \to W$, stated by

$$Qy = \int_0^b S_2(b - \Phi)Gy(\Phi)d\Phi$$

has an induced inverse operator $\tilde{Q}^{-1}$, which takes values in $L^2(H, U)/KerQ$ and there exists $F_1 > 0$ such that $\|G\tilde{Q}^{-1}\| \le F_1$.

(A_5) $2M(1 + MF_1b)[L_0 + (q + 1)F_0b + 2F_i^*] < 1$,
where

$$M = sup\{\|S_1'(\xi)\| + \|S_1(\xi)\| + \|S_2(\xi)\| + \|S_2'(\xi)\|\}, \xi \in (0, b].$$

Definition 2.3. The ISFDE of second order evolution equations (1) is called as controllable on H, if $v_0 \in V$ and $v_1, v_b \in W$ then we have a control $y \in L^2(H, U)$ in such a manner that the solution $v(.)$ of equation (1) fulfils $v(b) = v_b$.

3. Main Result

Theorem 3.1. *If assumptions* $(A_1) - (A_5)$ *hold. Then system* (1) *are controllable on* H.

Proof. By referring (A_4), for any arbitrary function $v(.)$, we describe the control formally as

$$y(\xi) = \tilde{Q}^{-1}\Bigg[v_b - S_1(b)v_0 - S_2(b)v_1 + S_2(b)h(v) - \int_0^b S_2(b - \Phi)K\Big(\Phi, v(\Phi), v\big(\gamma_1(\Phi)\big), ..., v\big(\gamma_q(\Phi)\big), v'(\Phi)\Big)d\Phi - \sum_{0<\varsigma_i<b} S_1(b - \varsigma_i)\ell_i\big(v(\varsigma_i)\big) - \sum_{0<\varsigma_i<b} S_2(b - \varsigma_i)\bar{\ell}_i\big(v'(\varsigma_i^+)\big)\Bigg](\xi).$$

Then, we must now demonstrate that when applying this control, the operator $\Theta : PC^1(H, W) \to PC^1(H, W)$ expressed as

$$(\Theta v)(\xi) = S_1(\xi)v_0 + S_2(\xi)v_1 - S_2(\xi)h(v) + \int_0^\xi S_2(\xi - \Phi)K\Big(\Phi, v(\Phi), v\big(\gamma_1(\Phi)\big), ..., v\big(\gamma_q(\Phi)\big), v'(\Phi)\Big)d\Phi + \sum_{0<\varsigma_i<\xi} S_1(\xi - \varsigma_i)\ell_i\big(v(\varsigma_i)\big) + \sum_{0<\varsigma_i<\xi} S_2(\xi - \varsigma_i)\bar{\ell}_i\big(v'(\varsigma_i^+)\big)$$

$$+\int_0^{\xi} S_2(\xi-\Phi)G\tilde{Q}^{-1}\Big[v_b - S_1(b)v_0 - S_2(b)v_1 + S_2(b)h(v) - \int_0^b S_2(b-\wp)\times$$

$$K\Big(\wp, v(\wp), v\big(\gamma_1(\wp)\big), ..., v\big(\gamma_q(\wp)\big), v'(\wp)\Big)d\wp - \sum_{0<\varsigma_i<b} S_1(b-\varsigma_i)\ell_i\big(v(\varsigma_i)\big)$$

$$-\sum_{0<\varsigma_i<b} S_2(b-\varsigma_i)\bar{\ell}_i\big(v'(\varsigma_i^+)\big)\Big](\Phi)d\Phi,\ \xi \in H, \quad (3)$$

has a fixed point. Obviously, this fixed point $v(.)$ is then a solution of (2).

Noticeably, $(\Theta v)(b) = v_b$, which implies that the control y steers the system from the initial function v_0 to v_b in time b, yielded that the nonlinear operator Θ admits a fixed point.

Further, it is mentioned that $PC^1(H,W)$ is a Banach space with the norm $\|v\|_1 = sup\{\|v(\xi)\| + \|v'(\xi)\| : \xi \in H\}$. Now, we shall show that Θ is a contraction on the Banach space $PC^1(H,W)$ together the norm $\|v\|_1$. Consider

$$\|(\Theta v)(\xi) - (\Theta v^*)(\xi)\| = \Big\|S_2(\xi)\big(h(v) - h(v^*)\big) + \int_0^{\xi} S_2(\xi-\Phi)\Big[K\Big(\Phi, v(\Phi), v\big(\gamma_1(\Phi)\big)$$

$$, ..., v\big(\gamma_q(\Phi)\big), v'(\Phi)\Big) - K\Big(\Phi, v^*(\Phi), v^*\big(\gamma_1(\Phi)\big), ..., v^*\big(\gamma_q(\Phi)\big), v^{*'}(\Phi)\Big)\Big]d\Phi$$

$$+\sum_{0<\varsigma_i<\xi} S_1(\xi-\varsigma_i)\Big(\ell_i\big(v(\varsigma_i)\big) - \ell_i\big(v^*(\varsigma_i)\big)\Big) + \sum_{0<\varsigma_i<\xi} S_2(\xi-\varsigma_i)\Big(\bar{\ell}_i\big(v'(\varsigma_i^+)\big) - \bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\Big)$$

$$+\int_0^{\xi} S_2(\xi-\Phi)G\tilde{Q}^{-1}\Big[S_2(b)\big(h(v) - h(v^*)\big) - \int_0^b S_2(b-\wp)\Big\{K\Big(\wp, v(\wp), v\big(\gamma_1(\wp)\big), ...,$$

$$v\big(\gamma_q(\wp)\big), v'(\wp)\Big) - K\Big(\wp, v^*(\wp), v^*\big(\gamma_1(\wp)\big), ..., v^*\big(\gamma_q(\wp)\big), v^{*'}(\wp)\Big)\Big\}d\wp - \sum_{0<\varsigma_i<b} S_1(b-\varsigma_i)$$

$$\Big(\ell_i\big(v(\varsigma_i)\big) - \ell_i\big(v^*(\varsigma_i)\big)\Big) - \sum_{0<\varsigma_i<b} S_2(b-\varsigma_i)\Big(\bar{\ell}_i\big(v'(\varsigma_i^+)\big) - \bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\Big)\Big](\Phi)d\Phi\Big\|$$

$$\le ML_0\|v - v^*\|_1 + \int_0^{\xi} \|S_2(\xi-\Phi)\|F_0\Big(\|v(\Phi) - v^*(\Phi)\| + \|v(\gamma_1(\Phi)) - v^*(\gamma_1(\Phi))\| + ...$$

$$+\|v(\gamma_q(\Phi)) - v^*(\gamma_q(\Phi))\| + \|v'(\Phi) - v^{*'}(\Phi)\|\Big)d\Phi$$

$$+\sum_{0<\varsigma_i<\xi} \|S_1(\xi-\varsigma_i)\|\big\|\ell_i\big(v(\varsigma_i)\big) - \ell_i\big(v^*(\varsigma_i)\big)\big\| + \sum_{0<\varsigma_i<\xi} \|S_2(\xi-\varsigma_i)\|\big\|\bar{\ell}_i\big(v'(\varsigma_i^+)\big) - \bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\big\|$$

$$+\int_0^{\xi} \|S_2(\xi-\Phi)\|F_1\Big[ML_0\|v - v^*\|_1 + \int_0^b \|S_2(b-\wp)\|F_0\Big(\|v(\wp) - v^*(\wp)\| + \|v(\gamma_1(\wp))$$

$$-v^*(\gamma_1(\wp))\| + ... + \|v(\gamma_q(\wp)) - v^*(\gamma_q(\wp))\| + \|v'(\wp) - v^{*'}(\wp)\|\Big)d\wp$$

$$+\sum_{0<\varsigma_i<b}\|S_1(b-\varsigma_i)\|\big\|\ell_i\big(v(\varsigma_i)\big)-\ell_i\big(v^*(\varsigma_i)\big)\big\|$$

$$+\sum_{0<\varsigma_i<b}\|S_2(b-\varsigma_i)\|\big\|\bar{\ell}_i\big(v'(\varsigma_i^+)\big)-\bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\big\|\Big]d\Phi$$

$$\leq ML_0\|v-v^*\|_1+MF_0\int_0^{\xi}(q+1)\|v(\Phi)-v^*(\Phi)\|_1 d\Phi+\sum_{0<\varsigma_i<\xi}MF_i\big\|v(\varsigma_i)-v^*(\varsigma_i)\big\|$$

$$+\sum_{0<\varsigma_i<\xi}M\bar{F}_i\big\|v'(\varsigma_i^+)-v^{*'}(\varsigma_i^+)\big\|+\int_0^{\xi}MF_1\Big[ML_0\|v-v^*\|_1+\int_0^b M(q+1)F_0\|v-v^*\|d\wp$$

$$+\sum_{0<\varsigma_i<b}MF_i\big\|v(\varsigma_i)-v^*(\varsigma_i)\big\|+\sum_{0<\varsigma_i<b}M\bar{F}_i\big\|v'(\varsigma_i^+)-v^{*'}(\varsigma_i^+)\big\|\Big]d\Phi$$

$$\leq ML_0\|v-v^*\|_1+(q+1)bMF_0\|v-v^*\|_1+MF_i^*\|v-v^*\|_1+MF_i^*\|v-v^*\|_1$$

$$+MF_1b\Big(ML_0\|v-v^*\|_1+(q+1)bMF_0\|v-v^*\|_1+MF_i^*\|v-v^*\|_1+MF_i^*\|v-v^*\|_1\Big)$$

$$\leq\Big[\big\{ML_0+(q+1)bMF_0+2MF_i^*\big\}+MF_1b\big\{ML_0+(q+1)bMF_0+2MF_i^*\big\}\Big]\|v-v^*\|_1$$

$$\leq\Big[ML_0+(q+1)bMF_0+2MF_i^*\Big](1+MF_1b)\|v-v^*\|_1$$

Now,

$$\|(\Theta v)'(\xi)-(\Theta v^*)'(\xi)\|=\Big\|S_2'(\xi)\big(h(v)-h(v^*)\big)+\int_0^{\xi}S_1(\xi-\Phi)\Big[K\Big(\Phi,v(\Phi),$$

$$v\big(\gamma_1(\Phi)\big),...,v\big(\gamma_q(\Phi)\big),v'(\Phi)\Big)-K\Big(\Phi,v^*(\Phi),v^*\big(\gamma_1(\Phi)\big),...,v^*\big(\gamma_q(\Phi)\big),v^{*'}(\Phi)\Big)\Big]d\Phi$$

$$+\sum_{0<\varsigma_i<\xi}S_1'(\xi-\varsigma_i)\Big(\ell_i\big(v(\varsigma_i)\big)-\ell_i\big(v^*(\varsigma_i)\big)\Big)+\sum_{0<\varsigma_i<\xi}S_1(\xi-\varsigma_i)\Big(\bar{\ell}_i\big(v'(\varsigma_i^+)\big)-\bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\Big)$$

$$+\int_0^{\xi}S_1(\xi-\Phi)G\tilde{Q}^{-1}\Big[S_2(b)\big(h(v)-h(v^*)\big)-\int_0^b S_2(b-\wp)\Big\{K\Big(\wp,v(\wp),v\big(\gamma_1(\wp)\big),...,$$

$$v\big(\gamma_q(\wp)\big),v'(\wp)\Big)-K\Big(\wp,v^*(\wp),v^*\big(\gamma_1(\wp)\big),...,v^*\big(\gamma_q(\wp)\big),v^{*'}(\wp)\Big)\Big\}d\wp$$

$$-\sum_{0<\varsigma_i<b}S_1(b-\varsigma_i)\Big(\ell_i\big(v(\varsigma_i)\big)-\ell_i\big(v^*(\varsigma_i)\big)\Big)$$

$$-\sum_{0<\varsigma_i<b}S_2(b-\varsigma_i)\Big(\bar{\ell}_i\big(v'(\varsigma_i^+)\big)-\bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\Big)\Big](\Phi)d\Phi\Big\|$$

$$\leq ML_0\|v-v^*\|_1+\int_0^{\xi}\|S_1(\xi-\Phi)\|F_0\Big(\|v(\Phi)-v^*(\Phi)\|+\|v(\gamma_1(\Phi))-v^*(\gamma_1(\Phi))\|+...$$

$$+\|v(\gamma_q(\Phi))-v^*(\gamma_q(\Phi))\|+\|v'(\Phi)-v^{*'}(\Phi)\|\Big)d\Phi$$

$$+\sum_{0<\varsigma_i<\xi}\|S_1'(\xi-\varsigma_i)\|\|\ell_i(v(\varsigma_i))-\ell_i(v^*(\varsigma_i))\|$$

$$+\sum_{0<\varsigma_i<\xi}\|S_1(\xi-\varsigma_i)\|\big\|\bar{\ell}_i\big(v'(\varsigma_i^+)\big)-\bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\big\|+\int_0^\xi\|S_1(\xi-\Phi)\|F_1\Big[ML_0\|v-v^*\|_1$$

$$+\int_0^b\|S_2(b-\wp)\|F_0\Big(\|v(\wp)-v^*(\wp)\|+\|v(\gamma_1(\wp))-v^*(\gamma_1(\wp))\|+...+\|v(\gamma_q(\wp))$$

$$-v^*(\gamma_q(\wp))\|+\|v'(\wp)-v^{*'}(\wp)\|\Big)d\wp+\sum_{0<\varsigma_i<b}\|S_1(b-\varsigma_i)\|\big\|\ell_i\big(v(\varsigma_i)\big)-\ell_i\big(v^*(\varsigma_i)\big)\big\|$$

$$+\sum_{0<\varsigma_i<b}\|S_2(b-\varsigma_i)\|\big\|\bar{\ell}_i\big(v'(\varsigma_i^+)\big)-\bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\big\|\Big]d\Phi$$

$$\leq ML_0\|v-v^*\|_1+MF_0\int_0^\xi(q+1)\|v-v^*\|_1d\Phi+\sum_{0<\varsigma_i<\xi}MF_i\|v(\varsigma_i)-v^*(\varsigma_i)\|$$

$$+\sum_{0<\varsigma_i<\xi}M\bar{F}_i\big\|\bar{\ell}_i\big(v'(\varsigma_i^+)\big)-\bar{\ell}_i\big(v^{*'}(\varsigma_i^+)\big)\big\|+\int_0^\xi MF_1\Big\{ML_0\|v-v^*\|_1+\int_0^b M(q+1)$$

$$\times F_0\|v-v^*\|d\wp+\sum_{0<\varsigma_i<b}MF_i\|v(\varsigma_i)-v^*(\varsigma_i)\|+\sum_{0<\varsigma_i<b}M\bar{F}_i\|v'(\varsigma_i^+)-v^{*'}(\varsigma_i^+)\|\Big\}d\Phi$$

$$\leq ML_0\|v-v^*\|_1+MF_0(q+1)b\|v-v^*\|_1+MF_i^*\|v-v^*\|_1+MF_i^*\|v-v^*\|_1+MF_1b$$

$$\Big\{ML_0\|v-v^*\|_1+MF_0(q+1)b\|v-v^*\|_1+MF_i^*\|v-v^*\|_1+MF_i^*\|v-v^*\|_1\Big\}$$

$$\leq M(1+MF_1b)[L_0+(q+1)F_0b+2F_i^*]\|v-v^*\|_1,\ \xi\in H.$$

Consequently,

$$\|\Theta v-\Theta v^*\|_1\leq 2M(1+MF_1b)[L_0+(q+1)F_0b+2F_i^*]\|v-v^*\|_1,$$

for $v,v^*\in PC^1(H,W)$.

Thus, in space $PC^1(H,W),\Theta$ is a contraction. Therefore, with the use of Banach contraction principle, only one fixed point of Θ is present there and this fixed point is the mild solution of ISFDE of second order (1) on H. Consequently, the system (1) is controllable on H. □

4. Example

To demonstrate our abstract theory, we take semilinear partial second order functional-differential equations of the following type:

$$\begin{cases} \frac{\partial}{\partial \xi}\left(\frac{\partial u(\xi,w)}{\partial \xi}\right) = \frac{\partial^2 u(\xi,w)}{\partial w^2} + \eta(\xi,w) \\ +C\Big(\xi, u(\xi,w), u(\alpha_1(\xi),w), ..., u(\alpha_q(\xi),w), \frac{\partial u(\xi,w)}{\partial \xi}\Big); w \in [0,\pi], \xi \in H, \end{cases} \tag{4}$$

subject to the conditions

$$u(\xi,0) = u(\xi,\pi) = 0, \xi \in H, \tag{5}$$

$$u(0,w) = u_0(w), \tag{6}$$

$$\frac{\partial u(0,w)}{\partial \xi} + \sum_{i=0}^{q} u(\xi_i,w) = u_0(w), 0 < \xi_1 < ... < \xi_q \leq b, w \in [0,\pi], \tag{7}$$

$$\Delta u(\xi_i)(w) = \int_0^{\xi_i} a_i(\xi_i - \Phi)u(\Phi,w)d\Phi, \tag{8}$$

$$\Delta u'(\xi_i)(w) = \int_0^{\xi_i} \bar{a}_i(\xi_i - \Phi)u(\Phi,w)d\Phi, \tag{9}$$

where $\eta(\xi,w) : H \times [0,\pi] \to [0,\pi]$ is continuous on $0 \leq w \leq \pi, \xi \in H$.

Let $W = L^2[0,\pi]$ and let $A : W \to W$ be expressed by $Au = u'', u \in dom(A)$. Here $dom(A) = \{u \in W : u, u'$ are absolutely continuous, $u'' \in W, u(0) = u(\pi) = 0\}$.

Subsequently $Au = \sum_{n=1}^{\infty} -p^2(u,u_p)u_p, u \in dom(A)$, where $u_p(\Phi) = \sqrt{\frac{2}{\pi}} \sin p\Phi,\ p = 1, 2, ...$ is the orthogonal set of eigenvalues of A. Explicitly, it is well known that A is the infinitesimal generator of cosine family $S_1(\xi), \xi \in H$, in W and it can be presented as

$$S_1(\xi)u = \sum_{n=1}^{\infty} \cos p\xi(u,u_p)u_p, u \in W.$$

The sine family that is associated to it, is presented by

$$S_2(\xi)u = \sum_{n=1}^{\infty} \frac{1}{p} \sin p\xi(u,u_p)u_p, u \in W.$$

Define the operator $K : H \times W^{q+2} \to W$ by

$$K\bigg(\Phi, v(\Phi), v\big(\gamma_1(\Phi)\big), ..., v\big(\gamma_q(\Phi)\big), v'(\Phi)\bigg)$$
$$= C\Big(\xi, u(\xi,w), u(\alpha_1(\xi),w), ..., u(\alpha_q(\xi),w), \frac{\partial u(\xi,w)}{\partial \xi}\Big)$$

Also, define the map $\ell_1, \bar{\ell}_i$ and G by

$$\ell_i(u)(w) = \int_0^{\pi} a_i(\Phi)u(\Phi,w)d\Phi,$$
$$\bar{\ell}_i(u)(w) = \int_0^{\pi} \bar{a}_i(\Phi)u(\Phi,w)d\Phi.$$

and satisfy the condition (A_1) and (A_2). Let $G : U \subset H \to W$ be expressed by $(Gy)(\xi)(w) = \eta(\xi,w), w \in (0,\pi)$ such that it satisfies condition (A_4). Then the above problem (4) – (9) can be formulated as (1). Then, all the assumptions presented in above theorem are fulfilled. So, the control system (4) – (9) is controllable on H.

In particular, we take $W = R^+, \xi \in [0,1]$ and so $b = 1$. Set

$$K(\xi, v(\xi), v(\gamma_1(\xi)), ..., v(\gamma_q(\xi)), v'(\xi)) = \frac{e^{-\xi}(v(\xi) + v'(\xi))}{(8 + e^{\xi})(1 + v(\xi) + v'(\xi))},$$
$$h(v) = \frac{v}{5+v},\ \gamma_i(\xi) = \xi, i = 1, 2, ..., q$$
$$\ell_i(v) = \frac{v}{6+v}, \bar{\ell}_i(v') = \frac{v'}{6+v'},$$

Let $v, r \in PC^1(H, W)$. Then, we have

$$\begin{aligned}\|K(\xi, v(\xi), v(\gamma_1(\xi)), ..., v(\gamma_q(\xi))v'(\xi)) - K(\xi, r(\xi), r(\gamma_1(\xi)), ..., r(\gamma_q(\xi))r'(\xi))\| \\ \leq \Big|\frac{e^{-\xi}}{(8+e^{\xi})}\Big|\Big\|\frac{v(\xi)+v'(\xi)}{(1+v(\xi)+v'(\xi))} - \frac{r(\xi)+r'(\xi)}{(1+r(\xi)+r'(\xi))}\Big\|, \\ \leq \frac{1}{9}\Big\|\frac{v(\xi)+v'(\xi)-(r(\xi)+r'(\xi))}{(1+v(\xi)+v'(\xi))(1+r(\xi)+r'(\xi))}\Big\|, \\ \leq \frac{1}{9}\|[v(\xi)-r(\xi)] + [v'(\xi)-r'(\xi)]\|, \\ \leq \frac{1}{9}\big[\|v(\xi)-r(\xi)\| + \|v'(\xi)-r'(\xi)\|\big].\end{aligned}$$

Hence, the assumption (A_1) holds with $F_0 = \frac{1}{9}$.

Now,

$$\|h(v) - h(r)\| = \Big\|\frac{v}{5+v} - \frac{r}{5+r}\Big\| = \Big\|\frac{5v-5r}{(5+v)(5+r)}\Big\| \leq \frac{1}{5}\|v-r\|.$$

Hence, the assumption (A_3) holds with $L_0 = \frac{1}{5}$.

Further,

$$\|\ell_i(v) - \ell_i(r)\| = \Big\|\frac{v}{6+v} - \frac{r}{6+r}\Big\| = \Big\|\frac{6v-6r}{(6+v)(6+r)}\Big\| \leq \frac{1}{6}\|v-r\|.$$

and

$$\|\bar{\ell}_i(v) - \bar{\ell}_i(r)\| = \Big\|\frac{v}{6+v} - \frac{r}{6+r}\Big\| = \Big\|\frac{6v-6r}{(6+v)(6+r)}\Big\| \leq \frac{1}{6}\|v-r\|.$$

Hence, the assumption (A_2) holds with $F_i = \bar{F}_i = \frac{1}{6}$.

Consider that the linear operator $Q : L^2(H, U) \to W$ defined by

$$Qy = \int_0^b S_2(b-\Phi)Gy(\Phi)d\Phi.$$

has an induced inverse operator $\tilde{Q}^{-1}$, which takes values in $L^2(H, U)/KerQ$. Taking $F_0 = \frac{1}{9}, L_0 = \frac{1}{5}, F_1 = 1, F_i = \bar{F}_i = \frac{1}{6}$ and by the choice of q, where $q \in \mathbb{N}$ and M, the inequality $2M(1+MF_1b)[L_0+(q+1)F_0b+2F_i^*] < 1$ can be satisfied. Thus the assumption (A_5) holds. Hence, all the assumptions of Theorem 3.1 are fulfilled. Therefore, the control system (1) is controllable on H.

5. Conclusions

We thus investigated the sufficient conditions for controllability of ISFDE of second order along with nonlocal condition in Banach spaces. To establish the result, we have implemented the Banach contraction principle and the concept of SCCF of linear operators. At last, we are presented an example to exhibit our abstract outcome and verify all the assumptions by taking a particular case. Moreover, the result which is derived in this manuscript is only theoretical. In future, numerical solution of equation (1) and (4) - (9) may also be derived. Moreover, the theory which is presented in this paper can be applied for second order fractional differential systems as well as for second order fractional delay differential systems with impulsive condition.

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