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ON THE BI-EXTREMAL BI-STABILIZATION OF A COOPERATIVE GAME IN PRODUCT SPACE $N_1 \times N_2$

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Abstract. In this research paper, we aim to expand the scope of cooperative game theory from its traditional domain in N to the more versatile cartesian product $N_1 \times N_2$. This extension allows us to explore the intriguing possibility of players collaborating in multiple games simultaneously. Within this context, we introduce several fundamental concepts in game theory that apply to the cartesian product $N_1 \times N_2$, such as coalition, cooperative games, core solution, and other essential notions. Additionally, we present the innovative bi-extremal bi-stabilization algorithm, a powerful computational tool designed to address maximization problems within the cartesian product $N_1 \times N_2$.

Keywords: Game theory, cooperative game, linear programming, algorithm, bi-stabilization.

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1. Introduction

Game theory comprises a collection of analytical tools designed to enhance comprehension in situations involving interactions among decision-makers, referred to as players. Initially conceptualized by von Neumann and Oskar Morgenstern [36], this framework addresses scenarios where players act autonomously, necessitating the management of their interactions, which may involve cooperation, competition, or both [6, 8, 9, 18, 19, 28]. This paper specifically focuses on the first category, namely cooperative games.

Cooperation is a fundamental and natural notion in our planet. For example, during the world wars, several countries formed coalitions to strengthen their military and political power. The researchers are collaborating to achieve strong and efficient results. Companies may also form agreements and collaborations with each other to ensure the production of high-quality products within an optimized timeframe [1, 21, 22, 37, 38]. A cooperative game on N is one in which each player will seek partners whose combined

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action allows him to win more than by himself, (i.e. the players play together to achieve a common goal without any competitive spirit). Numerous articles and books in the literature discuss cooperative games (For example: [4, 15, 17, 20, 23, 25]). In addition to its theoretical studies explored, there are different applications of cooperative games in economics, physics, mathematical finance, political science, computer science, mathematical programming, etc [5, 13, 16, 24, 26, 30, 31, 32, 33].

All the preceding theoretical studies and applications have primarily focused on traditional cooperative games within the set N. However, there is untapped potential in extending these investigations to higher dimensions, promising a novel perspective on game theory and enhancing its utility. Notably, in the year 2000, Bilbao et al. [2] introduced a groundbreaking category of cooperative games known as bi-cooperative games. This concept serves as an expansive generalization of traditional cooperative games, encompassing scenarios where each player may contribute positively, negatively, or not at all. The innovation lies in the coalition function's definition over pairs (S, T) of disjoint coalitions, where members of S act as positive contributors, and those in T act as negative contributors. Bilbao et al. departed from the convention of functions defined from 2^N to $\mathbb{R}$ in traditional cooperative games, opting instead for functions defined from 3^N to $\mathbb{R}$. In this context, our analysis takes a distinctive approach. We explore cooperative scenarios where players make decisions on the cartesian product of two sets, establishing our functions from $2^{N_1\times N_2}$ to $\mathbb{R}$. The overarching objective is to extend the notion of cooperative games to the cartesian product $N_1 \times N_2$, presenting an expansion of cooperative game concepts beyond the confines of the original set N. In other words, in our case, players can make coalitions not only in one game but they can make coalitions in two games at the same time. So, we will move from the set N to the cartesian product $N_1 \times N_2$ and the goal will always be the maximization of the gains.

Mathematical programming, particularly linear programming, which constitutes a substantial portion of operational research, has evolved parallel to game theory. Both fields were shaped by the challenges posed by economic problems, specifically those involving the optimization of objectives, whether it be maximization or minimization [11, 12, 17, 34, 35], and the main goal is the possibility of solving them as quickly as possible. In 2020, El Kamli et al. [10] introduced the extremal stabilization algorithm. They first established a first-generation algorithm which is time efficient when it succeeds, then they established a second-generation algorithm, which always converges in the case of a non-empty core. In this paper, we directly extend the second-generation algorithm. We use the polars of the bi-stable cooperative game v and a fundamental function noted $v_{C_1\times C_2}$ (which is the smallest cooperative game majorant v exact and admitting a node in $C_1 \times C_2$) for the bi-stabilization of the algorithm (paragraphs 2.2 and 2.3), and we highlight a family of simple inequalities necessary for the non-vacuity of the core (remark 3.2). When the cardinality of $N_1 \times N_2$ is relatively small, all methods converge rapidly. However, as the cardinality of $N_1 \times N_2$ grows, we encounter a challenge related to memory size. For instance, the bi-extremal bi-stabilization algorithm occupies a memory space on the order of 2^{nm}, whereas the simplex method utilizes memory space at a scale of $nm2^{nm}$, with $|N_1 \times N_2| = nm$.

This paper is divided into four sections. In the next section, we will define several notions of game theory on the cartesian product $N_1 \times N_2$ instead of the set N and we will discuss some of their basic properties. In section 3, we will establish the bi-extremal bi-stabilization algorithm that we can use to solve maximization problems in the product space $N_1 \times N_2$. And we conclude in the last section.

2. Results and notations

This section will introduce various concepts in game theory within $N_1 \times N_2$, encompassing definitions of key terms like coalitions and core. Additionally, we will establish the foundations of cooperative games, convex games, constant-sum games, and more.

2.1. Definitions.

In the forthcoming discussion, we take into account two sets, N_1 and N_2, each characterized by cardinalities of n and m, respectively [29].

Definition 2.1. ***(coalition)***

A "coalition" $A_1 \times A_2$ is defined as a subset of $N_1 \times N_2$, and the set of all coalitions is quite simply the set $2^{N_1 \times N_2}$ of cardinal 2^{nm}.

As a convention, we may refer to the empty set as a coalition, which we will term "the empty coalition." Additionally, the set $N_1 \times N_2$ is recognized as a coalition and is denoted as "the grand coalition".

Definition 2.2. ***(cooperative game)***

A "cooperative game" on $N_1 \times N_2$ is defined by a finite set of pairs of players $N_1 \times N_2 = \{(a_i; a'_j); a_i \in N_1; a'_j \in N_2 \text{ and } (i;j) \in \{1;2;...;n\} \times \{1;2;...;m\}\}$, and a real-valued v, defined on all subsets of $N_1 \times N_2$, (with $v(\emptyset) = 0$).

Example: (The gloves example).

Consider two sellers, denoted as A and B. Seller A possesses a left glove, which can be sold independently for 8 MAD (i.e. $v(A_1) = 8$), while seller B has a right glove, valued at 12 MAD (i.e. $v(B_1) = 12$). When these sellers collaborate to form a pair of gloves, their joint revenue amounts to 50 MAD (i.e. $v(A_1 \sqcup B_1) = 50$). This serves as a basic illustration of a cooperative game within the set N.

Now, let's extend the scenario to include another type of gloves, labeled as Type 2. Seller A can sell the left glove Type 2 for 11 MAD, and seller B can sell the right glove Type 2 for 10 MAD. When cooperating, they can generate a total revenue of 40 MAD. In this expanded context, the valuations are as follows: $v(A_1 \times A_2) = 19$, $v(B_1 \times B_2) = 22$, and $v[(A_1 \times A_2) \sqcup (B_1 \times B_2)] = 90$. This example illustrates a cooperative game within the cartesian product set $N_1 \times N_2$.

Remark 2.1.

Drawing an analogy to von Neumann and Morgenstern's work in Roth's book [27]*, we stipulate that v must be superadditive, i.e. if $A_1 \times A_2$ and $B_1 \times B_2$ are two disjoint subsets of $N_1 \times N_2$, then $v[(A_1 \times A_2) \sqcup (B_1 \times B_2)] \geq v(A_1 \times A_2) + v(B_1 \times B_2)$, where $(A_1 \times A_2) \sqcup (B_1 \times B_2)$ designates the disjoint reunion of $(A_1 \times A_2)$ and $(B_1 \times B_2)$. This implies that the value of the coalition $(A_1 \times A_2) \sqcup (B_1 \times B_2)$ is no less than the aggregate value of its individual parts acting independently.*

Definition 2.3. ***(dual game)***

A "dual game" of the cooperative game v is the function denoted by $v^\times$ defined by:

$$\begin{aligned} v^\times : \quad 2^{N_1 \times N_2} &\longrightarrow \mathbb{R} \\ (A_1 \times A_2) &\longmapsto v(N_1 \times N_2) - v[(A_1 \times A_2)^c] \end{aligned}$$

Corollary 2.1. *Let v and u be two cooperative games. We have the following properties:*

1. *v is an increasing function on $2^{N_1 \times N_2}$.*
2. *$v^\times$ is an increasing function on $2^{N_1 \times N_2}$.*
3. *$\forall (A_1 \times A_2) \in 2^{N_1 \times N_2}$, $v(A_1 \times A_2) \leq v^\times(A_1 \times A_2)$.*

(4) *If* $v(N_1 \times N_2) = u(N_1 \times N_2)$*, then,* $v \leq u \Longrightarrow u^\times \leq v^\times$.

Proof.
Let $(A_1 \times A_2), (B_1 \times B_2) \in 2^{N_1 \times N_2}$.

(1) If $A_1 \times A_2 \subset B_1 \times B_2$, then, $B_1 \times B_2 = (A_1 \times A_2) \sqcup (B_1 \times B_2 \setminus A_1 \times A_2)$ and $(A_1 \times A_2) \cap (B_1 \times B_2 \setminus A_1 \times A_2) = \emptyset$, [3, 14], therefore

$$\begin{aligned} v(B_1 \times B_2) &= v[(A_1 \times A_2) \sqcup (B_1 \times B_2 \setminus A_1 \times A_2)] \\ &\geq v(A_1 \times A_2) + v(B_1 \times B_2 \setminus A_1 \times A_2) \\ &\geq v(A_1 \times A_2). \end{aligned}$$

Since we have: $A_1 \times A_2 \subset B_1 \times B_2 \Longrightarrow v(A_1 \times A_2) \leq v(B_1 \times B_2)$, then, v is an increasing function.

(2) We suppose that $A_1 \times A_2 \subset B_1 \times B_2$.

$$\begin{aligned} v^\times(A_1 \times A_2) &= v(N_1 \times N_2) - v[(A_1 \times A_2)^c] \\ &\leq v(N_1 \times N_2) - v[(B_1 \times B_2)^c] \\ &\leq v^\times(B_1 \times B_2). \end{aligned}$$

Hence $v^\times$ is an increasing function.

(3) For all $(A_1 \times A_2)$ in $2^{N_1 \times N_2}$, we have:

$$\begin{aligned} v^\times(A_1 \times A_2) &= v(N_1 \times N_2) - v[(A_1 \times A_2)^c] \\ &= v[(A_1 \times A_2) \sqcup (A_1 \times A_2)^c] - v[(A_1 \times A_2)^c] \\ &\geq v(A_1 \times A_2) + v[(A_1 \times A_2)^c] - v[(A_1 \times A_2)^c] \\ &\geq v(A_1 \times A_2). \end{aligned}$$

(4) We suppose that: $v(N_1 \times N_2) = u(N_1 \times N_2)$, and that: $v \leq u$, then:

$$\begin{aligned} v^\times(A_1 \times A_2) &= v(N_1 \times N_2) - v[(A_1 \times A_2)^c] \\ &\geq u(N_1 \times N_2) - u(A_1 \times A_2)^c \\ &\geq u^\times(A_1 \times A_2). \end{aligned}$$

Hence the result.

Definition 2.4. *Let* v *be a cooperative game.*

(1) v *is said "constant-sum" if:*

$$\forall (A_1 \times A_2) \in 2^{N_1 \times N_2},\ v^\times(A_1 \times A_2) = v(A_1 \times A_2).$$

(2) v *is said "bi-stable" if its dual* $v^\times$ *is sub-additive on* $2^{N_1 \times N_2}$*, then, we have:*

$$(A_1 \times A_2) \cap (B_1 \times B_2) = \emptyset \Longrightarrow v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)] \leq v^\times(A_1 \times A_2) + v^\times(B_1 \times B_2).$$

Remark 2.2. *A game* v *characterized by constant-sum, bistability, and cooperation can be categorized as a probability.*

Indeed, if v is a constant-sum, bi-stable, and cooperative game with the condition $v(N \times N') = 1$, it follows that v is both superadditive and subadditive, implying that v is additive.

($i.e.$ $(A_1 \times A_2) \cap (B_1 \times B_2) = \emptyset \Longrightarrow v[(A_1 \times A_2) \sqcup (B_1 \times B_2)] = v(A_1 \times A_2) + v(B_1 \times B_2)$.)

Definition 2.5. ***(convex game)***
We said that v *is "convex" if for all parts* $(A_1 \times A_2)$ *and* $(B_1 \times B_2)$ *in* $2^{N_1 \times N_2}$*, we have:*

$$v(A_1 \times A_2) + v(B_1 \times B_2) \leq v[(A_1 \times A_2) \cup (B_1 \times B_2)] + v[(A_1 \times A_2) \cap (B_1 \times B_2)].$$

Definition 2.6. ***(core)***
The "core" of the cooperative game v is the set C_v defined by:

$$C_v = \{P \, probability \; : \; v \leq P\}.$$

Remark 2.3. .

(1) *The elements of the core of v are the bi-stable and constant-sum majorants of v.*
(2) *In a convex game, the core is never empty.*

Definition 2.7. *We call "bi-stabilized closure" or "bi-stability" of v denoted by $\widehat{v}$, the smallest bi-stable, cooperative game majorant of v if it exists.*

2.2. **Polar and bipolar of a cooperative game.**

In this part, we broaden the scope of the polarity concept within the framework of $N \times N'$. The inclusion of this fundamental concept is crucial for establishing the bi-stabilization of the algorithm [7, 10].

Definition 2.8. *Let v be a cooperative game and let $v^\times$ its dual.*

- *The "polar" of v denoted by v^*, is the sub-additive lower bound of $v^\times$, such that: $v^* = \underline{v^\times}$.*
- *If $v^*(N_1 \times N_2) = 1$, the "bipolar" of v denoted by v^{**}, is the super-additive upper bound of $v^{*\times}$, (i.e. $v^{**} = \overline{v^{*\times}}$).*

By recurrence we define $v^{(2n+1)^*}$ and $v^{(2n+2)^*}$ as following:

- For all $k \leq n$, if $v^{(2k)^*}(N_1 \times N_2) = 1$, then, $v^{(2n+1)^*} := (v^{(2n)^*})^* = \underline{v^{(2n)^{*\times}}}$.
- And for all $k \leq n$, if $v^{(2k+1)^*}(N_1 \times N_2) = 1$, then, $v^{(2n+2)^*} := (v^{(2n+1)^*})^* = \overline{v^{(2n+1)^{*\times}}}$.

And since v is a cooperative game with a non-empty core, then for all probability P in C_v, we have:

$$v \leq P \leq v^\times.$$

Lemma 2.1. *Let v be a cooperative game and let $v^\times$ its dual.*

(1) *If v is bi-stable on $2^{N_1 \times N_2}$, then, the cooperative game v has for bi-stabilized $\widehat{v}$ such that, $\widehat{v} = v$.*
Otherwise, we have, $v < v^{*\times} \leq P \leq v^* < v^\times$.
(2) *If $v^{*\times}$ is a super-additive function on $2^{N_1 \times N_2}$, then the cooperative game v has for bi-stabilized, $\widehat{v} = v^{*\times}$.*
Otherwise, we have, $v < v^{*\times} < v^{**} \leq P \leq v^{**\times} < v^* < v^\times$.
(3) *In the same way we obtain by recurrence the following inequalities:*

$$v < v^{**} < ... < v^{(2p)^*} \leq P = v^{(2p+1)^*} < ... < v^* < v^\times.$$

Proof.

(1) We know that: $v \leq v$, so if v is bi-stable, then $\widehat{v} = v$.
Otherwise, v is not bi-stable then $v^\times$ is not a sub-additive function on $2^{N_1 \times N_2}$, then we have: $v \leq P \leq v^* < v^\times$.
Since, $P \leq v^* \Longrightarrow v^{*\times} \leq P^\times = P$, then we have: $v < v^{*\times} \leq P \leq v^* < v^\times$.
(2) If $v^{*\times}$ is a super-additive function, then $\widehat{v} = v^{*\times}$, because $v^{*\times\times} = v^*$ and v^* is sub-additive.
Otherwise, we have, $v < v^{*\times} < v^{**} \leq P \leq v^* < v^\times$, then:

$$v < v^{*\times} < v^{**} \leq P \leq v^{**\times} < v^* < v^\times.$$

Remark 2.4. *From above, we can deduce that:*

(1) $v^{(2p)^*}$ *is an increasing sequence.*
(2) $v^{(2p+1)^*}$ *is a decreasing sequence.*

Proposition 2.1. *(Sufficient conditions for the non-vacuity of the core)*
If the core is non-empty, then, the bi-stabilized closure of v *exists and has the same core as* v*.*

Proof.
If the core is non-empty, then it exists an element P such that: $v \leq P \leq v^{\times}$.
Therefore, the set of bi-stable, cooperative game majorant of v is non empty, since it contains P.
According to the previous lemma, we can deduce that the core of v is equal to the core of $\widehat{v}$.

Definition 2.9. *We call "bi-stabilization" of* v*, the operation of calculating* $v^{(2p)^*}$ *until the sequence becomes stationary. We have reached the bi-stabilized* v*.*

The question that arises is the following: Does $v^{(2p)^*}$ always become stationary from a certain rank? The answer to this question is the objective of the following properties.

Proposition 2.2. *(Necessary conditions for the non-vacuity of the core)*
(1) (NC_1), $v \leq v^* \Longleftrightarrow v(N_1 \times N_2) = v^*(N_1 \times N_2)$.
(2) (NC_2), $v^{**} \leq v^* \Longleftrightarrow v(N_1 \times N_2) = v^{**}(N_1 \times N_2)$.
In the same way, we obtain (CN_{2k-1}) *and* (CN_{2k})*.*

Proof.
$\Longrightarrow$) This implication is evident.
Indeed, if $v \leq v^*$, then $v(N_1 \times N_2) \leq v^*(N_1 \times N_2) \leq v^{\times}(N_1 \times N_2) = v(N_1 \times N_2)$.
$\Longleftarrow$) By contraposed, we suppose that there exists a part $A_1 \times A_2$ of $N_1 \times N_2$ such that: $v(A_1 \times A_2) > v^*(A_1 \times A_2)$, then, $v(A_1 \times A_2) + v^{\times}[(A_1 \times A_2)^c] > v^*(A_1 \times A_2) + v^{\times}[(A_1 \times A_2)^c]$.
Since $v(A_1 \times A_2) + v^{\times}[(A_1 \times A_2)^c] = v(N_1 \times N_2)$,
and $v^*(A_1 \times A_2) + v^{\times}[(A_1 \times A_2)^c] \geq v^*(A_1 \times A_2) + v^*(A_1 \times A_2)^c \geq v^*(N_1 \times N_2)$, then we have, $v(N_1 \times N_2) > v^*(N_1 \times N_2)$. Hence the result.
In the same way, we establish the other conditions.

Remark 2.5. .
(1) *If there exists an integer* q *such as,* $v^{q^*}(N_1 \times N_2) \neq v(N_1 \times N_2)$*, then the core is empty.*
(2) *In practice, we calculate the successive polars of* v*, as long as:* $v^{(2p)^*} \neq v^{(2p-1)^*}$*, with* $v^{(2p)^*}(N_1 \times N_2) = v(N_1 \times N_2)$*.*

2.3. Nodes of a bi-stable cooperative game.

Definition 2.10. *We call "node" of a bi-stable cooperative game* v *any element* $C_1 \times C_2$ *of* $2^{N_1 \times N_2}$ *such that:* $v(C_1 \times C_2) = v^{\times}(C_1 \times C_2)$*.*

Remark 2.6. *If* v *is a cooperative constant-sum game, so any element* $C_1 \times C_2$ *of* $2^{N_1 \times N_2}$ *is a node of* v*.*

The following lemma will be useful to deduce characteristic properties of the nodes of a cooperative game.

Lemma 2.2.
(1) v *is super-additive if and only if, for all disjoint elements* $A \times A'$ *and* $B \times B'$ *of* $2^{N \times N'}$*, we have:* $v(A \times A') + v^{\times}(B \times B') \leq v^{\times}[(A \times A') \sqcup (B \times B')]$*.*

(2) *$v^\times$ is sub-additive if and only if, for all disjoint elements $A \times A'$ and $B \times B'$ of $2^{N \times N'}$, we have: $v[(A \times A') \sqcup (B \times B')] \leq v(A \times A') + v^\times(B \times B')$.*

Proof.

(1) We show that:
$$v(A_1 \times A_2) + v(B_1 \times B_2) \leq v[(A_1 \times A_2) \sqcup (B_1 \times B_2)]$$
$$\iff v(A_1 \times A_2) + v^\times(B_1 \times B_2) \leq v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)].$$
$\Longrightarrow$) Suppose that: $v(A_1 \times A_2) + v(B_1 \times B_2) \leq v[(A_1 \times A_2) \sqcup (B_1 \times B_2)]$.
We know that: $v^\times[(A_1 \times A_2) \sqcup (B1 \times B_2)] = v(N_1 \times N_2) - v[(A_1 \times A_2) \sqcup (B_1 \times B_2)]^c$ and $v^\times(B_1 \times B_2) = v(N_1 \times N_2) - v[(B_1 \times B_2)^c]$ then we have:
$$\begin{aligned} &v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)] - v(A_1 \times A_2) - v^\times(B_1 \times B_2) \\ &= -\{v[(A_1 \times A_2) \sqcup (B_1 \times B_2)]^c + v(A_1 \times A_2)\} + v[(B_1 \times B_2)^c] \\ &= -\{v[(A_1 \times A_2)^c \cap (B_1 \times B_2)^c] + v(A_1 \times A_2)\} + v[(B_1 \times B_2)^c] \\ &\geq -v\{[(A_1 \times A_2)^c \cap (B_1 \times B_2)^c] \sqcup (A_1 \times A_2)\} + v[(B_1 \times B_2)^c] \\ &\geq -v[(B_1 \times B_2)^c] + v[(B_1 \times B_2)^c] = 0, \end{aligned}$$
because $(A_1 \times A_2) \cap (B_1 \times B_2) = \emptyset$. And consequently, we have:
$$v(A_1 \times A_2) + v^\times(B_1 \times B_2) \leq v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)].$$
$\Longleftarrow$) Reciprocally, suppose that:
$v(A_1 \times A_2) + v^\times(B_1 \times B_2) \leq v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)]$, then, we have:
$$\begin{aligned} &v[(A_1 \times A_2) \sqcup (B_1 \times B_2)] - v(A_1 \times A_2) - v(B_1 \times B_2) \\ &= v(N_1 \times N_2) - v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)]^c - v(A_1 \times A_2) - v(B_1 \times B_2) \\ &= -\{v^\times[(A_1 \times A_2) \sqcup (B_1 \times B_2)]^c + v(A_1 \times A_2)\} + v(N_1 \times N_2) - v(B_1 \times B_2) \\ &\geq -v^\times[(B_1 \times B_2)^c] + v^\times[(B_1 \times B_2)^c] = 0. \end{aligned}$$
Hence, we have: $v(A_1 \times A_2) + v(B_1 \times B_2) \leq v[(A_1 \times A_2) \sqcup (B_1 \times B_2)]$.

(2) In the same way we establish the second proposition.

Using the previous lemma, it is easy to verify the following result.

Corollary 2.2. *Let v be a bi-stable cooperative game and $C_1 \times C_2$ an element of $2^{N_1 \times N_2}$. Then, the following properties are equivalent:*

1. *$C_1 \times C_2$ is a node of v.*
2. *$v(C_1 \times C_2) + v[(C_1 \times C_2)^c] = 1$.*
3. *For all element $A_1 \times A_2$ of $2^{N_1 \times N_2}$, if $(A_1 \times A_2) \cap (C_1 \times C_2) = \emptyset$, then we have: $v[(A_1 \times A_2) \sqcup (C_1 \times C_2)] = v(A_1 \times A_2) + v(C_1 \times C_2)$.*
4. *For all element $A_1 \times A_2$ of $2^{N_1 \times N_2}$, if $(A_1 \times A_2) \cap (C_1 \times C_2) = \emptyset$, then we have: $v^\times[(A_1 \times A_2) \sqcup (C_1 \times C_2)] = v^\times(A_1 \times A_2) + v^\times(C_1 \times C_2)$.*

Remark 2.7. *From the equivalent "(1) $\iff$ (2)", we can deduce that the complement of a node is also a node, and any element P of the core of v must pass through the value $P(C \times C') = v(C \times C')$, and this justifies the name of the node.*

Definition 2.11. *Let v be a bi-stable cooperative game and $C \times C'$ an element of $2^{N \times N'}$. Let v' a cooperative game such that: $v \leq v'$.*
We say that v' is a "exact majorante" in $C \times C'$ if, $v(C \times C') = v'(C \times C')$. We also say that $C \times C'$ is a contact point of v' with v.

Remark 2.8. *$N_1 \times N_2$ is always a contact point of v' with v, so, we have:*
$$v'(N_1 \times N_2) = v'^\times(N_1 \times N_2) \leq v^\times(N_1 \times N_2) = v(N_1 \times N_2).$$

This remark is immediate but it is important to show the following result.

Corollary 2.3. *Let v be a bi-stable cooperative game and $C_1 \times C_2$ an element of $2^{N_1 \times N_2}$. We consider the function $v_{C_1 \times C_2}$ defined by:*

$$v_{C_1\times C_2}(A_1 \times A_2) = \begin{cases} v^{\times}[(C_1 \times C_2)^c] + v(A_1 \times A_2 \cap C_1 \times C_2)\,; & \textit{if } (A_1 \times A_2) \cup (C_1 \times C_2) = N_1 \times N_2 \\ v(A_1 \times A_2) & ;\ \textit{if } (A_1 \times A_2) \cup (C_1 \times C_2) \neq N_1 \times N_2 \end{cases}$$

$v_{C_1\times C_2}$ is the smallest cooperative game majorant v exact and admitting a node in $C_1 \times C_2$.

Proof.

First, we show that: $v \leq v_{C_1\times C_2}$.
Let $A_1 \times A_2$ an element of $2^{N_1\times N_2}$.
a) If $(A_1 \times A_2) \cup (C_1 \times C_2) \neq N_1 \times N_2$, then, $v_{C_1\times C_2}(A_1 \times A_2) = v(A_1 \times A_2)$, hence, $v(A_1 \times A_2) \leq v_{C_1\times C_2}(A_1 \times A_2)$.
b) If $(A_1 \times A_2) \cup (C_1 \times C_2) = N_1 \times N_2$, then, $(C_1 \times C_2)^c \subset (A_1 \times A_2)$ [3, 14].
Hence, $v(A_1 \times A_2) = v[(C_1 \times C_2)^c \sqcup (A_1 \times A_2 \cap C_1 \times C_2)]$, so, according to the lemma 2.2, we have: $v(A_1 \times A_2) \leq v^{\times}[(C_1 \times C_2)^c] + v(A_1 \times A_2 \cap C_1 \times C_2)$, then,
$v(A_1 \times A_2) \leq v_{C_1\times C_2}(A_1 \times A_2)$, therefore, $v \leq v_{C_1\times C_2}$.

We show that: $v_{C_1\times C_2}$ is a cooperative game.
Let, $A_1 \times A_2$ and $B_1 \times B_2$ be two elements of $2^{N_1\times N_2}$, such that: $(A_1 \times A_2) \cap (B_1 \times B_2) = \emptyset$.
a) If $(A_1 \times A_2) \cup (C_1 \times C_2) \neq N_1 \times N_2$ and $(B_1 \times B_2) \cup (C_1 \times C_2) \neq N_1 \times N_2$, then,

$$\begin{aligned} v_{C_1\times C_2}(A_1 \times A_2) + v_{C_1\times C_2}(B_1 \times B_2) &= v(A_1 \times A_2) + v(B_1 \times B_2) \\ &\leq v(A_1 \times A_2 \sqcup B_1 \times B_2), (v \text{ is super-additive}) \\ &\leq v_{C_1\times C_2}(A_1 \times A_2 \sqcup B_1 \times B_2), (v \leq v_{C_1\times C_2}) \end{aligned}$$

b) We suppose that: $(A_1 \times A_2) \cup (C_1 \times C_2) = N_1 \times N_2$ $\overline{or}$ $(B_1 \times B_2) \cup (C_1 \times C_2) = N_1 \times N_2$.
In this case, the or (that we denote $\overline{or}$) is exclusive, because $(A_1 \times A_2) \cap (B_1 \times B_2) = \emptyset$.
If $(A_1 \times A_2) \cup (C_1 \times C_2) = N_1 \times N_2$, so, $(B_1 \times B_2) \subset (A_1 \times A_2)^c \subset (C_1 \times C_2)$, [3, 14], then,

$$\begin{aligned} & v_{C_1\times C_2}(A_1 \times A_2) + v_{C_1\times C_2}(B_1 \times B_2) \\ &= v^{\times}[(C_1 \times C_2)^c] + v(A_1 \times A_2 \cap C_1 \times C_2) + v(B_1 \times B_2) \\ &\leq v^{\times}[(C_1 \times C_2)^c] + v[(A_1 \times A_2 \cap C_1 \times C_2) \sqcup B_1 \times B_2] \\ &\leq v^{\times}[(C_1 \times C_2)^c] + v[(A_1 \times A_2 \sqcup B_1 \times B_2) \cap (C_1 \times C_2 \cup B_1 \times B_2)] \\ &\leq v^{\times}[(C_1 \times C_2)^c] + v[(A_1 \times A_2 \sqcup B_1 \times B_2) \cap C_1 \times C_2] \\ &\leq v_{C_1\times C_2}(A_1 \times A_2 \sqcup B_1 \times B_2). \end{aligned}$$

Therefore, $v_{C_1\times C_2}$ is a cooperative game.

Now we show that: $v_{C_1\times C_2}$ is exact in $C_1 \times C_2$.
a) If $C_1 \times C_2 = N_1 \times N_2$, then, according to the previous remark
$v_{C_1\times C_2}(C_1 \times C_2) = v(C_1 \times C_2)$.
b) Otherwise, i.e. $(C_1 \times C_2) \cup (C_1 \times C_2) \neq N_1 \times N_2$, then by definition of $v_{C_1\times C_2}$ we have: $v_{C_1\times C_2}(C_1 \times C_2) = v(C_1 \times C_2)$. As a result $v_{C_1\times C_2}$ is exact in $C_1 \times C_2$.

Now, we show that: $v_{C_1\times C_2}$ admits a node in $C_1 \times C_2$,

$$\begin{aligned} v^{\times}_{C_1\times C_2}(C_1 \times C_2) &= v_{C_1\times C_2}(N_1 \times N_2) - v_{C_1\times C_2}[(C_1 \times C_2)^c] \\ &= v(N_1 \times N_2) - v^{\times}[(C_1 \times C_2)^c] - v[(C_1 \times C_2)^c \cap (C_1 \times C_2)] \\ &= v(N_1 \times N_2) - v^{\times}[(C_1 \times C_2)^c] \\ &= v(C_1 \times C_2) \\ &= v_{C_1\times C_2}(C_1 \times C_2), \text{ since } v_{C_1\times C_2} \text{is exact in } C_1 \times C_2. \end{aligned}$$

Hence, $v_{C_1\times C_2}$ admits a node in $C_1 \times C_2$.

Finally, we show that: $v_{C_1\times C_2}$ is the smallest game majoring v.
Let v' be a cooperative game majoring v exact in $C_1 \times C_2$ admitting a node in $C_1 \times C_2$.

a) If $(A_1 \times A_2) \cup (C_1 \times C_2) \neq N_1 \times N_2$, then, $v_{C_1 \times C_2}(A_1 \times A_2) = v(A_1 \times A_2) \leq v'(A_1 \times A_2)$.
b) If $(A_1 \times A_2) \cup (C_1 \times C_2) = N_1 \times N_2$, then,

$$\begin{aligned}
v'(A_1 \times A_2) &\geq v'[(C_1 \times C_2)^c] + v'[(A_1 \times A_2) \cap (C_1 \times C_2)] \\
&\geq v'(N_1 \times N_2) - v'^{\times}(C_1 \times C_2) + v[(A_1 \times A_2) \cap (C_1 \times C_2)] \\
&\geq v(N_1 \times N_2) - v'(C_1 \times C_2) + v[(A_1 \times A_2) \cap (C_1 \times C_2)] \\
&\geq v(N_1 \times N_2) - v(C_1 \times C_2) + v[(A_1 \times A_2) \cap (C_1 \times C_2)] \\
&\geq v^{\times}[(C_1 \times C_2)^c] + v[(A_1 \times A_2) \cap (C_1 \times C_2)] \\
&\geq v_{C_1 \times C_2}(A_1 \times A_2).
\end{aligned}$$

Hence the result.

3. Bi-extremal bi-stabilization algorithm

In this section, we present an algorithm that involves addressing the maximization problem in two distinct games concurrently.

3.1. **Definitions and propositions.**

Definition 3.1. *Let* $C_1 \times C_2$ *be an element of* $2^{N_1 \times N_2}$.
$C_1 \times C_2$ *is a "regular point" of* v *if* $v_{C_1 \times C_2}$ *stabilizes. We noted by* $\widehat{v}_{C_1 \times C_2}$ *its bi-stabilized.*

Corollary 3.1. *Let* v *be a bi-stable, cooperative game and* N_v *be the set of its nodes.*

(1) *If* $C_1 \times C_2$ *is a regular point for* v*, then,* $N_v \subset N_{v_{C_1 \times C_2}} \subset N_{\widehat{v}_{C_1 \times C_2}}$.
(2) *For all* $S_1 \times S_2$ *of* N_v*,* $v(S_1 \times S_2) = v_{C_1 \times C_2}(S_1 \times S_2) = \widehat{v}_{C_1 \times C_2}(S_1 \times S_2)$.

Proof.

(1) First, we show that: $N_v \subset N_{v_{C \times C'}}$.
Let $S \times S'$ be an element of N_v, then: $v(S \times S') = v^{\times}(S \times S')$.
Let us show that: $S \times S' \in N_{v_{C \times C'}}$.
We know that: $v \leq v_{C \times C'}$, so $v^{\times}_{C \times C'} \leq v^{\times}$, then,

$$\forall S \times S' \in N_v, v^{\times}_{C \times C'}(S \times S') \leq v^{\times}(S \times S') = v(S \times S') \leq v_{C \times C'}(S \times S').$$

Since $v_{C \times C'}$ is superadditive, then, for all $S \times S' \in N_v$:

$$v_{C \times C'}(S \times S') \leq v^{\times}_{C \times C'}(S \times S').$$

Consequently, $v^{\times}_{C \times C'}(S \times S') = v_{C \times C'}(S \times S')$, i.e. $S \times S' \in N_{v_{C \times C'}}$.
Hence the result.
We follow the same process to establish $N_{v_{C \times C'}} \subset N_{\widehat{v}_{C \times C'}}$.

(2) Now, we show that, for all $S \times S' \in N_v$:

$$v(S \times S') = v_{C \times C'}(S \times S') = \widehat{v}_{C \times C'}(S \times S').$$

Let $S \times S'$ an element of N_v, then $v(S \times S') + v[(S \times S')^c] = 1$,
$v_{C \times C'}(S \times S') + v_{C \times C'}[(S \times S')^c] = 1$ and $\widehat{v}_{C \times C'}(S \times S') + \widehat{v}_{C \times C'}[(S \times S')^c] = 1$.
On the other hand, we know that, $v(S \times S') \leq v_{C \times C'}(S \times S') \leq \widehat{v}_{C \times C'}(S \times S')$ and $v[(S \times S')^c] \leq v_{C \times C'}[(S \times S')^c] \leq \widehat{v}_{C \times C'}[(S \times S')^c]$.
Then, we have: $v(S \times S') = v_{C \times C'}(S \times S') = \widehat{v}_{C \times C'}(S \times S')$.

Corollary 3.2. *Let* $C \times C'$ *be a regular point of* v*, and let* $\widehat{v}_{C \times C'}$ *be the bi-stability closure of the node* $C \times C'$*. The core of* $\widehat{v}_{C \times C'}$ *is defined as the set of elements within the core of* v *that are exact in* $C \times C'$.

Proof.
Let $P \in C_{\widehat{v}_{C\times C'}}$, then: $v \leq \widehat{v}_{C\times C'} \leq P$ and $P \in C_v$.
In addition to this, since $C \times C' \in N_{v_{C\times C'}} \subset N_{\widehat{v}_{C\times C'}}$ and according to the previous corollary, we have: $\widehat{v}_{C\times C'}(C \times C') = P(C \times C')$.
Reciprocally, if $P \in C_v$ such that: $v(C \times C') = P(C \times C')$, then,
$v^{\times}[(C_1 \times C_2)^c] = P[(C_1 \times C_2)^c]$, so for all $B_1 \times B_2$ such that: $(B_1 \times B_2) \cap (C_1 \times C_2)^c = \emptyset$,

$$\begin{aligned} v(B_1 \times B_2) + v^{\times}[(C_1 \times C_2)^c] &= v_{C_1\times C_2}[B_1 \times B_2 \sqcup (C_1 \times C_2)^c] \\ &\leq P[B_1 \times B_2 \sqcup (C_1 \times C_2)^c] \\ &\leq P(B_1 \times B_2) + P[(C_1 \times C_2)^c] \end{aligned}$$

and as a result: $v(B_1 \times B_2) \leq P(B_1 \times B_2)$ for all $B_1 \times B_2$, hence: $v \leq \widehat{v}_{C_1\times C_2} \leq P$ (i.e. $P \in C_{\widehat{v}_{C_1\times C_2}}$).

Definition 3.2. *We call "bi-extremal element" of the core of v, all point P of C_v which cannot be expressed as a convex combination of other points of C_v.*

Remark 3.1.

(1) *All the points of contact between v and a bi-extremal of the core are regular points.*
(2) *The progressive formation of an admissible base within the meaning of linear programming justifies the following denomination.*

Definition 3.3. *We call "admissible point" of v, all part $A_1 \times A_2$ of $N_1 \times N_2$ which is linearly independent of the nodes of v.*

3.2. **Diagram.**

Let v be a bi-stable cooperative game.
The algorithm is founded on the identification of a bi-extremal element within the core of v, assuming it is not empty. This process involves leveraging the function $v_{C_1\times C_2}$ and systematically constructing nodes at admissible points of v anticipated to exhibit regular behavior. The goal is to iteratively navigate through these nodes, ultimately determining a bi-extremal element within the core of v.

Step 1. Let $C_1^1 \times C_2^1$ be an admissible point of v.
We calculate $v_{C_1^1\times C_2^1}$ and we analyze the regularity of $C_1^1 \times C_2^1$ by calculating the bi-stability closure $\widehat{v}_{C_1^1\times C_2^1}$ of $v_{C_1^1\times C_2^1}$ if it exists.
We will have the following equality: $\widehat{v}_{C_1^1\times C_2^1}(C_1^1 \times C_2^1) = v(C_1^1 \times C_2^1)$ and we go to the second stage (by replacing the cooperative game v by the cooperative game $\widehat{v}_{C_1^1\times C_2^1}$). Otherwise, we change the admissible point until a regular point is obtained.

Step 2. Let $C_1^2 \times C_2^2$ be an admissible point of $v_{C_1^1\times C_2^1}$.
We calculate $v_{C_1^2\times C_2^2}$ and we stabilize it if $C_1^2 \times C_2^2$ is regular.
We will then have the following equality: $\widehat{v}_{C_1^2\times C_2^2}(C_1^i \times C_2^i) = v(C_1^i \times C_2^i)$ for all $i \in \{1; 2\}$ and we move on to the third stage (by replacing the cooperative game $\widehat{v}_{C_1^1\times C_2^1}$ by the cooperative game $\widehat{v}_{C_1^2\times C_2^2}$).
Otherwise, we change the admissible point until a regular point is obtained.
In the same way and on a recurring basis, we have:

(p+1)th step. Let $C_1^{(p+1)} \times C_2^{(p+1)}$ be an admissible point of $v_{C_1^p\times C_2^p}$.
We calculate $v_{C_1^{(p+1)}\times C_2^{(p+1)}}$ and we stabilize it, if $C_1^{(p+1)} \times C_2^{(p+1)}$ is regular, even if it means changing the admissible point.
We will have, $\widehat{v}_{C_1^{(p+1)}\times C_2^{(p+1)}}(C_1^i \times C_2^i) = v(C_1^i \times C_2^i)$ for all $i \in \{1; 2; 3; ...; p+1\}$.
And so the sequence continues, repeating in a similar manner.

Remark 3.2. *The efficiency of the algorithm is reduced by the fact that the regular points are only known after the bi-stabilization, it is therefore necessary to limit as much as possible the set of admissible points to be analyzed. We highlight a family of simple inequalities necessary for the non-vacuity of the core.*
Indeed, for all points $C_1 \times C_2$ *and* $D_1 \times D_2$ *and all additive* P*, we have:*

$$P(C_1 \times C_2) + P(D_1 \times D_2) = P[(C_1 \times C_2 \cap D_1 \times D_2) \cup (A_1 \times A_2)] + P[(C_1 \times C_2 \cap D_1 \times D_2) \cup (B_1 \times B_2)]$$

for all parties $A_1 \times A_2$ *and* $B_1 \times B_2$ *of* $(C_1 \times C_2)\Delta(D_1 \times D_2)$*. Therefore, if the cooperative game* v *bounded above by* P *with two points of contact* $C_1 \times C_2$ *and* $D_1 \times D_2$*, we have:*

$$v(C_1 \times C_2) + v(D_1 \times D_2) \geq v[(C_1 \times C_2 \cap D_1 \times D_2) \cup (A_1 \times A_2)] + v[(C_1 \times C_2 \cap D_1 \times D_2) \cup (B_1 \times B_2)].$$

If we apply this to a cooperative game v*,*

a) The preceding inequalities are always checked for any pair of nodes.

b) A non-node point $C_1 \times C_2$ *is a point which does not satisfy these inequalities with the node* $D_1 \times D_2$ *of* v *and an element* P *of node.*

The admissible candidates for additional nodes must therefore verify all these inequalities with any node of v. Hence the following definition.

Definition 3.4. *We define a point as a candidate at contact if it satisfies the preceding inequalities with all nodes of* v*.*

3.3. **Bi-extremal bi-stabilisation algorithm.**

```
Beginning
$\mathcal{B} \leftarrow$ Base    (nodes of $v$)
$\overline{v} \leftarrow v$
$\overline{A} \leftarrow \{A \times B\}$ ($A \times B$ admissible for $v$)
while $\overline{A} \neq \emptyset$ and $|\mathcal{B}| < nm$, make choose $C_1 \times C_2$ from $\overline{A}$
    If $v_{C_1 \times C_2}$ then stabilizes,
    $\mathcal{B} \leftarrow$ Base    (nodes of $v_{C_1 \times C_2}$)
    $\overline{A} \leftarrow \{A \times B\}$ [$A \times B$ admissible for $\widehat{v}_{C_1 \times C_2}$ and $\widehat{v}_{C_1 \times C_2}(A \times B) = v(A \times B)$]
    $\overline{v} \leftarrow \widehat{v}_{C_1 \times C_2}$
    Else $\overline{A} \leftarrow \overline{A} \setminus \{C_1 \times C_2\}$
    End if
End while
End.
```

Remark 3.3.

1. *This method is an algorithm because it ends in a finite number of stages (less than or equal to* 2^{nm} *with* $nm = |N_1 \times N_2|$*).*
   Indeed, at each iteration of while, the cardinal of $\overline{A}$ *decrease by at least 1 (if* $v_{C_1 \times C_2}$ *stabilize, then* $C_1 \times C_2$ *is no longer admissible for* $\widehat{v}_{C_1 \times C_2}$*).*
2. *The algorithm's difficulty and efficiency hinge on selecting the appropriate base* $\mathcal{B}$ *and choosing admissible points wisely.*
3. *For the bi-stabilization of the algorithm (which is a necessary condition of non-vacuity of the core), we used the polars of the bi-stable cooperative game* v *and the fundamental function* $v_{C_1 \times C_2}$ *which is the smallest cooperative game majorant* v *exact and admitting a node in* $C_1 \times C_2$*.*

Corollary 3.3. *If the algorithm gives a solution (probability)* P *from the core of* $\overline{v}$*, then this solution is a bi-extremal element of* C_v*.*

Proof.
To have a solution is to have, $|\mathcal{B}| = nm$, and since, $v \leq \overline{v} \leq P$, then, P is in C_v.
P is bi-extremal, since we have a base of contact points (i.e. the elements of $\mathcal{B}$).

4. Conclusion

In this paper, we expanded upon various concepts in game theory by transitioning from N to the cartesian product $N_1 \times N_2$, offering a novel perspective. Additionally, we introduced the bi-extremal bi-stabilization algorithm, a two-dimensional extension inspired by [10]. Drawing parallels with this previous work, we can conclude that if $nm = |N_1 \times N_2|$, the algorithm of the bi-extremal bi-stabilization of a cooperative game can solve in only 2^{nm} - instead of $nm2^{nm}$ for simplex - a system of the form:

$$\begin{cases} MaxP(N_1 \times N_2) \\ P(S_1 \times S_2) \leq v^{\times}(S_1 \times S_2) \quad S_1 \times S_2 \in 2^{N_1 \times N_2} \\ P \geq 0 \end{cases}$$

For minimization problems, we use the initial problem of this problem (dual):

$$\begin{cases} MinP(N_1 \times N_2) \\ P(S_1 \times S_2) \geq v(S_1 \times S_2) \quad S_1 \times S_2 \in 2^{N_1 \times N_2} \\ P \geq 0 \end{cases}$$

Consequently, we believe that the extensions of this type may have a wide application area, and may serve as a useful reference for further studies. While our paper has primarily focused on two-dimensional extension, further research can explore the possibilities of higher-dimensional extension.

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