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MODIFIED NEWTON METHOD IN RIEMANNIAN MANIFOLDS

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Abstract. In this article, we present the semilocal convergence analysis of the modified Newton method in Riemannian manifolds. We establish the Newton–Kantorovich convergence theorem for the modified Newton method in Riemannian manifolds by using majorizing function. Finally, two numerical examples are given to show the application of our theorem.

Keywords: Vector fields; Riemannian manifolds; Lipschitz condition; Modified Newton method.

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1. Introduction

Solving a nonlinear equation is an important issue in the field of material science, civil engineering, chemical engineering, mechanics, numerical optimization etc. There are various methods to find the solution of a nonlinear equation. Iterative methods are often used to solve a nonlinear equation. The Newton's method is a very important method to find the approximate roots of a nonlinear equation. To improve the convergence order many iterative methods have been presented. Some famous third order iterative methods in Banach spaces are Halley's method, Chebyshev method, super-Halley method and modified Newton method etc. The modified Newton method in Banach space [17] to solve a nonlinear equation is defined as:

$$\left.\begin{aligned} y_n &= x_n - \mathfrak{A}'(x_n)^{-1}\mathfrak{A}(x_n), \\ x_{n+1} &= x_n - 2[\mathfrak{A}'(x_n) + \mathfrak{A}'(y_n)]^{-1}\mathfrak{A}(x_n), \text{ for each } n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{1}$$

where $\mathfrak{A}'(x_n)$ is first Fréchet derivative of $\mathfrak{A}$ at x_n. Recently, there has been a growing interest in studying iterative methods in Riemannian manifolds, since there are many numerical problems in manifolds that arise in many contexts [6, 7]. Some higher order

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iterative methods in manifolds have been studied in [12, 16, 18]. In this study, we consider to establish the Newton–Kantorovich convergence theorem for the modified Newton method in Riemannian manifolds by using majorizing function which is used to find the approximate zeros of a vector field.

The article is organized as follows: Section 1 is the introduction. In Section 2, we present some basic results of differential geometry. In Section 3, we establish the semilocal convergence analysis of the modified Newton method in Riemannian manifolds by using majorizing function. In Section 4, two numerical examples are given. Finally, conclusions form the Section 5.

2. Notions and Preliminaries

In this section, we discuss some basic results of differential geometry (for more details see [12, 13, 14, 15]).

Let $\mathbb{K}$ be a real n dimensional Riemannian manifold. The tangent space of $\mathbb{K}$ at a is denoted by $T_a\mathbb{K}$. The inner product $\langle .,. \rangle_a$ on $T_a\mathbb{K}$ induces the norm $\|.\|_a$. The tangent bundle of $\mathbb{K}$ is denoted by $T\mathbb{K}$ and is defined by

$$T\mathbb{K} := \{(a,v); a \in \mathbb{K} \text{ and } v \in T_a\mathbb{K}\} = \bigcup_{a\in\mathbb{K}} T_a\mathbb{K}.$$

Let $a, b \in \mathbb{K}$, and $\varrho : [0,1] \to \mathbb{K}$ be a piecewise smooth curve joining a and b. Then the arc length of ϱ is defined by $l(\varrho) = \int_0^1 \|\varrho'(x)\| dx$, and the Riemannian distance from a to b is defined by $d(a,b) = \inf_\varrho l(\varrho)$, where the infimum is taken over all the piecewise smooth curves ϱ connecting a and b. Let $\chi(\mathbb{K})$ be the set of all vector fields of class C^∞ on $\mathbb{K}$ and $D(\mathbb{K})$ the ring of real-valued functions of class C^∞ defined on $\mathbb{K}$. An affine connection ∇ on $\mathbb{K}$ is a mapping

$$\nabla : \chi(\mathbb{K}) \times \chi(\mathbb{K}) \to \chi(\mathbb{K})$$
$$(X, \mathfrak{F}) \mapsto \nabla_X \mathfrak{F}.$$

Let $\mathfrak{F}$ be a vector field of class C^1 on $\mathbb{K}$, the covariant derivative of $\mathfrak{F}$ is determined by the connection ∇ which defines on each $a \in \mathbb{K}$, a linear application of $T_a\mathbb{K}$ itself

$$D\mathfrak{F}(a) : T_a\mathbb{K} \to T_a\mathbb{K}$$
$$v \mapsto D\mathfrak{F}(a)(v) = \nabla_X \mathfrak{F}(a),$$

where X is a vector field satisfying $X(a) = v$. A parametrized curve $\varrho : I \subseteq \mathbb{R} \to \mathbb{K}$ is a geodesic at $p_0 \in I$, if $\nabla_{\varrho'(p)}\varrho'(p) = 0$ at the point p_0. If ϱ is a geodesic for all $p \in I$, we say that ϱ is a geodesic. If $[x,y] \subseteq I$, then ϱ is called a geodesic segment joining $\varrho(x)$ to $\varrho(y)$. Since $\varrho'(p)$ is parallel along $\varrho(p)$ therefore $\|\varrho'(p)\|$ is constant. Let $U(a,s)$ and $U[a,s]$ be an open and a closed geodesic ball with centre a and radius s respectively. By the Hopf-Rinow theorem, if $\mathbb{K}$ is a complete metric space, then for any $a, b \in \mathbb{K}$ there exists a geodesic ϱ called the minimizing geodesic joining a to b with

$$l(\varrho) = d(a,b).$$

Here we have assumed that $\mathbb{K}$ is complete. Therefore, if $v \in T_a\mathbb{K}$ then there exists a unique minimizing geodesic ϱ such that $\varrho(0) = a$ and $\varrho'(0) = v$. The point $\varrho(1)$ is called the image of v by the exponential map at a, i.e.

$$\exp_a : T_a\mathbb{K} \to \mathbb{K},$$

such that $\exp_a(v) = \varrho(1)$ and $\varrho(p) = \exp_a(pv)$, for any $p \in [0,1]$. Let ϱ be a piecewise smooth curve. Then for any $x, y \in \mathbb{R}$, the parallel transport along ϱ is denoted by $R_{\varrho,.,.}$ and given by

$$R_{\varrho,x,y} : T_{\varrho(x)}\mathbb{K} \to T_{\varrho(y)}\mathbb{K}$$
$$v \mapsto V(\varrho(y)),$$

where V is the unique vector field along ϱ such that $\nabla_{\varrho'(p)}V = 0$ and $V(\varrho(x)) = v$. It can be easily proved that $R_{\varrho,x,y}$ is linear and one-to-one. Therefore $R_{\varrho,x,y} : T_{\varrho(x)}\mathbb{K} \to T_{\varrho(y)}\mathbb{K}$ is an isomorphism and $R_{\varrho,y,x}$ is the inverse of parallel transport along the reversed portion of ϱ from $V(\varrho(y))$ to $V(\varrho(x))$. Thus $R_{\varrho,x,y}$ is an isometry between $T_{\varrho(x)}\mathbb{K}$ and $T_{\varrho(y)}\mathbb{K}$. For $i \in \mathbb{N}$, we define R^i_ϱ as

$$R^i_{\varrho,x,y} : (T_{\varrho(x)}\mathbb{K})^i \to (T_{\varrho(y)}\mathbb{K})^i,$$

where

$$R^i_{\varrho,x,y}(v_1, v_2, ..., v_i) = (R_{\varrho,x,y}(v_1), R_{\varrho,x,y}(v_2), ..., R_{\varrho,x,y}(v_i).$$

It has the important properties:

$$R^{-1}_{\varrho,y,x} = R_{\varrho,x,y},$$
$$R_{\varrho,x,y} \circ R_{\varrho,y,z} = R_{\varrho,x,z}.$$

Let $j \in \mathbb{N}$ and $\mathfrak{F}$ be a vector field of class C^k. Then the covariant derivative of order j of $\mathfrak{F}$ is denoted by $D^j\mathfrak{F}$ and defined as the multilinear map

$$D^j\mathfrak{F} : \underbrace{C^k(T\mathbb{K}) \times C^k(T\mathbb{K}) \times \cdots \times C^k(T\mathbb{K})}_{\text{j-times}} \to C^{k-j}(T\mathbb{K})$$

which is given by

$$D^j\mathfrak{F}(A_1, A_2, \dots, A_{j-1}, A) = \nabla_A D^{j-1}\mathfrak{F}(A_1, A_2, \dots, A_{j-1}) - \sum_{i=1}^{j-1} D^{j-1}\mathfrak{F}(A_1, A_2, \dots, \nabla_A A_i, \dots, A_{j-1}),$$

for all $A_1, A_2, \dots, A_{j-1} \in C^k(T\mathbb{K})$.

Definition 2.1. *Let $\mathbb{K}$ be a Riemannian manifold, $\Omega \subseteq \mathbb{K}$ be an open convex set, and $\mathfrak{F} \in \chi(\mathbb{K})$. Then $D\mathfrak{F} = \nabla_{(.)}\mathfrak{F}$ is Lipschitz with constant $\boldsymbol{W} > 0$, if for any geodesic ϱ and $x, y \in \mathbb{R}$ such that $\varrho[x,y] \subseteq \Omega$, it holds the inequality*

$$\|R_{\varrho,y,x}D\mathfrak{F}(\varrho(y))R_{\varrho,x,y} - D\mathfrak{F}(\varrho(x))\| \le \boldsymbol{W}\int_x^y \|\varrho'(p)\|dp.$$

We will write $D\mathfrak{F} \in Lip_{\boldsymbol{W}}(\Omega)$. If $\mathbb{K} = \mathbb{R}^n$, the above definition coincides with the usual Lipschitz definition for the operator $D\mathfrak{F} : \mathbb{K} \to \mathbb{K}$.

Proposition 1. *Let ϱ be a curve in $\mathbb{K}$ and $\mathfrak{F}$ be a C^1 vector field on $\mathbb{K}$, then the covariant derivative of $\mathfrak{F}$ in the direction of $\varrho'(t)$ is defined as*

$$D\mathfrak{F}(\varrho(t))\varrho'(t) = \nabla_{\varrho'(t)}\mathfrak{F}_{\varrho(t)} = \lim_{r\to 0}\frac{1}{r}\big(R_{\varrho,t+r,t}\mathfrak{F}(\varrho(t+r)) - \mathfrak{F}(\varrho(t))\big).$$

If $\mathbb{K} = \mathbb{R}^n$, the above proposition coincides with the definition of directional derivative in $\mathbb{R}^n$.

Next, we take some theorems from [12] that are useful to prove our convergence theorem. The proofs are given there.

Theorem 2.1. *Let ϱ be a geodesic in $\mathbb{K}$ and let $\mathfrak{F}$ be a C^1-vector field on $\mathbb{K}$. Then,*

$$R_{\varrho,t,0}\mathfrak{F}(\varrho(t)) = \mathfrak{F}(\varrho(0)) + \int_0^t R_{\varrho,\theta,0}D\mathfrak{F}(\varrho(\theta))\varrho'(\theta)d\theta.$$

Theorem 2.2. *Let ϱ be a geodesic in $\mathbb{K}$ and let $\mathfrak{F}$ be a C^2-vector field on $\mathbb{K}$. Then,*

$$R_{\varrho,t,0}D\mathfrak{F}(\varrho(t))\varrho'(t) = D\mathfrak{F}(\varrho(0))\varrho'(0) + \int_0^t R_{\varrho,\theta,0}D^2\mathfrak{F}(\varrho(\theta))(\varrho'(\theta), \varrho'(\theta))d\theta.$$

Theorem 2.3. *Let ϱ be a geodesic in $\mathbb{K}$ such that $[0, 1] \subseteq Dom(\varrho)$ and let $\mathfrak{F}$ be a C^2-vector field on $\mathbb{K}$. Then,*

$$R_{\varrho,1,0}\mathfrak{F}(\varrho(1)) = \mathfrak{F}(\varrho(0)) + D\mathfrak{F}(\varrho(0))\varrho'(0) + \int_0^1 (1-\theta)R_{\varrho,\theta,0}D^2\mathfrak{F}(\varrho(\theta))(\varrho'(\theta), \varrho'(\theta))d\theta.$$

3. Modified Newton method in Riemannian manifolds

In this section, we will prove convergence and uniqueness of modified Newton method in Riemannian manifolds to find the singularity of a vector field $\mathfrak{F}$. The modified Newton method (1) in $\mathbb{K}$ has the form:

$$\left.\begin{aligned} g_n &= -D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n), \\ b_n &= \exp_{a_n}(g_n), \\ \alpha(t) &= \exp_{a_n}(tg_n), \\ h_n &= -2\big[R_{\alpha,0,1}D\mathfrak{F}(a_n) + D\mathfrak{F}(b_n)R_{\alpha,0,1}\big]^{-1}\mathfrak{F}(a_n), \\ a_{n+1} &= \exp_{a_n}(h_n), \text{ for each } n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{2}$$

where $D\mathfrak{F}(a_n) = \nabla_{(.)}\mathfrak{F}(a_n)$. Let $a_0 \in \Omega \subseteq \mathbb{K}$ and assume that

(1) $\|D\mathfrak{F}(a_0)^{-1}\| \leq \varepsilon,\ \varepsilon > 0$,
(2) $\|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\| \leq \varphi,\ \varphi > 0$,
(3) $\|D^2\mathfrak{F}(a)\| \leq \varpi$, for all $a \in \Omega,\ \varpi > 0$,
(4) $\|R_{\varrho,b,a}D^2\mathfrak{F}(\varrho(b))R^2_{\varrho,a,b} - D^2\mathfrak{F}(\varrho(a))\| \leq K\int_a^b \|\varrho'(x)\|dx,\ K > 0$, where ϱ is a geodesic such that $\varrho[a, b] \subseteq \Omega$,
(5) $\varpi\Big[1 + \frac{5K}{3\varpi^2\varepsilon}\Big] \leq M,\ M > 0$,
(6) $e = M\varepsilon\varphi \leq \frac{1}{2},\ U[a_0, z^*] \in \Omega$.

We define the polynomial

$$m(z) = \frac{1}{2}Mz^2 - \frac{z}{\varepsilon} + \frac{\varphi}{\varepsilon}.$$

Let z^* and z^{**} be the two positive roots of $m(z)$. Now, we define the sequences for $n \geq 0$,

$$\left.\begin{aligned} y_n &= z_n - m'(z_n)^{-1}m(z_n),\ z_0 = 0, \\ z_{n+1} &= z_n - 2\big(m'(z_n) + m'(y_n)\big)^{-1}m(z_n). \end{aligned}\right\} \tag{3}$$

Next, we give some Lemmas to prove the convergence of (2).

Lemma 3.1. *The sequences generated by* (3) *are increasing, bounded above by* z^*. *Hence it converges to* z^* *and for* $n \in \mathbb{N}$,

$$z^* - z_{n+1} = \frac{(z^* - z_n)^3}{(z^* - z_n + z^{**} - z_n)^2 - (z^* - z_n)(z^{**} - z_n)}, \tag{4}$$

$$z^{**} - z_{n+1} = \frac{(z^{**} - z_n)^3}{(z^* - z_n + z^{**} - z_n)^2 - (z^* - z_n)(z^{**} - z_n)}, \tag{5}$$

$$0 \le z_n \le y_n \le z_{n+1} < z^*. \tag{6}$$

Proof. As $m(z) = \frac{M}{2}(z^* - z)(z^{**} - z)$, $m'(z) = -\frac{M}{2}[(z^* - z) + (z^{**} - z)]$, denote $r_n = (z^* - z_n)$, $s_n = (z^{**} - z_n)$, then we get

$$y_n - z_n = -\frac{m(z_n)}{m'(z_n)} = \frac{r_n s_n}{r_n + s_n}, \tag{7}$$

$$\begin{aligned} z_{n+1} - z_n &= -\frac{2m(z_n)}{m'(z_n) + m'(y_n)} = \frac{2\frac{M}{2} r_n s_n}{\frac{M}{2}(z^* - z_n + z^{**} - z_n + z^* - y_n + z^{**} - y_n)} \\ &= \frac{r_n s_n}{r_n + s_n - (y_n - z_n)} = \frac{r_n s_n (r_n + s_n)}{(r_n + s_n)^2 - r_n s_n}. \end{aligned}$$

Then, we can get

$$\begin{aligned} z_{n+1} - y_n = z_{n+1} - z_n - (y_n - z_n) &= \frac{r_n s_n (r_n + s_n)}{(r_n + s_n)^2 - r_n s_n} - \frac{r_n s_n}{r_n + s_n} \\ &= \frac{r_n^2 s_n^2}{(r_n^2 + s_n^2 + r_n s_n)(r_n + s_n)}. \end{aligned} \tag{8}$$

Also, we can obtain

$$r_{n+1} = z^* - z_{n+1} = z^* - z_n - (z_{n+1} - z_n) = r_n - \frac{r_n s_n (r_n + s_n)}{(r_n + s_n)^2 - r_n s_n} = \frac{r_n^3}{(r_n + s_n)^2 - r_n s_n}$$

and

$$s_{n+1} = z^{**} - z_{n+1} = z^{**} - z_n - (z_{n+1} - z_n) = s_n - \frac{r_n s_n (r_n + s_n)}{(r_n + s_n)^2 - r_n s_n} = \frac{s_n^3}{(r_n + s_n)^2 - r_n s_n}.$$

By (4),(5),(7) and (8), $z_0 = 0 < z^*$, and by the principle of mathematical induction, (6) holds. Therefore the sequences $\{y_n\}$, $\{z_n\}$ are increasing and bounded above by z^*. Hence it converges to z^*. □

Lemma 3.2. *Let* $\mathfrak{F}$ *be a* C^2 *vector field and let* $\alpha(t)$ *be given as above. Then, for all* $n \ge 0$, *we have*

$$R_{\alpha,1,0}\mathfrak{F}(b_n) = \int_0^1 (1-t) R_{\alpha,t,0} D^2\mathfrak{F}(\alpha(t))(R_{\alpha,0,t}g_n, R_{\alpha,0,t}g_n)dt.$$

Proof. By Theorem 2.3 and (2), we have

$$
\begin{aligned}
P_{\alpha,1,0}\mathfrak{F}(b_n) &= \mathfrak{F}(a_n) + D\mathfrak{F}(a_n)g_n + \int_0^1 (1-t)R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))(\alpha'(t),\alpha'(t))dt \\
&= \mathfrak{F}(a_n) + D\mathfrak{F}(a_n)(-D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n)) \\
&+ \int_0^1 (1-t)R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))(\alpha'(t),\alpha'(t))dt \\
&= \int_0^1 (1-t)R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))(\alpha'(t),\alpha'(t))dt.
\end{aligned}
$$

Since α is a geodesic, then $\alpha'(t)$ is parallel and $\alpha'(t) = R_{\alpha,0,t}\alpha'(0),\ \alpha'(0) = g_n$. we have

$$
R_{\alpha,1,0}\mathfrak{F}(b_n) = \int_0^1 (1-t)R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))(R_{\alpha,0,t}g_n, R_{\alpha,0,t}g_n)dt.
$$

□

Lemma 3.3. *Let $\mathfrak{F}$ be a C^2 vector field on $\mathbb{K}$ and let $\alpha(t)$ be given as above and let $\mu(t)$ be given as $\mu(t) = \exp_{b_n}(tl_n)$, where $\mu(0) = b_n$, $\mu(1) = a_{n+1}$ and $l_n = R_{\alpha,0,1}\Big[D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n) - 2\big(R_{\alpha,0,1}D\mathfrak{F}(a_n) + D\mathfrak{F}(b_n)R_{\alpha,0,1}\big)^{-1}R_{\alpha,0,1}\mathfrak{F}(a_n)\Big]$. Then, for all $n \geq 0$, we have*

$$
\begin{aligned}
R_{\mu,1,0}\mathfrak{F}(a_{n+1}) &= \int_0^1 (1-t)R_{\mu,t,0}D^2\mathfrak{F}(\mu(t))R^2_{\mu,0,t}(l_n,l_n)dt \\
&+ \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, R_{\alpha,1,0}l_n)dt \\
&+ R_{\alpha,0,1}\int_0^1 (1-t)R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n,g_n)dt \\
&- \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n,g_n)dt.
\end{aligned}
$$

Proof. Since $\alpha(0) = a_n$ and $\alpha(1) = b_n$, by Theorem 2.2, we have

$$\begin{aligned}
D\mathfrak{F}(b_n)l_n &= \frac{1}{2}[D\mathfrak{F}(b_n)R_{\alpha,0,1} - R_{\alpha,0,1}D\mathfrak{F}(a_n)]R_{\alpha,1,0}l_n \\
&+ \frac{1}{2}[D\mathfrak{F}(b_n)R_{\alpha,0,1} + R_{\alpha,0,1}D\mathfrak{F}(a_n)]R_{\alpha,1,0}l_n \\
&= \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, R_{\alpha,1,0}l_n)dt \\
&+ \frac{1}{2}[D\mathfrak{F}(b_n)R_{\alpha,0,1} + R_{\alpha,0,1}D\mathfrak{F}(a_n)] \times R_{\alpha,1,0}R_{\alpha,0,1}\Big[D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n) \\
&- 2\big(R_{\alpha,0,1}D\mathfrak{F}(a_n) + D\mathfrak{F}(b_n)R_{\alpha,0,1}\big)^{-1}R_{\alpha,0,1}\mathfrak{F}(a_n)\Big] \\
&= \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, R_{\alpha,1,0}l_n)dt \\
&+ \Big[\frac{1}{2}[D\mathfrak{F}(b_n)R_{\alpha,0,1} + R_{\alpha,0,1}D\mathfrak{F}(a_n)] - R_{\alpha,0,1}D\mathfrak{F}(a_n)\Big]D\mathfrak{F}(a_n)^{-1}\mathfrak{F}(a_n) \\
&= \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, R_{\alpha,1,0}l_n)dt \\
&- \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, g_n)dt.
\end{aligned}$$

By using Theorem 2.3 and Lemma 3.2, we get

$$\begin{aligned}
R_{\mu,1,0}\mathfrak{F}(a_{n+1}) &= R_{\mu,1,0}\mathfrak{F}(a_{n+1}) - \mathfrak{F}(b_n) - D\mathfrak{F}(b_n)l_n + \mathfrak{F}(b_n) + D\mathfrak{F}(b_n)l_n \\
&= \int_0^1 (1-t)R_{\mu,t,0}D^2\mathfrak{F}(\mu(t))R^2_{\mu,0,t}(l_n, l_n)dt \\
&+ \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, R_{\alpha,1,0}l_n)dt \\
&+ R_{\alpha,0,1}\int_0^1 (1-t)R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, g_n)dt \\
&- \frac{1}{2}R_{\alpha,0,1}\int_0^1 R_{\alpha,t,0}D^2\mathfrak{F}(\alpha(t))R^2_{\alpha,0,t}(g_n, g_n)dt.
\end{aligned}$$

□

Theorem 3.1. *Let $\mathbb{K}$ be a complete Riemannian manifold, $\Omega \subseteq \mathbb{K}$ be an open convex set and $\mathfrak{F} \in \chi(\mathbb{K})$ satisfies the conditions (1) – (6) with:*

$$\varepsilon\varpi(3z^* + z^{**}) < 2.$$

Then, the method given by (2) is well defined, $a_n \in U[a_0, z^*]$ and converges to the unique singular point a^* of $\mathfrak{F}$ in $U[a_0, z^{**}]$.

Proof. Firstly, we shall prove that the following statements hold for $n \geq 0$, by mathematical induction :

- $a_n \in U[a_0, z_n]$,
- $d(b_n, a_n) \leq y_n - z_n$,
- $\|D\mathfrak{F}(a_n)^{-1}R_{\mathfrak{P},0,1}D\mathfrak{F}(a_0)\| \leq -m'(z_n)^{-1}$,
- $\left\|\left[\frac{R_{\alpha,0,1}D\mathfrak{F}(a_n)+D\mathfrak{F}(b_n)R_{\alpha,0,1}}{2}\right]^{-1}R_{\alpha,0,1}R_{\mathfrak{P},0,1}D\mathfrak{F}(a_0)\right\| \leq -2\big(m'(z_n) + m'(y_n)\big)^{-1}$,
- $d(a_{n+1}, b_n) \leq z_{n+1} - y_n$,

- $d(a_{n+1}, a_n) \leq z_{n+1} - z_n$.

By Lemma 3.1, we know that for any natural number n, $z_n < z^*$. Therefore it can be easily proved that the above statements hold for $n = 0$. Suppose that the above statements hold for $n > 0$. Then, we get

$$d(a_{n+1}, a_0) \leq d(a_{n+1}, a_n) + d(a_n, a_0) \leq z_{n+1} - z_n + z_n = z_{n+1}.$$

Let $\mathfrak{P}$ be a geodesic such that $\mathfrak{P}(0) = a_0$, $\mathfrak{P}(1) = a_{n+1}$, and $\|\mathfrak{P}'(0)\| = d(a_{n+1}, a_0)$. By Theorem 2.2, we have

$$\begin{aligned}\|R_{\mathfrak{P},1,0} D\mathfrak{F}(a_{n+1}) R_{\mathfrak{P},0,1} - D\mathfrak{F}(a_0)\| \leq& \varpi d(a_{n+1}, a_0) \leq \varpi z_{n+1} < M z^* \\ =& M\varphi \frac{1 - \sqrt{1 - 2e}}{e} \leq \frac{1}{\varepsilon} \leq \frac{1}{\|D\mathfrak{F}(a_0)^{-1}\|}.\end{aligned}$$

By Banach's Lemma [19], $R_{\mathfrak{P},1,0} D\mathfrak{F}(a_{n+1}) R_{\mathfrak{P},0,1}$ is invertible and

$$\begin{aligned}\|R_{\mathfrak{P},1,0} D\mathfrak{F}(a_{n+1})^{-1} R_{\mathfrak{P},0,1}\| =& \|D\mathfrak{F}(a_{n+1})^{-1}\| \\ \leq& \frac{\|D\mathfrak{F}(a_0)^{-1}\|}{1 - \|D\mathfrak{F}(a_0)^{-1}\| \|R_{\mathfrak{P},1,0} D\mathfrak{F}(a_{n+1}) R_{\mathfrak{P},0,1} - D\mathfrak{F}(a_0)\|} \\ \leq& \frac{\varepsilon}{1 - \varepsilon \varpi d(a_{n+1}, a_0)} \leq \frac{\varepsilon}{1 - \varepsilon M z_{n+1}} = -m'(z_{n+1})^{-1}.\end{aligned}$$

Now, we have

$$\begin{aligned}&\left\| R_{\alpha,0,1} \Big(\int_0^1 R_{\alpha,t,0} D^2\mathfrak{F}(\alpha(t)) R^2_{\alpha,0,t} dt - \frac{1}{2} \int_0^1 R_{\alpha,t,0} D^2\mathfrak{F}(\alpha(t)) R^2_{\alpha,0,t} dt \Big) \right\| \\ \leq& \left\| \int_0^1 \Big(R_{\alpha,t,0} D^2\mathfrak{F}(\alpha(t)) R^2_{\alpha,0,t} - D^2\mathfrak{F}(a_n) \Big)(1 - t) dt \right\| \\ +& \left\| \frac{1}{2} \int_0^1 \Big(R_{\alpha,t,0} D^2\mathfrak{F}(\alpha(t)) R^2_{\alpha,0,t} - D^2\mathfrak{F}(a_n) \Big) dt \right\| \\ \leq& \frac{K}{6} d(b_n, a_n) + \frac{K}{4} d(b_n, a_n) = \frac{5K}{12} d(b_n, a_n).\end{aligned}$$

From (7) and (8), we have

$$\frac{(y_n - z_n)^2}{z_{n+1} - y_n} = \frac{(r_n s_n)^2}{(r_n + s_n)^2} . \frac{(r_n^2 + s_n^2 + r_n s_n)(r_n + s_n)}{r_n^2 s_n^2} \leq r_n + s_n \leq z^* + z^{**} = \frac{2}{M\varepsilon} \leq \frac{2}{\varpi\varepsilon}.$$

By Lemma 3.3, we get

$$\begin{aligned}\|R_{\mu,1,0} \mathfrak{F}(a_{n+1})\| =& \|\mathfrak{F}(a_{n+1})\| \\ \leq& \frac{\varpi}{2} d(a_{n+1}, b_n)^2 + \frac{\varpi}{2} d(a_n, b_n) d(a_{n+1}, b_n) + \frac{5K}{12} d(b_n, a_n)^3 \\ \leq& \frac{\varpi}{2} (z_{n+1} - y_n)^2 + \frac{\varpi}{2} (y_n - z_n)(z_{n+1} - y_n) + \frac{5K}{12} (y_n - z_n)^3 \\ =& \frac{\varpi}{2} (z_{n+1} - y_n)^2 + \Big[\varpi + \frac{5K}{12} . \frac{2(y_n - z_n)^2}{z_{n+1} - y_n} \Big] \frac{1}{2} (y_n - z_n)(z_{n+1} - y_n) \\ \leq& \frac{\varpi}{2} (z_{n+1} - y_n)^2 + \Big[\varpi + \frac{5K}{3\varpi\varepsilon} \Big] \frac{1}{2} (y_n - z_n)(z_{n+1} - y_n) \\ \leq& \frac{M}{2} (z_{n+1} - y_n)^2 + \frac{M}{2} (y_n - z_n)(z_{n+1} - y_n) = m(z_{n+1}).\end{aligned}$$

Therefore

$$d(b_{n+1},a_{n+1}) = \|-D\mathfrak{F}(a_{n+1})^{-1}\mathfrak{F}(a_{n+1})\| \leq -m'(z_{n+1})^{-1}m(z_{n+1}) = y_{n+1}-z_{n+1}.$$

Now,

$$\begin{aligned}
&\left\|\frac{R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})R_{\alpha,1,0}+D\mathfrak{F}(b_{n+1})}{2} - R_{\alpha,0,1}R_{\mathfrak{P},0,1}D\mathfrak{F}(a_0)R_{\mathfrak{P},1,0}R_{\alpha,1,0}\right\|\\
=&\Bigg\|\Bigg[\frac{D\mathfrak{F}(b_{n+1})-R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})R_{\alpha,1,0}}{2}+R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})R_{\alpha,1,0}\\
&-R_{\alpha,0,1}R_{\mathfrak{P},0,1}D\mathfrak{F}(a_0)R_{\mathfrak{P},1,0}R_{\alpha,1,0}\Bigg]\Bigg\|\\
\leq&\frac{\varpi}{2}d(b_{n+1},a_{n+1})+\varpi d(a_{n+1},a_0) \leq \frac{\varpi}{2}(y_{n+1}-z_{n+1})+\varpi z_{n+1}\\
\leq&\frac{\varpi}{2}d(y_{n+1}+z_{n+1}) < Mz^* \leq \frac{1}{\varepsilon} \leq \frac{1}{\|D\mathfrak{F}(a_0)^{-1}\|}.
\end{aligned}$$

By Banach's Lemma [19], $\left[\frac{R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})R_{\alpha,1,0}+D\mathfrak{F}(b_{n+1})}{2}\right]^{-1}$ exists, and

$$\begin{aligned}
\left\|\left[\frac{R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})R_{\alpha,1,0}+D\mathfrak{F}(b_{n+1})}{2}\right]^{-1}\right\| =& \left\|\left[\frac{R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})+D\mathfrak{F}(b_{n+1})R_{\alpha,0,1}}{2}\right]^{-1}\right\|\\
\leq&\frac{\varepsilon}{1-\frac{\varepsilon M}{2}(y_{n+1}+z_{n+1})}\\
\leq&\frac{2}{\frac{1}{\varepsilon}-Mz_{n+1}+\frac{1}{\varepsilon}-My_{n+1}}\\
\leq&-2[m'(z_{n+1})+m'(y_{n+1})]^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&d(a_{n+2},b_{n+1})\\
=&\left\|R_{\alpha,0,1}\Big[D\mathfrak{F}(a_{n+1})^{-1}\mathfrak{F}(a_{n+1})-2\big(R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})+D\mathfrak{F}(b_{n+1})R_{\alpha,0,1}\big)^{-1}R_{\alpha,0,1}\mathfrak{F}(a_{n+1})\Big]\right\|\\
=&\left\|D\mathfrak{F}(a_{n+1})^{-1}\mathfrak{F}(a_{n+1})-2\big(R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})+D\mathfrak{F}(b_{n+1})R_{\alpha,0,1}\big)^{-1}R_{\alpha,0,1}\mathfrak{F}(a_{n+1})\right\|\\
=&\Bigg\|\left[\frac{R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})+D\mathfrak{F}(b_{n+1})R_{\alpha,0,1}}{2}\right]^{-1}\times\\
&\left[\frac{R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})+D\mathfrak{F}(b_{n+1})R_{\alpha,0,1}}{2}-R_{\alpha,0,1}D\mathfrak{F}(a_{n+1})\right]D\mathfrak{F}(a_{n+1})^{-1}\mathfrak{F}(a_{n+1})\Bigg\|\\
\leq&-2[m'(z_{n+1})+m'(y_{n+1})]^{-1}\frac{\varpi}{2}(y_{n+1}-z_{n+1})^2\\
\leq&-M[m'(z_{n+1})+m'(y_{n+1})]^{-1}(y_{n+1}-z_{n+1})^2\\
=&2[m'(z_{n+1})+m'(y_{n+1})]^{-1}\left[\frac{[m'(z_{n+1})+m'(y_{n+1})]}{2}-m'(z_{n+1})\right]m'(z_{n+1})^{-1}m(z_{n+1})\\
=&m'(z_{n+1})^{-1}m(z_{n+1})-2[m'(z_{n+1})+m'(y_{n+1})]^{-1}m(z_{n+1})=z_{n+2}-y_{n+1}.
\end{aligned}$$

Hence, $d(a_{n+2},a_{n+1}) \leq d(a_{n+2},b_{n+1})+d(b_{n+1},a_{n+1}) \leq z_{n+2}-z_{n+1}$. Thus the sequence generated by (2) is well defined, $a_n \in U[a_0,z^*]$ and converges to the singular point of $\mathfrak{F}$ in $U[a_0,z^*]$. Now, we will prove the singularity is unique. Let b^* be the another singularity of $\mathfrak{F}$ in $U[a_0, z^{**}]$ and let $\vartheta : [0,1] \to Z$ be the minimizing geodesic such that $\vartheta(0) = a^*$ and $\vartheta(1) = b^*$. We have

$$\begin{aligned}\|R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t} - D\mathfrak{F}(a^*)\| \leq& \varpi \int_0^t \|\vartheta'(0)\|ds \\ =& \varpi t d(a^*, b^*) \leq \varpi t\Big(d(a_0, a^*) + d(a_0, b^*)\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\|D\mathfrak{F}(a^*)^{-1}\| \int_0^1 \|R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t}dt - D\mathfrak{F}(a^*)\|dt \\ \leq& \frac{1}{\frac{1}{\varepsilon} - \varpi z^*} \int_0^1 \varpi t\Big(d(a_0, a^*) + d(a_0, b^*)\Big)dt \\ \leq& \frac{1}{\frac{1}{\varepsilon} - \varpi z^*} \times \frac{\varpi}{2}(z^* + z^{**}) < 1.\end{aligned}$$

By Banach's Lemma [19], the operator $\int_0^1 R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t}dt$ is invertible and we have

$$0 = R_{\vartheta,1,0}\mathfrak{F}(b^*) - \mathfrak{F}(a^*) = \int_0^1 R_{\vartheta,t,0}D\mathfrak{F}(\vartheta(t))R_{\vartheta,0,t}(\vartheta'(0))dt.$$

Therefore $\vartheta'(0) = 0$. As $0 = \|\vartheta'(0)\| = d(a^*, b^*)$, we get $a^* = b^*$. Hence the proof is complete. □

Theorem 3.2. *If $e < \frac{1}{2}$ and $U(a_0, p) \subseteq \Omega$, where $z^* < p \leq z^{**}$, then a^* is the unique singular point of $\mathfrak{F}$ in $U(a_0, p)$.*

Proof. Let b^* be the singularity of $\mathfrak{F}$ in $U(a_0, p)$ and let $\mathfrak{H}$ be the minimizing geodesic such that $\mathfrak{H}(0) = a_0$, $\mathfrak{H}(1) = b^*$, and $\|\mathfrak{H}'(0)\| = d(a_0, b^*)$. By Theorem 2.1, we have

$$\begin{aligned}R_{\mathfrak{H},1,0}\mathfrak{F}(b^*) =& R_{\mathfrak{H},1,0}\mathfrak{F}(b^*) - \mathfrak{F}(a_0) + \mathfrak{F}(a_0) + D\mathfrak{F}(a_0)\mathfrak{H}'(0) - D\mathfrak{F}(a_0)\mathfrak{H}'(0) \\ =& \int_0^1 R_{\mathfrak{H},t,0}D\mathfrak{F}(\mathfrak{H}(t))R_{\mathfrak{H},0,t}\mathfrak{H}'(0)dt - D\mathfrak{F}(a_0)\mathfrak{H}'(0) + \mathfrak{F}(a_0) + D\mathfrak{F}(a_0)\mathfrak{H}'(0) \\ =& \int_0^1 \big(R_{\mathfrak{H},t,0}D\mathfrak{F}(\mathfrak{H}(t))R_{\mathfrak{H},0,t} - D\mathfrak{F}(a_0)\big)\mathfrak{H}'(0)dt + \mathfrak{F}(a_0) + D\mathfrak{F}(a_0)\mathfrak{H}'(0).\end{aligned}$$

By using Theorem 2.2, we have

$$\begin{aligned}\frac{Md(a_0, b^*)^2}{2} \geq& \frac{\varpi d(a_0, b^*)^2}{2} \geq \|\mathfrak{F}(a_0) + D\mathfrak{F}(a_0)\mathfrak{H}'(0)\| \\ \geq& \frac{1}{\|D\mathfrak{F}(a_0)^{-1}\|}\|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0) + \mathfrak{H}'(0)\| \\ \leq& \frac{1}{\varepsilon}\|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0) + \mathfrak{H}'(0)\| \\ \geq& \frac{1}{\varepsilon}\Big(\|\mathfrak{H}'(0)\| - \|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\|\Big) \geq \Big(\frac{d(a_0, b^*)}{\varepsilon} - \frac{\varphi}{\varepsilon}\Big).\end{aligned}$$

Therefore

$$m(d(a_0, b^*)) = \frac{Md(a_0, b^*)^2}{2} - \frac{d(a_0, b^*)}{\varepsilon} + \frac{\varphi}{\varepsilon} \geq 0.$$

Since $d(a_0, b^*) \leq p \leq z^{**}$, we have $d(a_0, b^*) \leq z^*$, hence by Theorem 3.1, $a^* = b^*$. □

4. Numerical examples

In this section, two numerical examples are given to show the application of our theorem.

Example 4.1. *Let us take the vector field Z from $\mathbb{R}^3$ to $\mathbb{R}^3$ which is given by*

$$Z(a) = Z\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_2 \\ a_1 - a_1 a_3^2 \\ a_1 a_2 a_3 \end{pmatrix} \tag{9}$$

with the Frobenius norm and let $\mathfrak{F} = Z|_{\boldsymbol{S}^2}$. Then it can be easily verified that

$$Z|_{\boldsymbol{S}^2}(a) \in T_a\boldsymbol{S}^2 \ \forall\ a \in \boldsymbol{S}^2.$$

From [16], *$D\mathfrak{F}(a)$ in the basis*

$$\beta_a = \left\{ \begin{pmatrix} -a_3 \\ 0 \\ a_1 \end{pmatrix}, \begin{pmatrix} 0 \\ -a_3 \\ a_2 \end{pmatrix} \right\}$$

of $T_a\boldsymbol{S}^2$ is given by

$$D\mathfrak{F}(a) = \begin{pmatrix} -\frac{1}{a_3}c_{1,1}(a) & -\frac{1}{a_3}c_{1,2}(a) \\ -\frac{1}{a_3}c_{2,1}(a) & -\frac{1}{a_3}c_{2,2}(a) \end{pmatrix},$$

where

$$c_{i,j}(a) = a_j\Big(f_{i,a_3}(a) - \sum_{m=1}^{3} a_m f_{m,a_3}(a)a_i\Big) - a_3\Big(f_{i,a_j}(a) - \sum_{m=1}^{3} a_m f_{m,a_j}(a)a_i\Big)$$

for $i,j = 1,2$, $f_{i,a_j} = \frac{\partial f_i}{\partial a_j}$, and $[\mathfrak{F}(a)]_{\beta_a} = \big(-f_1(a)/a_3, -f_2(a)/a_3\big)^T$. Now, we define

$$u_1 = \frac{f_2(a)c_{1,2}(a) - f_1(a)c_{2,2}(a)}{c_{1,1}(a)c_{2,2}(a) - c_{1,2}(a)c_{2,1}(a)},\ u_2 = \frac{f_1(a)c_{2,1}(a) - f_2(a)c_{1,1}(a)}{c_{1,1}(a)c_{2,2}(a) - c_{1,2}(a)c_{2,1}(a)},$$

$$g_1(a) = u_1c_{1,1}(a) + u_2c_{1,2}(a),\ g_2(a) = u_1c_{2,1}(a) + u_2c_{2,2}(a),\ g_3(a) = -\frac{a_1}{a_3}g_1(a) - \frac{a_2}{a_3}g_2(a).$$

Therefore from [16] *$D^2\mathfrak{F}(a)$ with the same basis β_a of $T_a\boldsymbol{S}^2$ is given by*

$$D^2\mathfrak{F}(a) = \begin{pmatrix} -\frac{1}{a_3}e_{1,1}(a) & -\frac{1}{a_3}e_{1,2}(a) \\ -\frac{1}{a_3}e_{2,1}(a) & -\frac{1}{a_3}e_{2,2}(a) \end{pmatrix},$$

where

$$e_{i,j}(a) = a_j\Big(g_{i,a_3}(a) - \sum_{m=1}^{3} a_m g_{m,a_3}(a)a_i\Big) - a_3\Big(g_{i,a_2}(a) - \sum_{m=1}^{3} a_m g_{m,a_j}(a)a_i\Big)$$

for $i,j = 1,2$, $g_{i,a_j} = \frac{\partial g_i}{\partial a_j}$. Therefore

$$D\mathfrak{F}(a) = \begin{pmatrix} -a_1a_2(a_1^2+1) & -1 - a_1^2(a_2^2 + a_3^2 - 1) \\ 1 - a_2^2 - a_1^2(-2 + a_2^2) - a_3^2 & -a_1a_2(a_2^2 + a_3^2 - 3) \end{pmatrix}.$$

Next, by using the method of Lagrange's multipliers, we get

$$\varpi = \sup\{D\mathfrak{F}(a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 = 5\} = 11$$

is a Lipschitz constant of $D\mathfrak{F}$. Also

$$D^2\mathfrak{F}(a) = \begin{pmatrix} 1 + a_1a_2 + a_1^3a_2 & 1 + a_1^2(-1 + a_2^2 + a_3^2) \\ a_2^2 + a_1^2(-2 + a_2^2) & a_1a_2(a_2^2 + a_3^2 - 3) \end{pmatrix}.$$

Again by using the method of Lagrange's multipliers, we get

$$K = \sup\{D^2\mathfrak{F}(a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 = 5\} = 11.33$$

is a Lipschitz constant of $D^2\mathfrak{F}$. *Initially for* $a_0 = (2, -0.0013091, 1)^T$, *we have*

$$\|D\mathfrak{F}(a_0)^{-1}\| = 1.00779 = \varepsilon,$$
$$\|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\| = 0.0013193 = \varphi.$$

Then, we easily get

$$M = 12.7034, \ e = 0.0168901 \leq \frac{1}{2}, \ z^* = 0.00133063,$$
$$z^{**} = 0.154891, \ \varepsilon\varpi(3z^* + z^{**}) = 1.76132 < 2.$$

That is, all the conditions of above Theorem are satisfied. By the modified Newton method on $\boldsymbol{S}^2$, *we get the solution. The results are in the Table 1, which shows that* (a_i) *converges to the singularity* $(2, 0, 1)^T$.

Table 1. Results of modified Newton method on $\mathbf{S}^2$:

Iterations	a_i	$\|\|\mathfrak{F}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 2 \\ -1.309100e-03 \\ 1 \end{pmatrix}$	*2.927237e-03*	*0*
1	$\begin{pmatrix} 1.999999e+00 \\ 6.356511e-09 \\ 9.999991e-01 \end{pmatrix}$	*2.927237e-03*	*1.309107e-03*
2	$\begin{pmatrix} 1.999999e+00 \\ -8.170206e-15 \\ 1.000000e+00 \end{pmatrix}$	*3.427543e-06*	*9.580408e-07*

Example 4.2. *Consider the vector field* $\mathfrak{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$\mathfrak{F}(a) = \mathfrak{F}(a_1, a_2)^T = (\frac{\cos a_1 + 20a_1}{20}, a_2)^T \tag{10}$$

with the max norm $\|.\| = \|.\|_\infty$. *The first, second, and third Fréchet derivatives of* $\mathfrak{F}$ *are respectively:*

$$D\mathfrak{F}(a) = \begin{bmatrix} \frac{-\sin a_1 + 20}{20} & 0 \\ 0 & 1 \end{bmatrix},$$

$$D^2\mathfrak{F}(a) = \left[\begin{array}{cc|cc} \frac{-\cos a_1}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$D^3\mathfrak{F}(a) = \left[\begin{array}{cc|cc|cc|cc} \frac{\sin a_1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $a_0 = (0, 0)^T$, *we have*

$$\|D\mathfrak{F}(a_0)^{-1}\| = 1 = \varepsilon, \ \|D\mathfrak{F}(a_0)^{-1}\mathfrak{F}(a_0)\| = 0.05 = \varphi,$$

$$\|D^2\mathfrak{F}(a)\| = \max(|\frac{-\cos a_1}{20}|, 0) = |\frac{\cos a_1}{20}| \leq \frac{1}{20} = \varpi,$$

$$\|D^3\mathfrak{F}(a)\| = \max(|\frac{\sin a_1}{20}|, 0) = |\frac{\sin a_1}{20}| \leq \frac{1}{20} = K.$$

Then, we easily get

$$M = 1.71667,\ e = 0.0858333 \leq \frac{1}{2},\ z^* = 0.052353,$$
$$z^{**} = 1.1127,\ \varepsilon\varpi(3z^* + z^{**}) = 0.063488 < 2.$$

That is, all the conditions of above Theorem are satisfied. By the modified Newton method on $\mathbb{R}^2$*, we get the solution. The results are in the Table 2, which shows that* (a_i) *converges to the singularity* $(-0.04994, 0)^T$.

Table 2. Results of modified Newton method on $\mathbb{R}^2$:

Iterations	a_i	$\|\|\mathfrak{F}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	5.000000e-02	0
1	$\begin{pmatrix} -4.993760e-02 \\ 0 \end{pmatrix}$	5.000000e-02	4.993760e-02
2	$\begin{pmatrix} -4.993767e-02 \\ 0 \end{pmatrix}$	6.484663e-08	6.468519e-08

5. Conclusions

In this article, we have extended the modified Newton method from Banach space to Riemannian manifolds to find the singularity of a vector field. We have studied the semilocal convergence analysis of the modified Newton method in Riemannian manifolds by using majorizing function and two numerical examples are given to show the application of our theorem.

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References

[1] Argyros, I. K., (2008), Convergence and Applications of Newton-Type Iterations, Springer, New York.

[2] Argyros, I. K., Cho, Y. J. and Hilout, S., (2012), Numerical Methods for Equations and Variational Inclusions, CRC Press, New York.

[3] Argyros, I. K., Hilout, S. and Tabatabai, M. A., (2011), Mathematical Modelling with Applications in Biosciences and Engineering, Nova Publishers, New York.

[4] Li, W., Szidarovszky, F. and Kuang, Y., (2000), Notes on the stability of dynamic economic systems, Appl. Math. Comput., 108, pp. 85-89.

[5] Tabatabai, M. A., Eby, W. M. and Singh, K. P., (2011), Hyperbolastic modeling of wound healing, Math. Comput. Modell, 53, pp. 755-768.

[6] Dedieu, J. P., Priouret, P. and Malajovich, G., (2003), Newton's method on Riemannian manifolds: covariant alpha theory, IMA J Numer Anal., 23, pp. 395-419.

[7] Ferreira, O. and Svaiter, B., (2002), Kantorovich's Theorem on Newton's Method in Riemannian manifolds, Journal of Complexity, 18, pp. 304-329.

[8] Argyros, I. K., (2007), An improved unifying convergence analysis of Newton's method in Riemannian manifolds, J. Appl. Math. Comput., 25, pp. 345-351.

[9] Argyros, I. K., (2009), Newton's method on Lie groups, Journal of Applied Mathematics and Computing, 31, pp. 217-228.

[10] He, J., Wang, J. and Yao, J.-C., (2013), Convergence criteria of Newton's method on Lie groups, Fixed Point Theory and Applications.

[11] Li, C. and Wang, J., (2006), Newton's method on Riemannian manifolds: Smale's point estimate theory under the γ-condition, IMA J.Numer. Anal., 26, pp. 228-251.

[12] Amat, S., Argyros, I. K., Busquier, S., Castro, R., Hilout, S. and Plaza, S., (2014), Traub-type high order iterative procedures on Riemannian manifolds, SeMA Journal, 63(1), pp. 27-52.

[13] Lang, S., (1995), Differential and Riemannian manifolds, volume 160 of Graduate Texts in Mathematics, Springer-Verlag, New York, third edition.
[14] Tu, L. W., (2011), An introduction to manifolds, Universitext, Springer-Verlag, New York, second edition.
[15] Sakai, T., (1996), Riemannian geometry, American Mathematical Society, Providence, RI.
[16] Castro, R. A., Rodríguez, J. C., Sierra, W. W., Di Giorgi, G. L. and Gomez, S. J., (2018), Chebyshev - Halley's method on Riemannian manifolds, Journal of Computational and Applied Mathematics, 336, pp. 30-53.
[17] Wu, Q. and Zhao, Y., (2006), Third-order convergence theorem by using majorizing function for a modified Newton method in Banach space, Applied Mathematics and Computation, 175(2), pp. 1515-1524.
[18] Amat, S., Busquier, S., Castro, R. and Plaza, S., (2014), Third-order methods on Riemannian manifolds under Kantorovich conditions, Journal of computational and applied mathematics, 255, pp. 106-121.
[19] Kantorovich, L. V. and Akilov, G. P., (1982), Functional analysis, Pergamon Press, Oxford.

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