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HARMONIC MEAN CORDIAL LABELING OF SOME WELL KNOWN GRAPHS

J. PAREJIYA[1*], P. T. LALCHANDANI[2], D. B. JANI[3], S. MUNDADIYA[4], §

Abstract. All the graphs considered in this article are simple and undirected. Let G = (V(G), E(G)) be a simple undirected Graph. A function $f : V(G) \to \{1, 2\}$ is called Harmonic Mean Cordial if the induced function $f^* : E(G) \to \{1, 2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1, 2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x. A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. In this article, we have discussed Harmonic Mean Cordial labeling of splitting graphs graph of some well known graphs.

Keywords:Harmonic Mean Cordial, Splitting graph, Corona Product, Path graph, Star Graph, Bistar Graph.

AMS Subject Classification: 05C78

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For basic terminology and notation not defined here we have followed Balakrishnan and Rangnathan [1]. In [5] J. Gowri and J. Jayapriya defined Harmonic Mean Cordial labeling of graph G. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f : V(G) \to \{1, 2\}$ is called *Harmonic Mean Cordial* if the induced function $f^* : E(G) \to \{1, 2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1, 2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number

[1] Governement Polytechnic, Rajkot, Gujarat- 360003, India.
e-mail: parejiyajay@gmail.com; ORCID: https://orcid.org/0000-0002-2072-2719.
[2] Dr. Subhash University, Junagadh, Gujarat- 362001, India.
e-mail: finiteuniverse@live.com; ORCID: https://orcid.org/0000-0001-8938-7552.
[3] Dr. Subhash University, Junagadh, Gujarat- 362001, India.
Darshan University, Rajkot, Gujarat- 360003, India.
e-mail: daxajani24@gmail.com; ORCID: https://orcid.org/0000-0002-1412-8270.
[4] Children's Univesity, Gandhinagar- 382020, India.
e-mail: snehamundadiya@gmail.com; ORCID: https://orcid.org/0000-0003-3318-1322.
∗ Corresponding author.
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of edges with label x respectively and $\lfloor x \rfloor$ is the floor function. A Graph G is called *Harmonic Mean Cordial graph* if it admits Harmonic Mean Cordial labeling. For the sake of convenience of the reader we use HMC for harmonic mean cordial labeling. Motivated by the interesting results proved on HMC labeling in [5, 6, 8, 9] and on interesting results proved on Root Cube Mean Cordial labeling by [10], in this article we have discussed HMC labeling of Splitting graph of Path graph, Star graph, Bistar graph, Comb graph, $P_n \odot \overline{K}_m$ also we have proved that if a connected graph G is HMC then $G \odot K_1$ is HMC. We have also shown a graph G which is not HMC but $G \odot K_1$ is HMC. Here, we have mentioned basic definitions to make this article self-contained.

Definition 1.1. [11] *The splitting graph $S^{'}(G)$ of a graph G is obtained by adding to each vertex v a new vertex $v^{'}$ such that $v^{'}$ is adjacent to every vertex that is adjacent to v in G.*

Definition 1.2. [1] *A walk in a graph G is a finite alternating sequence of vertices and edges. A walk is called a trail if all the edges are distinct.*

Definition 1.3. [1] *A trail is called a Path if all the vertices are distinct. It is denoted by P_n, where $n > 1$.*

Definition 1.4. [1] *A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. It is denoted by K_n, where $n > 2$.*

Definition 1.5. [1] *Let G_1 and G_2 be two vertex-disjoint graphs. Then the join of two graphs G_1 and G_1 denoted as $G_1 \vee G_2$ is a supergraph of $G_1 + G_2$ in which every vertex of G_1 is adjacent to each vertex of G_2.*

Definition 1.6. [1] *A graph is bipartite if its vertex set can be partitioned into two non empty subsets V_1 and V_2 such that each edge of G has one end in V_1 and other in V_2. The pair (V_1, V_2) is called bipartition of a bipartite graph. It is denoted by $G(V_1, V_2)$. A simple bipartite graph $G(V_1, V_2)$ is complete if each vertex of V_1 is adjacent to all the vertices of V_2. If $G(V_1, V_2)$ is complete with $|V_1| = m$ and $|V_2| = n$ then $G(V_1, V_2)$ is denoted by $K_{m,n}$.*

Definition 1.7. [1] *A complete bipartite graph of the form $K_{1,n}$ is called a star, where $n > 1$.*

Definition 1.8. [4] *Bistar is the graph obtained by joining the apex vertices of two copies of star graph $K_{1,n}$, where $n > 1$.*

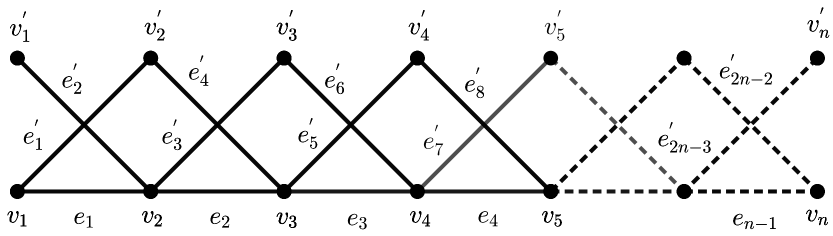
Definition 1.9. [7] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the Corona of G_1 and G_2 is denoted as $G_1 \odot G_2$ is a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 .*

Definition 1.10. [1] *Let P_n be a path with n vertices. The Comb graph is defined $P_n \odot K_1$. It has $2n$ vertices and $2n - 1$ edges, where $n > 2$.*

2. Main Results

Theorem 2.1. *$S^{'}(P_n)$ is HMC if n is even.*

Proof. Note that $G = (V, E) = S^{'}(P_n)$ be the graph with $|V(G)| = 2n$ and $|E(G)| = 3n-3$. Let $V(P_n) = \{v_i | 1 \leq i \leq n\}$ and $E(P_n) = \{e_i | 1 \leq i \leq n-1\}$. Let $v_i^{'}$ be the corresponding vertex of v_i in $S^{'}(P_n)$ for $1 \leq i \leq n$ and $e_i^{'}$ be the a corresonding edge of e_i in $S^{'}(P_n)$ for $1 \leq i \leq n-1$.



$Figure\ 1 : S^{'}(P_n)$

In this graph we have $(n-2)$ vertices with degree 4, n vertices with degree 2 and 2 vertices with degree 1. Suppose that $S^{'}(P_n)$ is HMC. Suppose that n is even. Since the graph $S^{'}(P_n)$ is HMC, we have $v_f(1) = n = v_f(2)$. Now, if we define a labeling function $f : V(G) \to \{1,2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & ; \quad 1 \le i \le \frac{n}{2}+1 \\ 1 & ; \quad \frac{n}{2}+1 < i \le n \end{cases}$$

$f(v_1^{'}) = 1$

and

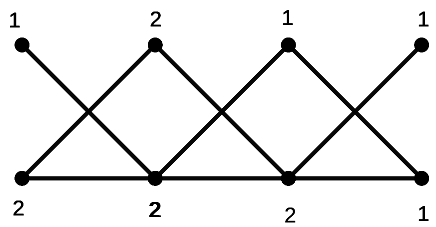
$$f(v_i^{'}) = \begin{cases} 2 & ; \quad 2 \le i \le \frac{n}{2} \\ 1 & ; \quad \frac{n}{2} < i \le n \end{cases}$$

Note that $e_{f(1)} = \frac{3n-2}{2}$ and $e_f(2) = \frac{3n-4}{2}$. So, $|e_f(1) - e_f(2)| = 1$.

Therefore, $S^{'}(P_n)$ is HMC when n is even.

Hence, $S^{'}(P_n)$ is HMC if n is even. □

Example 2.1. *HMC labeling of* $S^{'}(P_4)$.



$Figure\ 2 : S^{'}(P_4)$

Theorem 2.2. $S^{'}(P_n)$ *is not HMC if* n *is odd.*

Proof. Note that $G = (V,E) = S^{'}(P_n)$ be the graph with $|V(G)| = 2n$ and $|E(G)| = 3n-3$. Let $V(P_n) = \{v_i | 1 \le i \le n\}$ and $E(P_n) = \{e_i | 1 \le i \le n-1\}$. Let $v_i^{'}$ be the corresponding vertex of v_i in $S^{'}(P_n)$ for $1 \le i \le n$ and $e_i^{'}$ be the a corresonding edge of e_i in $S^{'}(P_n)$ for $1 \le i \le n-1$ as shown in Figure 1. In this graph we have $(n-2)$ vertices with degree 4, n vertices with degree 2 and 2 vertices with degree 1. Suppose that $S^{'}(P_n)$ is HMC.

Suppose that n is odd. Since a graph $S^{'}(P_n)$ is HMC, $v_f(1) = n = v_f(2)$. Note that if we give vetrex label as shown in pattern 1 or pattern 2, then it generates maximum no. of edges with label 2 and it is $\frac{3n-5}{2}$ and minimum no. of edges with label 1 and it is $\frac{3n-1}{2}$. So, labeling functions shown in pattern 1 and pattern 2 is the best possible labeling function for $|e_f(1) - e_f(2)|$ to be minimum.

pattern 1:

We have defined a labeling function $f : V(G) \to \{1,2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & ; \quad 1 \le i \le \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \le n \end{cases}$$

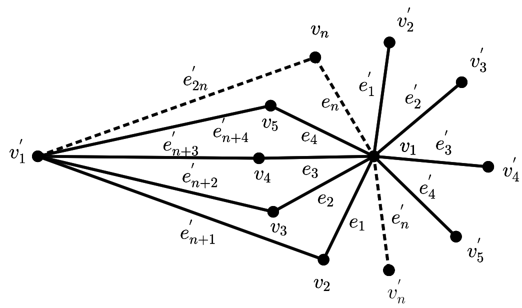
$f(v_1^{'}) = 1$
and
$f(v_i^{'}) = \begin{cases} 2 & ; \quad 2 \leq i \leq \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \leq n \end{cases}$
pattern 2:
We have defined a labeling function $f : V(G) \rightarrow \{1, 2\}$ as follows,
$f(v_i) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \leq n \end{cases}$
and
$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n-1}{2} \\ 1 & ; \quad \frac{n-1}{2} < i \leq n \end{cases}$
But in this case also we have $|e_f(1) - e_f(2)| = 2 > 1$. So, $S^{'}(P_n)$ is not HMC, when n is odd. Hence, $S^{'}(P_n)$ is not HMC if n is odd. □

Corollary 2.1. *$S^{'}(P_n)$ is HMC if and only if n is even.*

Proof. Proof follows from Theorem 2.1 and Theorem 2.2. □

Theorem 2.3. *$S^{'}(K_{1,n})$ is HMC.*

Proof. Note that $G = (V, E) = S^{'}(K_{1,n})$ be the graph with $|V(G)| = 2n + 2$ and $|E(G)| = 3n$. Let $V(K_{1,n}) = \{v_i | 1 \leq i \leq n + 1\}$ and $E(K_{1,n}) = \{e_i | 1 \leq i \leq n\}$. Let $v_i^{'}$ be the corresponding vertex of v_i in $S^{'}(K_{1,n})$ for $1 \leq i \leq n + 1$ and $e_i^{'}$ be the a corresonding edge of e_i in $S^{'}(K_{1,n})$ for $1 \leq i \leq n$.



$Figure\,3 : S^{'}(K_{1,n})$

In this graph we have one vertex with degree $2n$, one vertex with degree n, n no. of vertices with degree 2 and n no. of vertices with degree 1. Suppose that $S^{'}(K_{1,n})$ is HMC.
Case 1: n is even
Since $S^{'}(K_{1,n})$ is HMC, we have $v_f(1) = n + 1 = v_f(2)$. Now, if we define a labeling function $f : V(G) \rightarrow \{1, 2\}$ as follows,
$f(v_i) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n}{2} + 2 \\ 1 & ; \quad \frac{n}{2} + 2 < i \leq n + 1 \end{cases}$
and
$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n}{2} - 1 \\ 1 & ; \quad \frac{n}{2} - 1 < i \leq n + 1 \end{cases}$
Note that $e_{f(1)} = \frac{3n}{2}$ and $e_f(2) = \frac{3n}{2}$. So, $|e_f(1) - e_f(2)| = 0$.
Case 2: n is odd
Since $S^{'}(K_{1,n})$ is HMC, we have $v_f(1) = n + 1 = v_f(2)$. Now, if we define a labeling function $f : V(G) \rightarrow \{1, 2\}$ as follows,
$f(v_i) = 2; 1 \leq i \leq n$

$f(v_{n+1}) = 1$
$f(v_1^{'}) = 2$
$f(v_1^{'}) = 1; 2 \leq i \leq n$
Note that $e_{f(1)} = \frac{3n-1}{2}$ and $e_f(2) = \frac{3n+1}{2}$. So, $|e_f(1) - e_f(2)| = 1$.
Hence, $S^{'}(K_{1,n})$ is HMC.

□

Example 2.2. *HMC labeling of $S^{'}(K_{1,4})$ and $S^{'}(K_{1,5})$.*

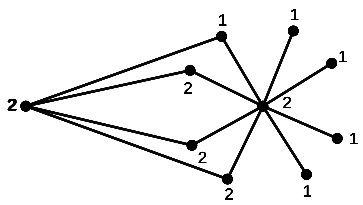


Figure $4(a) : S^{'}(K_{1,4})$

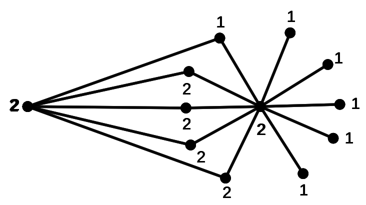


Figure $4(b) : S^{'}(K_{1,5})$

Theorem 2.4. *$S^{'}(B_{n,n})$ is HMC.*

Proof. Note that $G = (V, E) = S^{'}(B_{n,n})$ be the graph with $|V(G)| = 4n+4$ and $|E(G)| = 6n + 3$. Let $V(B_{n,n}) = \{v_i | 1 \leq i \leq 2n + 2\}$ and $E(B_{n,n}) = \{e_i | 1 \leq i \leq 2n + 1\}$. Let $v_i^{'}$ be the corresponding vertex of v_i in $S^{'}(B_{n,n})$ for $1 \leq i \leq 2n+2$ and $e_i^{'}$ be the a corresonding edge of e_i in $S^{'}(B_{n,n})$ for $1 \leq i \leq 2n + 1$.

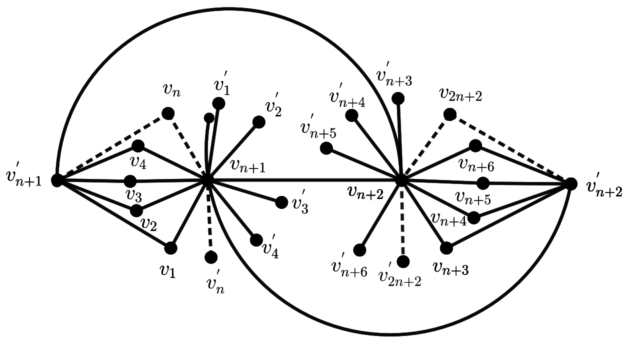


Figure $5 : S^{'}(B_{n,n})$

In this graph we have 2 apex vertices with degree $(2n + 2)$, 2 vertices corresponding to apex vertices with degree $(n + 1)$, $2n$ no. of vertices with degree 1 and $2n$ no. of vertices with degree 2. Suppose that $S^{'}(B_{n,n})$ is HMC. So, we have $v_f(1) = 2n + 2 = v_f(2)$. Now, if we define a labeling function $f : V(G) \rightarrow \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 1 & ; \quad 1 \leq i < n+1 \\ 2 & ; \quad n+1 \leq i \leq 2n+2 \end{cases}$$

and

$$f(v_i^{'}) = \begin{cases} 1 & ; \quad 1 \leq i \leq n+1 \\ 2 & ; \quad n+1 < i \leq 2n+1 \end{cases}$$

$f(v_{2n+2}^{'}) = 1$
Note that, $e_f(1) = 3n+2$ and $e_f(2) = 3n+1$. Then $|e_f(1) - e_f(2)| = 3n+2-3n-1 = 1$.
Hence, $S^{'}(B_{n,n})$ is HMC. □

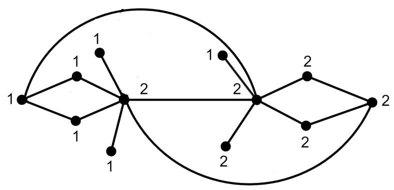


Figure $6(a) : S^{'}(B_{2,2})$

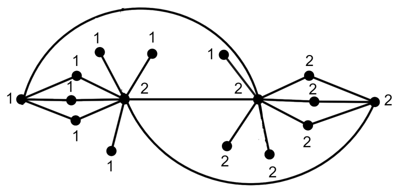


Figure $6(b) : S^{'}(B_{3,3})$

Example 2.3. *HMC labeling of* $S^{'}(B_{2,2})$ *and* $S^{'}(B_{3,3})$.

Theorem 2.5. $S^{'}(P_n \odot K_1)$ *is HMC.*

Proof. Note that $G = (V, E) = S^{'}(P_n \odot K_1)$ be the graph with $|V(G)| = 4n$ and $|E(G)| = 6n - 3$. Let $V(P_n \odot K_1) = \{v_i | 1 \leq i \leq 2n\}$ and $E(P_n \odot K_1) = \{e_i | 1 \leq i \leq 2n - 1\}$. Let $\{v_i^{'}\}$ be the corresponding vertex of $\{v_i\}$ in $S^{'}(P_n \odot K_1)$ for $1 \leq i \leq 2n$ and $e_i^{'}$ be the a corresonding edge of e_i in $S^{'}(P_n \odot K_1)$ for $1 \leq i \leq 2n - 1$.

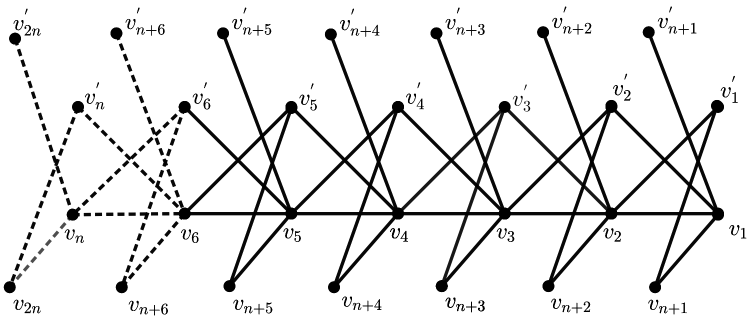


Figure $7 : S^{'}(P_n \odot k_1)$

In this graph we have $(n - 2)$ no. of vertices with degree 6, two vertices with degree 4, $(n + 2)$ no. of vertices with degree 2, $(n - 2)$ vertices with degree 3 and n vertices with degree 1. Suppose that $S^{'}(P_n \odot K_1)$ is HMC.

Case 1: n is even

As the graph $S^{'}(P_n \odot K_1)$ is HMC, we have $v_f(1) = 2n = v_f(2)$. Now, if we define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{3n}{2} \\ 1 & ; \quad \frac{3n}{2} < i \leq 2n \end{cases}$$

and

$$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n}{2} \\ 1 & ; \quad \frac{n}{2} < i \leq 2n \end{cases}$$

Note that, $e_f(1) = 3n - 1$ and $e_f(2) = 3n - 2$. Then $|e_f(1) - e_f(2)| = 1$.

Case 2: n is odd

As the graph $S^{'}(P_n \odot K_1)$ is HMC, we have $v_f(1) = 2n = v_f(2)$. Now, if we define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

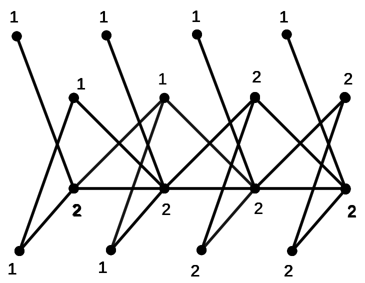
$$f(v_i) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{3n-1}{2} \\ 1 & ; \quad \frac{3n-1}{2} < i \leq 2n \end{cases}$$

and

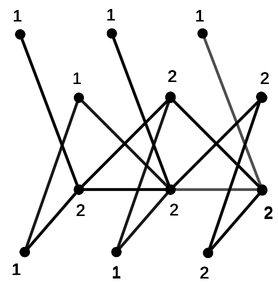
$$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \leq 2n \end{cases}$$

Note that, $e_f(1) = 3n - 1$ and $e_f(2) = 3n - 2$. Then $|e_f(1) - e_f(2)| = 1$.
Hence, $S^{'}(P_n \odot K_1)$ is HMC. $\square$

Example 2.4. *HMC labeling of $S^{'}(P_4 \odot k_1)$ and $S^{'}(P_3 \odot k_1)$.*



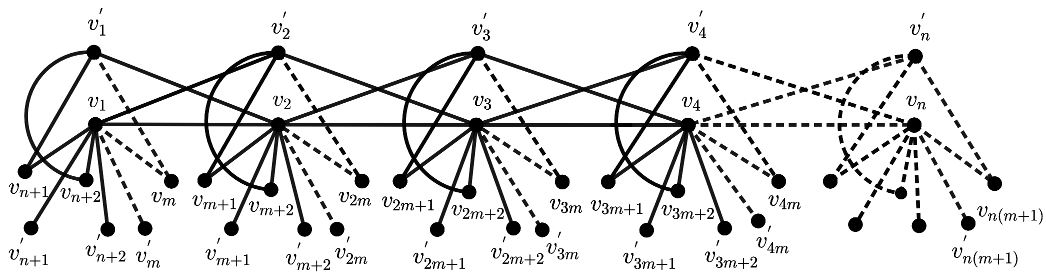
$Figure\ 8(a) : S^{'}(P_4 \odot k_1)$



$Figure\ 8(b) : S^{'}(P_3 \odot k_1)$

Theorem 2.6. *$S^{'}(P_n \odot \overline{K}_m)$ is HMC.*

Proof. Note that $G = (V, E) = S^{'}(P_n \odot \overline{K}_m)$ be the graph with $|V(G)| = 2n(m + 1)$ and $|E(G)| = 3(n + nm - 1)$. Let $V(P_n \odot \overline{K}_m) = \{v_i | 1 \leq i \leq n(m + 1)\}$ and $E(P_n \odot \overline{K}_{2m}) = \{e_i | 1 \leq i \leq mn + n - 1\}$. Let $v_i^{'}$ be the corresponding vertex of v_i in $(P_n \odot \overline{K}_m)$ for $1 \leq i \leq 3n$ and $e_i^{'}$ be the a corresonding edge of e_i in $S^{'}(P_n \odot \overline{K}_m)$ for $1 \leq i \leq mn + n - 1$.



$Figure\ 9 : S^{'}(P_n \odot \overline{K}_m)$

In this graph we have $(n - 2)$ vertices with degree $(2m + 4)$, two vertices with degree $(2m + 2)$, $(n - 2)$ no. of vertices with degree $(m + 2)$, 2 vertices with degree $(m + 1)$, nm no. of vertices with degree 2 and nm no. of vertices with degree 1. Suppose that $S^{'}(P_n \odot \overline{K}_m)$ is HMC.

Case 1: n is even

Since $S^{'}(P_n \odot \overline{K}_m)$ is HMC, we have $v_f(1) = n(m + 1) = v_f(2)$. Now, if we define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n}{2} + 1 \\ 1 & ; \quad \frac{n}{2} + 1 < i \leq n \\ 2 & ; \quad n + 1 \leq i \leq n + \frac{nm}{2} \\ 1 & ; \quad n + \frac{nm}{2} < i \leq n(1 + m) \end{cases}$$

and

$$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \leq i \leq \frac{n}{2} + 1 \\ 1 & ; \quad \frac{n}{2} + 1 < i \leq n \\ 2 & ; \quad n + 1 \leq i \leq \frac{nm}{2} - 2 + n \\ 1 & ; \quad \frac{nm}{2} - 2 + n < i \leq n(1 + m) \end{cases}$$

Note that $e_{f(1)} = \frac{3n+3mn-2}{2}$ and $e_f(2) = \frac{3n+3mn-4}{2}$. So, $|e_f(1) - e_f(2)| = 1$.

Case 2: n **is odd**

Since $S^{'}(P_n \odot \overline{K}_m)$ is HMC, we have $v_f(1) = n(m+1) = v_f(2)$.

In this Case we have two possibilities.

Subase 1: m is even

If we define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & ; \quad 1 \le i \le \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \le n \\ 2 & ; \quad n+1 \le i \le n+3+m(\frac{n-1}{2}) \\ 1 & ; \quad n+3+m(\frac{n-1}{2}) < i \le n(m+1) \end{cases}$$

and

$$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \le i \le \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \le n \\ 2 & ; \quad n+1 \le i \le n+m(\frac{n-1}{2}) \\ 1 & ; \quad n+m(\frac{n-1}{2}) < i \le n(m+1) \end{cases}$$

Note that $e_{f(1)} = \frac{3}{2}(n + nm - 1)$ and $e_f(2) = \frac{3}{2}(n + nm - 1)$. So, $|e_f(1) - e_f(2)| = 0$.

Subase 2: m is odd

If we define a labeling function $f : V(G) \to \{1, 2\}$ as follows,

$$f(v_i) = \begin{cases} 2 & ; \quad 1 \le i \le \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \le n \\ 2 & ; \quad n+1 \le i \le n+m(\frac{n+1}{2}) \\ 1 & ; \quad n+m(\frac{n+1}{2}) < i \le n(m+1) \end{cases}$$

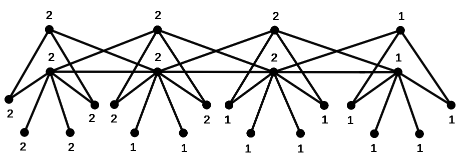
and

$$f(v_i^{'}) = \begin{cases} 2 & ; \quad 1 \le i \le \frac{n+1}{2} \\ 1 & ; \quad \frac{n+1}{2} < i \le n \\ 2 & ; \quad n+1 \le i \le n+2+m(\frac{n-3}{2}) \\ 1 & ; \quad n+2+m(\frac{n-3}{2}) < i \le n(m+1) \end{cases}$$

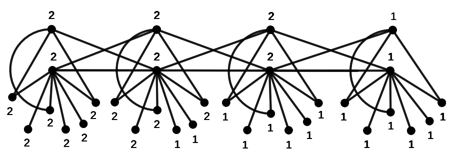
Note that $e_{f(1)} = \frac{3n+3mn-4}{2}$ and $e_f(2) = \frac{3n+3mn-2}{2}$. So, $|e_f(2) - e_f(1)| = 1$.

Hence, $S^{'}(P_n \odot \overline{K}_m)$ is HMC. □

Example 2.5. *HMC labeling of* $S^{'}(P_4 \odot \overline{k}_2)$ *and* $S^{'}(P_4 \odot \overline{k}_3)$.

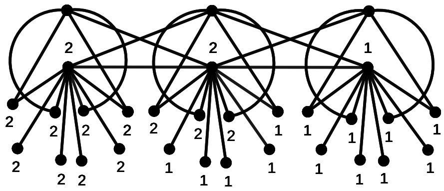


$Figure\ 10(a) : S^{'}(P_4 \odot \overline{k}_2)$

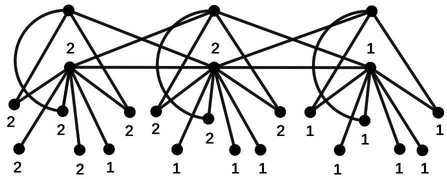


$Figure\ 10(b) : S^{'}(P_4 \odot \overline{k}_3)$

Example 2.6. *HMC labeling of* $S^{'}(P_3 \odot \overline{k}_4)$ *and* $S^{'}(P_3 \odot \overline{k}_3)$.



$Figure\ 11(a) : S^{'}(P_3 \odot \overline{k}_4)$



$Figure\ 11(b) : S^{'}(P_3 \odot \overline{k}_3)$

Theorem 2.7. *Let G be a connected graph. If G is HMC then $G \odot K_1$ is HMC.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and let $f : V(G) \to \{1, 2\}$ be the HMC labelling function. Let u_i be the pendant vertex adjacent to v_i in $G \odot K_1$ for each $i \in \{1, 2, ..., n\}$.
Since G is *HMC*, then we have $v_f(1) = v_f(2)$.
Case 1: $v_f(1) = v_f(2)$
Since, G is *HMC*, we have $|e_f(1) - e_f(2)| \le 1$.
Define $f^* : V(G \odot K_1) \to \{1, 2\}$ as follows
For any $i \in \{1, 2, ..., n\}$
$f^*(v_i) = 1$ if $f(v_i) = 1$
$f^*(v_i) = 2$ if $f(v_i) = 2$
$f^*(u_i) = 1$ if $f(v_i) = 1$
$f^*(u_i) = 2$ if $f(v_i) = 2$
Note that, $v^*_f(0) = v^*_f(1) = 2v_f(1)$ and $e_{f^*}(1) = e_f(1) + \frac{n}{2}$ and $e_{f^*}(2) = e_f(2) + \frac{n}{2}$. Now, $|e_{f^*}(1) - e_{f^*}(2)| = |e_f(1) + \frac{n}{2} - e_f(2) - \frac{n}{2}| = |e_f(1) - e_f(2)| \le 1$.

□

Theorem 2.8. *The Graphs $P_n \odot K_1$,$K_{1,n} \odot K_1$ and $B_{n,n} \odot K_1$ are HMC.*

Proof. By [5],Theorems 3.2,3.3 and 3.4, we have P_n, $K_{1,n}$ and $B_{n,n}$ are HMC. So, by Theorem 2.7 it follows that $P_n \odot K_1$,$K_{1,n} \odot K_1$ and $B_{n,n} \odot K_1$ are HMC.

□

In the next Reamrk 2.1 we have provided an example of a connect graph G which is not *HMC* but $G \odot K_1$ is *HMC*.

Remark 2.1. *By Theorem 2.2, it is clear that $S^{'}(P_3)$ is not HMC if n is odd. Let $G = S^{'}(P_n)$. Let $V(G \odot K_1) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$ where v_i are the vertices of $S^{'}(P_n)$ and u_i is the pendent vertex adjacent to v_i in $G \odot K_1$ for $i = 1, 2, 3$. Now define f : $V(G \odot K_1) \to \{1, 2\}$ as follows*
$f(v_i) = 2$ for $1 \le i \le 6$
$f(u_i) = 1$ for $1 \le i \le 6$.
Note that $v_f(1) = v_f(2) = 6 = e_f(1) = e_f(2)$. So, $G \odot K_1$ is HMC.

3. Conclusion

In this article we have proved that Splitting graph of Path graph $S^{'}(P_n)$ is HMC if and only if n is even. Splitting graph of Star graph $S^{'}(K_{1,n})$, Splitting graph of Bistar graph $S^{'}(B_{n,n})$, Splitting graph of Comb graph $S^{'}(P_n \odot K_1)$ and $S^{'}(P_n \odot \overline{K}_m)$ are HMC. We have also proved that If a connected graph G is HMC then $G \odot K_1$ is HMC. We have also shown a graph G which is not HMC but $G \odot K_1$ is HMC.

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Jaydeep Parejiya is presently working as a lecturer in Mathematics in Government Polytechnic, Rajkot, Gujarat, India. He has done his Ph.D. from Saurashtra University, Rajkot under the guidance of Dr. S. Visweswaran on Commutative Ring Theory. The author has published 18 International research papers and his 3 Books are published by International/National Level Publishers. He has also cracked NET (National Eligibility Test) with All India 7th Rank. The author has total 12 years of extensive academic experience and his main areas of Research are Commutative Ring Theory and Graph Theory.

Premkumar T. Lalchandani is working as an Associate Professor of Mathematics at Dr. Subhash University, Junagadh. He has completed his Ph.D. from Saurashtra University, Rajkot under the guidance of Dr. S. Visweswaran. He has published 11 Research Papers in reputed international and national journals. He has 10 years of experience. He qualified for the GSET examination. His research interests are Commutative Ring Theory and Algebraic Graph Theory.

Daxa B. Jani is presently working as an Assistant Professor at Darshan University, Rajkot, Gujarat, India. Her total teaching experience is of seven years. She has obtained her B.Sc. Degree in Mathematics from Shree M. and N. Virani Science College, Saurashtra University, Rajkot, Gujarat with first class in 2010 and her M. Sc. Degree in Applied Mathematics from Faculty of Technology and Engineering, The Maharaja Sayajirao University of Baroda, Gujarat with first class in the year 2015. Her fields of interest include Graph Theory, Numerical Analysis and Complex Analysis. The author has also published 2 research papers in reputed International Journals.

Sneha Mundadiya is presently working as an Assistant Professor in Mathematics at Marwadi University, Rajkot, Gujarat, India. She has completed her M.Sc. from M.S. University, Baroda, Gujarat, India with first class. She also has pursuing her Ph.D. under the guidance of Dr. M M Jariya from Children's University, Gandhinagar, Gujarat-India. She has total 5 years of teaching experience. Her main areas of Research are Commutative Ring Theory and Graph Theory.