TWMS J. App. and Eng. Math. V.15, N.3, 2025, pp. 675-687

HOMOCLINIC ORBITS IN GENERALIZED PLANAR SYSTEM OF LIÉNARD TYPE

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ABSTRACT. In this manuscript a generalized Liénard system will be considered. First, the existence and uniqueness of the solutions of the related initial value problem will be proven. Given some definitions, a necessary and sufficient condition for property (Z_1^+) will be presented. Some explicit conditions will also be given for the system to have or fail to have properties (Z_1^+). These results are very sharp and extend and improve the previous results in this subject. Finally, a necessary and sufficient condition will be presented about the existence and nonexistence of homoclinic orbits in the upper or lower half-plane. At the end, some examples will be provided to illustrate our results.

Keywords: Liénard System, Homoclinic Orbit, Planar System, Stable Manifold.

AMS Subject Classification: 34C37, 34A12, 34C10

1. INTRODUCTION

This paper is a study of one of the most beautiful phenomena in dynamical systems; homoclinic orbit. The existence of homoclinic orbits in the Liénard-type systems has a close relation with the stability of the zero solution, the center problem, the global attractivity of the origin and oscillation of solutions and so on. A homoclinic orbit divides the plane into two stable and unstable manifolds. Consider the following autonomous planar system

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{a(x)}\big[K(H(y) - F(x))\big] \\ \frac{dy}{dt} &= -a(x)g(x), \end{aligned} \tag{1}$$

which is a generalized Liénard system, where a, K, H, F, and g are continuous functions which ensure the existence of a unique solution to the corresponding initial value problem. This system includes the classical Liénard system as a special case which is of great

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§ Manuscript received: October 17, 2023; accepted: February 12, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.3;

importance in various applications (see [1-21]). Moreover, suppose that the following assumptions hold.

A$_1$**:** $F(0) = 0$, $a(x) > 0$ for $x \in \mathbf{R}$, $xg(x) > 0$ for $x \neq 0$,

A$_2$**:** $K(u)$ and $H(y)$ are continuously differentiable and strictly increasing with $K(0) = H(0) = 0$, $yH(y) > 0$ for $y \neq 0$ and $K(\pm\infty) = \pm\infty$.

Under these assumptions, the origin is the unique critical point for system (1).

In [21], the authors proved a proposition about the existence of a unique solution for initial value problem corresponding to classical Liénard system. In the following, we prove the same proposition for system (1).

Proposition 1.1. *If* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *hold, then for any initial point* $p(x_0, y_0)$, *system* (1) *has a unique orbit passing through* p.

Proof: By Peano's Theorem (see [10] p. 10]), (1) has at least one solution $(x(t), y(t))$ satisfying $x(0) = x_0$ and $y(0) = y_0$. Along such a solution, we have

$$\begin{aligned} &\frac{dy}{dx} = -\frac{a^2(x)g(x)}{K(H(y) - F(x))} \\ &y(x_0) = y_0. \end{aligned} \tag{2}$$

In order to prove this proposition, it is only needed to prove that if $p \neq O = (0, 0)$, then the initial value problem (2) has a unique solution. consider the following two cases.

(i) Suppose $p \notin \Gamma = \{(x, y) \in \mathbb{R}^2 : y = H^{-1}(F(x))\}$, that is, $y_0 \neq H^{-1}(F(x_0))$. Then there exists a rectangle E : $|x - x_0| \leq a \ and \ |y - y_0| \leq b$ such that E does not intersect Γ. Therefore, $\mathbf{A}_1$ and $\mathbf{A}_2$ imply that $\frac{\partial}{\partial y}(\frac{a^2(x)g(x)}{K(H(y)-F(x))})$ is continuous on E. Applying the Picard-Lindelöf Theorem, obviously the initial value problem (2) has a unique solution on E.

(ii) Suppose $p \in \Gamma$, that is $y_0 = H^{-1}(F(x_0))$, for example, $x_0 > 0$. If the conclusion is not true in this case, then (2) has two solutions $y = y_i(x)$ with $y_i(x_0) = y_0$, for $i = 1, 2$ and $y_1(x) \not\equiv y_2(x)$ for $x_1 \leq x < x_0$. We may assume $y = y_i(x)$ $([x_1, x_0])$ is under the characteristic curve Γ for $i = 1, 2$. Thus, there is an $x^* \in [x_1, x_0)$ with $y_1(x^*) > y_2(x^*)$. Set

$$\bar{x} = \sup \{x : x \in [x^*, x_0) \ \ such \ \ that \ \ y_1(s) > y_2(s) \ \ for \ \ any \ \ s \in [x^*, x]\}.$$

Then, $y_1(x) > y_2(x)$ for $x \in [x^*, \bar{x}]$ and $y_1(\bar{x}) = y_2(\bar{x})$. This shows that (2) has two solutions passing through the point $(\bar{x}, y_1(\bar{x}))$. The case (i) implies that $(\bar{x}, y_1(\bar{x})) \in \Gamma$. Hence, $\bar{x} = x_0$. Thus, (2) yields

$$\frac{d(y_1(x) - y_2(x))}{dx} = \frac{a^2(x)g(x)(K(H(y_1(x)) - F(x)) - K(H(y_2(x)) - F(x)))}{K(H(y_1(x)) - F(x))K(H(y_2(x)) - F(x))}. \tag{3}$$

It follows from $\mathbf{A}_2$ that $K(u)$ is strictly increasing with $uK(u) > 0$ for $u \neq 0$. Therefore, from $y_i(x) - F(x) < 0$ for $x \in [x^*, x_0)$ and $y_1(x) > y_2(x)$ and (3), it can be concluded that

$$\frac{d(y_1(x) - y_2(x))}{dx} > 0 \ \ for \ \ x \in [x^*, x_0).$$

This implies that $y_1(x) - y_2(x)$ is strictly increasing on $[x^*, x_0]$. Thus, $y_1(x) - y_2(x) < y_1(x_0) - y_2(x_0) = 0$, that is $y_1(x) < y_2(x)$ for $x \in [x^*, x_0)$ which is a contradiction. This completes the proof.

In applied sciences many practical problems concerning physics, mechanics and the engineering technique fields associated with differential equations and system of differential equations. In mathematics, in the phase portrait of a dynamical system, a heteroclinic orbit (sometimes called a heteroclinic connection) is a path in phase space which joins two different equilibrium points. If the two equilibrium points be the same, the orbit is called a homoclinic orbit. Homoclinic and heteroclinic orbits arise in the study of bifurcation and chaos phenomena as well as their applications in mechanics, biomathematics and chemistry.

In recent years, many authors have studied the existence of homoclinic orbits for various systems of differential equation such as Lagrangian and Hamiltonian systems, the Lorenz system, the Schrödinger systems and Predator-Prey systems (see [4, 5, 7, 15, 20] and the references cited therein).

The existence of homoclinic orbits in the Liénard-type systems (see [3, 17, 19]) is in conjunction with the stability of the zero solution and the center problem. If system (1) has a homoclinic orbit, then the zero solution is no longer stable. A homoclinic orbit and a center cannot exist together in system (1). This subject also has a close relation with the global attractivity of the origin and oscillation of solutions and so on (see [8, 12, 18]).

To state our main results, we need the following definitions.

Definition 1.1. *System* (1) *has property* (Z_1^+) *(resp.,* (Z_3^+)*) if there exists a point* $P(x_0, y_0)$ *with* $H(y_0) = F(x_0)$ *and* $x_0 > 0$ *(resp.,* $x_0 < 0$*) such that the positive semi-orbit of* (1) *starting at* P *tends to the origin through only the first (resp., third) quadrant.*

Definition 1.2. *System* (1) *has property* (Z_2^-) *(resp.,* (Z_4^-)*) if there exist a point* $P(x_0, y_0)$ *with* $H(y_0) = F(x_0)$ *and* $x_0 < 0$ *(resp.,* $x_0 > 0$*) such that the negative semi-orbit of* (1) *starting at* P *tends to the origin through only the second (resp., fourth) quadrant.*

If system (1) has both properties (Z_1^+) and (Z_2^-), a homoclinic orbit exists in the upper half-plane. Similarly, if system (1) has both properties (Z_3^+) and (Z_4^-), a homoclinic orbit exists in the lower half-plane. This paper will study the existence of homoclinic orbits for system (1). In this work, we will extend and improve the results presented in [1, 8, 17] and we will extend the results in [1].

2. Necessary and Sufficient Conditions for Property (Z_1^+)

In this section a necessary and sufficient condition and some sufficient conditions will be given for system (1) to have property (Z_1^+). First, consider the following lemmas about asymptotic behavior of solutions of (1).

Lemma 2.1. *For each point* $P(c, H^{-1}(F(c)))$ *with* $c > 0$ *(resp.,* $c < 0$*), the positive (resp., negative) semi-orbit of* (1) *starting at* P *crosses the negative* y*-axis if the following condition hold.*

$\mathbf{B}_1$**:** *There exists a* $\delta > 0$ *such that* $F(x) < 0$ *for* $0 < x < \delta$ *(resp.,* $-\delta < x < 0$*) or* $F(x)$ *has an infinite number of positive zeroes clustering at* $x = 0$.

Lemma 2.2. *For each point* $P(c, H^{-1}(F(c)))$ *with* $c > 0$ *(resp.,* $c < 0$*), the negative (resp., positive) semi-orbit of* (1) *starting at* P *crosses the positive* y*-axis if the following condition hold.*

$\mathbf{B}_2$**:** *There exists a* $\delta > 0$ *such that* $F(x) > 0$ *for* $0 < x < \delta$ *(resp.,* $-\delta < x < 0$*) or* $F(x)$ *has an infinite number of positive zeroes clustering at* $x = 0$.

From Lemmas 2.1 and 2.2, it can be concluded that system (1) fails to have properties (Z_1^+) and (Z_2^-) if $\mathbf{B}_1$ holds and fails to have properties (Z_3^+) and (Z_4^-) if $\mathbf{B}_2$ holds. To state our results about property (Z_1^+), we assume that there exists a $\delta > 0$ such that $F(x) > 0$ for $0 < x < \delta$. The positive and negative semi-orbits of (1) passing through $p \in R^2$ are shown by $O^+(p)$ and $O^-(p)$, respectively.

Theorem 2.1. *System (1) has property (Z_1^+) if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \le \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-a^2(\eta)g(\eta)}{K(\phi(\eta) - F(\eta))} d\eta \le H^{-1}(\phi(x)) \tag{4}$$

for $0 < x < \delta$.

Proof: Note that the positive semi-orbit of (1) starting at $P(x_0, H^{-1}(F(x_0)))$ is considered as a solution $y(x)$ of

$$\frac{dy}{dx} = \frac{-a^2(x)g(x)}{K(H(y) - F(x))}, \tag{5}$$

with $y(x_0) = H^{-1}(F(x_0))$.
Sufficiency. Suppose that system (1) fails to have property (Z_1^+). Thus, there exist a point $P(x_0, H^{-1}(F(x_0)))$ and $x_0 > 0$ such that the positive semi-orbit of (1) starting at P does not tend to the origin through the first quadrant. Taking the vector field of (1) into account, we see that the positive semi-orbit rotates clockwise direction about the origin. For this reason, it crosses the curve $y = H^{-1}(\phi(x))$ and meets the y-axis at a point $(0, y_1)$ with $y_1 < 0$. Let

$$x_1 = \inf\{x : 0 < x < \delta \ \text{ and } \ y(x) > H^{-1}(\phi(x))\}.$$

Then, $(x_1, y(x_1))$ is the intersection point of $O^+(P)$ and the curve $y = H^{-1}(\phi(x))$ nearest to the origin; that is, $y(x_1) = H^{-1}(\phi(x_1))$ and $y < H^{-1}(\phi(x))$ for $0 < x < x_1$. Hence, from (4) we have

$$\begin{aligned} H^{-1}(\phi(x_1)) < y(x_1) - y_1 &= \int_0^{x_1} \frac{-a^2(\eta)g(\eta)}{K(H(y(\eta)) - F(\eta))} d\eta \\ &< \int_0^{x_1} \frac{-a^2(\eta)g(\eta)}{K(\phi(\eta) - F(\eta))} d\eta \le H^{-1}(\phi(x_1)), \end{aligned}$$

which is a contradiction.
Necessity. Suppose that $O^+(P)$ approaches the origin through the first quadrant. Then its corresponding solution $y(x)$ satisfies

$$y(x) \to 0^+ \quad as \quad x \to 0. \tag{6}$$

Let $\delta = x_0$ and $\phi(x) = H(y(x))$ for $0 < x < \delta$. It is obvious that $\phi(x) \ge 0$. Thus,

$$H^{-1}(\phi(x)) = y(x) < H^{-1}(F(x)),$$

and therefore, $\phi(x) < F(x)$ for $0 < x < \delta$. Also, from (6) we get

$$\int_0^x \frac{-a^2(\eta)g(\eta)}{K(\phi(\eta) - F(\eta))} d\eta = \int_0^x \frac{-a^2(\eta)g(\eta)}{K(H(y(\eta)) - F(\eta))} d\eta = y(x) - \lim_{\epsilon \to 0} y(\epsilon) = H^{-1}(\phi(x)).$$

Thus, (4) holds. The proof is complete.

As it is presented, condition (4) is an implicit necessary and sufficient for system (1) to have property (Z_1^+). In some cases, however, it is not easy to find a suitable function $\phi(x)$ with a constant δ satisfying (4). Here, some explicit conditions will be given for system (1) to have properties (Z_1^+). The results can be formulated for the properties (Z_3^+), (Z_2^-) and (Z_4^-) easily and so the proofs are omitted. To state our results, we need the following definitions.

Definition 2.1. *Suppose K satisfies condition $\mathbf{A}_2$. For every $s \in (0,1)$ and $r > 0$ define:*

$$\begin{aligned} m_s^r(K) &= \inf_{u\in(-r,0)} \frac{K(su)}{K(u)} \quad \textit{and} \\ m_s(K) &= \lim_{r\to 0^+} m_s^r(K) = \liminf_{u\to 0^-} \frac{K(su)}{K(u)}. \end{aligned} \tag{7}$$

Definition 2.2. *Suppose K satisfies condition $\mathbf{A}_2$. For every $s \in (0,1)$ and $r > 0$ define:*

$$\begin{aligned} M_s^r(K) &= \sup_{u\in(-r,0)} \frac{K(su)}{K(u)} \quad \textit{and} \\ M_s(K) &= \lim_{r\to 0^+} M_s^r(K) = \limsup_{u\to 0^-} \frac{K(su)}{K(u)}. \end{aligned} \tag{8}$$

Remark 2.1. *Notice that if K satisfies $\mathbf{A}_2$, then for every fixed $r > 0$, $m_s^r(K)$ is an increasing function of s, a decreasing function of r for every fixed $0 < s < 1$, and $0 \le m_s^r(K) \le 1$ for $r > 0$, $0 < s < 1$. Also, $m_s(K)$ is an increasing function of s and $0 \le m_s(K) \le 1$ for every $0 < s < 1$.*

Remark 2.2. *Notice that if K satisfies $\mathbf{A}_2$, then for every fixed $r > 0$, $M_s^r(K)$ is an increasing function of s, an increasing function of r for every fixed $0 < s < 1$, and $0 < M_s^r(K) \le 1$ for $r > 0$, $0 < s < 1$. Also, $M_s(K)$ is an increasing function of s and $0 \le M_s(K) \le 1$ for every $0 < s < 1$.*

The following conditions are needed to state our results about property (Z_1^+).

$\mathbf{B}_3$: For every $t \in (0,1)$ and $y > 0$, assume that H satisfies

$$H^{-1}(ty) \ge tH^{-1}(y). \tag{9}$$

$\mathbf{B}_4$: For every $t \in (0,1)$ and $y > 0$, there exists a $0 < \gamma_t < 1 - t$ such that H satisfies

$$H^{-1}((1-t+\gamma_t)y) \le (1-t+\gamma_t))H^{-1}(y). \tag{10}$$

$\mathbf{B}_5$: For every $u > 0$ and $v > 0$, assume that H satisfies

$$H^{-1}(u) \le H^{-1}\big(\frac{u}{v}\big)H^{-1}(v). \tag{11}$$

Theorem 2.2. *Suppose that $\mathbf{B}_3$ holds and there exist $s \in (0,1)$ and $r > 0$ such that $m_s^r(K) \neq 0$. If there exists $\delta > 0$ such that*

$$\frac{1}{H^{-1}(F(x))} \int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))} d\eta \le (1-s)m_s^r(K) \quad for \quad 0 < x < \delta, \tag{12}$$

then system (1) has property (Z_1^+).

Proof: Suppose (12) holds. From the definition of $m_s^r(K)$ it is clear that

$$K(su) \le m_s^r(K)K(u) \quad for \quad u \in (-r, 0). \tag{13}$$

Since F is continuous and $F(0) = 0$, there exists a δ_1 such that $0 < \delta_1 < \min\{r, \delta\}$ and $0 < F(x) < r$ for $0 < x < \delta_1$. Now let $\phi(x) = (1-s)F(x)$. Then, condition (4) is satisfied. In fact, it is obvious that $\phi(x) < F(x)$ for $0 < x < \delta_1$. Now from (12) and (13) it can be written that

$$\int_0^x \frac{-a^2(\eta)g(\eta)}{K(H(\phi(\eta))-F(\eta))}d\eta = \int_0^x \frac{-a^2(\eta)g(\eta)}{K(-sF(\eta))}d\eta \le \int_0^x \frac{-a^2(\eta)g(\eta)}{m_s^r(K)K(-F(\eta))}d\eta$$
$$\le \frac{1}{m_s^r(K)}(1-s)H^{-1}(F(x))m_s^r(K)$$
$$\le H^{-1}((1-s)F(x)) = H^{-1}(\phi(x)),$$

for $0<x<\delta_1$. Thus, by Theorem 2.1 system (1) has property (Z_1^+).

Corollary 2.1. *Suppose that* $\mathbf{B}_3$ *holds. Also, assume that* $\lambda = \max_{s\in(0,1)}(1-s)m_s^r(K)$ *exists and is not equal to zero for some* $r>0$. *If there exists* $\delta>0$ *such that*

$$\frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta \le \lambda \quad for \quad 0<x<\delta, \tag{14}$$

then system (1) *has property* (Z_1^+).

Example 2.1. *Let* $K(u)=u^n$ *where* n *is odd. Then,* K *satisfies* $\mathbf{A}_2$ *and for every* $s\in(0,1)$ *and* $r>0$

$$m_s^r(K) = \inf_{u\in(-r,0)} \frac{K(su)}{K(u)} = s^n.$$

Hence,

$$\lambda = \max_{s\in(0,1)}(1-s)m_s^r(K) = \max_{s\in(0,1)}(1-s)s^n = \frac{n^n}{(n+1)^{(n+1)}}.$$

Therefore, if there exists $\delta>0$ *such that*

$$\frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta \le \frac{n^n}{(n+1)^{(n+1)}} \quad for \quad 0<x<\delta, \tag{15}$$

then by Corollary 2.1 system (1) *has property* (Z_1^+).

Remark 2.3. *For* $n=1$, $H(y)=y$ *and* $a(x)=1$, *Example 2.1 gives the corresponding result of Hara and Yaneyama in* [8] *(condition* (2)*).*

Theorem 2.3. *Suppose that* $\mathbf{B}_3$ *holds and* $\lambda=\sup_{s\in(0,1)}(1-s)m_s(K)\ne 0$. *Then, system* (1) *has property* (Z_1^+) *if*

$$\limsup_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta < \lambda. \tag{16}$$

Proof: Suppose (16) holds. Then, by definition of λ and Remark 2.1 there exist $s\in(0,1)$ and $r>0$ such that

$$\limsup_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta < (1-s)m_s^r(K) \quad and \quad m_s^r(K)\ne 0.$$

Therefore, there exists $\delta>0$ such that

$$\frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta \le (1-s)m_s^r(K) \quad for \quad 0<x<\delta.$$

Hence, by Theorem 2.2, system (1) has property (Z_1^+).

Corollary 2.2. *Suppose that* $\mathbf{B}_3$ *holds. Also, assume that for any fixed number* $s \in (0,1)$

$$K(su) \leq sK(u), \tag{17}$$

for $u < 0$ *and* $|\, u \,|$ *sufficiently small. If there exists* $\delta > 0$ *such that*

$$\frac{1}{H^{-1}(F(x))} \int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))} d\eta < \frac{1}{4} \quad for \quad 0 < x < \delta, \tag{18}$$

then system (1) *has property* (Z_1^+).

Proof: By (17) it is obvious that $m_s(K) \geq s$ for every $s \in (0,1)$. Thus,

$$\sup_{s\in(0,1)} (1-s)m_s(K) \geq \max_{s\in(0,1)} (1-s)s = \frac{1}{4}.$$

Therefore, by Theorem 2.3 system (1) has property (Z_1^+).

In the following, the lack of property (Z_1^+) will be investigated and some results will be given.

Theorem 2.4. *Suppose that* $\mathbf{B}_4$ *and* $\mathbf{B}_5$ *hold and* $r > 0$. *If for every* $s \in (0,1)$ *there exists a constant* $0 < \gamma_s < 1 - s$ *such that*

$$\liminf_{x\to 0^+} \frac{1}{H^{-1}(F(x))} \int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))} d\eta \geq (1 - s + \gamma_s) M^r_{s+\gamma_s}(K), \tag{19}$$

then system (1) *fails to have property* (Z_1^+).

Proof: Suppose that system (1) has property (Z_1^+). From the definition of $M^r_s(K)$ it is clear that

$$\forall s \in (0,1), \quad K(su) \geq M^r_s(K)K(u) \quad for \quad u \in (-r, 0). \tag{20}$$

Now suppose that there exist a constant $0 < \delta < 1 - s$ and a continuous function ϕ such that condition (4) holds. Define $k' = 1 - \liminf_{x\to 0^+} \frac{\phi(x)}{F(x)}$. Then, $0 \leq k' \leq 1$.

If $k' \neq 1$, the continuity of F, $F(0) = 0$ and the definition of k' imply that for every $0 < \varepsilon < 1 - k'$, there exist r_0 and a sequence $\{x_n\}$ with $0 < r_0 < \min\{\delta, r\}$, $0 < x_n \leq r_0$, and $x_n \to 0$ as $n \to +\infty$ such that $0 < F(x) < r$ for $0 < x < r_0$, and

$$\begin{aligned} &\frac{\phi(x)}{F(x)} > 1 - k' - \varepsilon \quad for \quad 0 < x \leq r_0 \quad and \\ &\frac{\phi(x_n)}{F(x_n)} < 1 - k' + \varepsilon \quad for \quad n \in \mathbb{N}. \end{aligned}$$

Hence,

$$\begin{aligned} &\varphi(x) > (1 - k' - \varepsilon)F(x) \quad for \quad 0 < x \leq r_0 \quad and \\ &\varphi(x_n) < (1 - k' + \varepsilon)F(x_n). \end{aligned}$$

Now from (4) and (19) it can be concluded that

$$\begin{aligned}
0 &\geq \int_0^{x_n} \frac{-a^2(\eta)g(\eta)}{K(\phi(\eta)-F(\eta))}d\eta - H^{-1}(\phi(x_n)) \\
&> \int_0^{x_n} \frac{-a^2(\eta)g(\eta)}{K((1-k'-\varepsilon)F(\eta)-F(\eta))}d\eta - H^{-1}((1-k'+\varepsilon)F(x_n)) \\
&= \int_0^{x_n} \frac{-a^2(\eta)g(\eta)}{K((k'+\varepsilon)(-F(\eta)))}d\eta - H^{-1}((1-k'+\varepsilon)F(x_n)) \\
&\geq \int_0^{x_n} \frac{-a^2(\eta)g(\eta)}{M^b_{k'+\varepsilon}(K)K(-F(\eta))}d\eta - (1-k'+\varepsilon)H^{-1}(F(x_n)).
\end{aligned}$$

Consequently, for $n \geq 1$, the following inequality holds.

$$\frac{1}{H^{-1}(F(x_n))}\int_0^{x_n} \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta < (1-k'+\varepsilon)M^b_{k'+\varepsilon}(K).$$

Thus, for $k' \in [0,1)$ the inequality (19) does not hold for any $\gamma_{k'} = \varepsilon > 0$ which is a contradiction.

If $k' = 1$, then by (4) we have

$$\begin{aligned}
\liminf_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^{x} \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta &\leq \liminf_{x\to 0^+} \frac{H^{-1}(\phi(x))}{H^{-1}(F(x))} \\
&\leq \liminf_{x\to 0^+} H^{-1}\Big(\frac{\phi(x)}{F(x)}\Big) = H^{-1}(0) = 0,
\end{aligned}$$

which contradicts (19). The proof is complete.

Theorem 2.5. *Suppose that* $\mathbf{B}_4$ *and* $\mathbf{B}_5$ *hold and* $\lambda = \sup_{s\in(0,1)}(1-s)M_s(K) \neq 0$. *Then, system* (1) *fails to have property* (Z_1^+) *if*

$$\liminf_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^{x} \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta > \lambda. \tag{21}$$

Proof: Let

$$\lambda_r = \sup_{s\in(0,1)} (1-s)M^r_s(K).$$

First, we show that $\lim_{r\to 0^+}\lambda_r = \lambda$. Note that λ_r is an increasing function of r and $\lambda_r \geq \lambda$. So, $\lim_{r\to 0^+}\lambda_r := \lambda_1$ exists and $\lambda_1 \geq \lambda$. Assume that $\{r_n\}$ is a sequence of positive real numbers such that $r_n \to 0$. Then, $\lambda_{r_n} \to \lambda_1$. Now suppose $0 < k < 1$. By the definition of λ_{r_n}, for every $n \in \mathbb{N}$ there is an $s_n \in (0,1)$ such that $k\lambda_{r_n} < (1-s_n)M^{r_n}_{s_n}(K)$. The sequence $\{s_n\}$ is bounded and has a convergent subsequence, call it again $\{s_n\}$ and let $\lim_{n\to+\infty} s_n := s_0 \in [0,1)$. In fact $s_0 \neq 1$. Otherwise,

$$0 < k\lambda \leq k\lambda_1 = \lim_{n\to+\infty} k\lambda_{r_n} \leq \lim_{n\to+\infty}(1-s_n) = 0$$

which is a contradiction.

Now let $0 < \varepsilon < 1 - s_0$, then for n sufficiently large $s_n < a_0 + \varepsilon$. Therefore,

$$\begin{aligned}
k\lambda_1 &\leq \limsup_{n\to+\infty}(1-s_n)M^{r_n}_{s_n}(K) \leq \limsup_{n\to+\infty}(1-s_n)M^{r_n}_{s_0+\varepsilon}(K) \\
&= (1-s_0)M_{s_0+\varepsilon}(K) = (1-(s_0+\varepsilon))M_{s_0+\varepsilon}(K) + \varepsilon M_{s_0+\varepsilon}(K) \\
&\leq \lambda + \varepsilon.
\end{aligned}$$

Hence, if $\varepsilon \to 0$, then $k\lambda_1 \le \lambda$ and since $k < 1$ was arbitrary, it can be obtained that $\lambda_1 \le \lambda$ and so $\lambda_1 = \lambda$. Now let

$$\alpha := \liminf_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta.$$

But since $\alpha > \lambda$, there exists a $r > 0$ such that

$$\alpha > \lambda_r = \sup_{s\in(0,1)} (1-s)M_s^r(K).$$

On the other hand,

$$\lim_{x\to 0^+} xM_{s+x}^r(K) = 0.$$

Hence, there exists $0 < \gamma_s < 1-s$ such that $\gamma_s M_{s+\gamma_s}^r(K) < \frac{\alpha-\lambda_r}{2}$. Thus,

$$(1-s+\gamma_s)M_{s+\gamma_s}^r(K) = (1-(s+\gamma_s))M_{s+\gamma_s}^r(K) + 2\gamma_s M_{s+\gamma_s}^r(K) \le \lambda_r + 2\frac{\alpha-\lambda_r}{2} = \alpha.$$

Therefore, by Theorem 2.4, system (1) fails to have property (Z_1^+).

Example 2.2. *Let $K(u) = u^n$ where n is odd. Then, for every $s \in (0,1)$*

$$M_s(K) = \limsup_{u\to 0^-} \frac{K(su)}{K(u)} = s^n.$$

Thus,

$$\lambda = \sup_{s\in(0,1)} (1-s)M_s(K) = \max_{s\in(0,1)} (1-s)s^n = \frac{n^n}{(n+1)^{(n+1)}}.$$

Therefore, if

$$\liminf_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta \ge \alpha > \frac{n^n}{(n+1)^{(n+1)}}, \tag{22}$$

then system (1) *fails to have property (Z_1^+).*

Remark 2.4. *for $n = 1$, $H(y) = y$ and $a(x) = 1$, Example 2.2 gives the corresponding result of Hara and Yaneyama in* [8] *(condition* (3)*).*

Corollary 2.3. *Suppose that for any fixed number $s \in (0,1)$*

$$K(su) \ge sK(u), \tag{23}$$

for $u < 0$ and $|u|$ sufficiently small. If there exists a constant $\alpha > \frac{1}{4}$ such that

$$\liminf_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta \ge \alpha, \tag{24}$$

then system (1) *fails to have property (Z_1^+).*

Proof: From (23) it is obvious that $M_s(K) \le s$ for every $s \in (0,1)$. Thus,

$$\sup_{s\in(0,1)} (1-s)M_s(K) \le \sup_{s\in(0,1)} (1-s)s = \frac{1}{4}.$$

Therefore, by Theorem 2.5 system (1) fails to have property (Z_1^+).

Remark 2.5. *From Definitions 2.1 and 2.2, it is easy to see that, if $\lim_{u\to 0^-} \frac{K(su)}{K(u)}$ exists for any $s \in (0,1)$, then*

$$M_s(K) = m_s(K) = \lim_{u\to 0^-} \frac{K(su)}{K(u)} \quad for \quad every \quad s \in (0,1).$$

The following comparison theorem is obtained by combining Theorems 2.3 and 2.5.

Theorem 2.6. *Suppose that K satisfies condition $\mathbf{A}_2$ and $\lim_{u\to 0^-}\frac{K(su)}{K(u)} := L_s(K)$ exists for every $s \in (0,1)$ and it is not equal to zero at least for one $s \in (0,1)$. Let $\lambda = \sup_{s\in(0,1)}(1-s)L_s(K)$. Then:*

i) System (1) has property (Z_1^+) if $\mathbf{B}_3$ holds and

$$\limsup_{x\to 0^+}\frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta < \lambda. \tag{25}$$

ii) System (1) fails to have property (Z_1^+) if $\mathbf{B}_4$ and $\mathbf{B}_5$ hold and

$$\liminf_{x\to 0^+}\frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta > \lambda. \tag{26}$$

As a result of Theorem 2.1, we have the following corollaries which are very useful in applications.

Corollary 2.4. *Suppose that system (1) with $K(u) = K_1(u)$ has (resp., fails to have) property (Z_1^+). If $K_2(u) \le K_1(u)$ (resp., $K_2(u) \ge K_1(u)$) for $u < 0$, then system (1) with $K(u) = K_2(u)$ has (resp., fails to have) property (Z_1^+).*

Corollary 2.5. *Suppose that system (1) with $H(x) = H_1(x)$ has (resp., fails to have) property (Z_1^+). If $H_2(x) \le H_1(x)$ (resp., $H_2(x) \ge H_1(x)$) for $x > 0$ sufficiently small, then system (1) with $H(x) = H_2(x)$ has (resp., fails to have) property (Z_1^+).*

3. Homoclinic Orbit

In this section a necessary and sufficient condition and some sufficient conditions are presented for system (1) to have homoclinic orbits in the upper half-plane. As we stated in section 1, if system has both properties (Z_1^+) and (Z_2^-), a homoclinic orbit exists in the upper half-plane. If we replace the interval $0 < x < \delta$ by $-\delta < x < 0$, all theorems and corollaries for the property (Z_1^+) of the previous section remain valid for the property (Z_2^-).

Theorem 3.1. *System (1) has a homoclinic orbit in the upper half-plane if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \le \varphi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-a^2(\eta)g(\eta)}{K(\phi(\eta)-F(\eta))}d\eta \le H^{-1}(\phi(x)), \tag{27}$$

for $0 <| x |< \delta$.

The following theorems are extracted from Theorem 3.1 which present explicit conditions for system (1) to have homoclinic orbit in upper half-plane.

Theorem 3.2. *Suppose that $\mathbf{B}_3$ holds. Also, assume that $\lambda = \max_{s\in(0,1)}(1-s)m_s^r(K)$ exists and is not equal to zero for some $r > 0$. If there exists $\delta > 0$ such that*

$$\frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta \le \lambda \quad for \quad 0 <| x |< \delta, \tag{28}$$

then system (1) has a homoclinic orbit in the upper half-plane.

Theorem 3.3. *Suppose that* $\mathbf{B}_3$ *holds and* $\lambda = \sup_{s\in(0,1)}(1-s)m_s(K) \neq 0$. *Then, system* (1) *has a homoclinic orbit in the upper half-plane if*

$$\limsup_{x\to 0} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta < \lambda. \tag{29}$$

Theorem 3.4. *Suppose that* $\mathbf{B}_4$ *and* $\mathbf{B}_5$ *hold. If* $\lambda = \sup_{s\in(0,1)}(1-s)M_s(K) \neq 0$ *and*

$$\liminf_{x\to 0} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta > \lambda, \tag{30}$$

then system (1) *has no homoclinic orbit in upper half-plane.*

Remark 3.1. *Suppose that* F *is an even and* g *is an odd function. It is easy to see that system* (1) *has property* (Z_1^+) *if and only if it has property* (Z_2^-). *So, if system* (1) *has property* (Z_1^+), *then it has a homoclinic orbit in the upper half-plane.*

Example 3.1. *Consider system* (1) *with*

$$F(x) = \sqrt[3]{\ln(|\, x\,|^\alpha + 1)}, \quad H(y) = y, \quad a(x) = \sqrt{1+x^2}, \quad g(x) = \frac{mx}{1+x^2},$$

$$\alpha > 0, \quad m > 0, \quad \textit{and} \quad K(u) = \begin{cases} e^{u^3} - 1 & u \geq 0 \\ 1 - e^{-u^3} & u < 0 \end{cases}.$$

We have

$$L_a(K) := \lim_{u\to 0^-} \frac{K(au)}{K(u)} = \lim_{u\to 0^-} \frac{1-e^{-a^3u^3}}{1-e^{-u^3}} = a^3.$$

Thus,

$$\lambda = \sup_{a\in(0,1)} (1-a)L_a(K) = \sup_{a\in(0,1)} (1-a)a^3 = \frac{27}{256}.$$

Hence, if $\alpha < 2$, then

$$\lim_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta = \lim_{x\to 0^+} \frac{m}{\sqrt[3]{\ln(|\, x\,|^\alpha+1)}}\int_0^x \eta^{1-\alpha}d\eta$$

$$= \lim_{x\to 0^+} \frac{mx^{2-\alpha}}{(2-\alpha)\sqrt[3]{\ln(|\, x\,|^\alpha+1)}} = \lim_{x\to 0^+} \frac{3m}{\alpha}(x^{2-\alpha}+x^{2-2\alpha})\sqrt[3]{(\ln(|\, x\,|^\alpha+1))^2} = A.$$

Therefore, the following results are obtained by Theorem 2.6:

i) If $0 < \alpha < 2$, then $A = 0$. So, this system has property (Z_1^+).
ii) If $\alpha \geq 2$, it is easy to see that

$$\lim_{x\to 0^+} \frac{1}{H^{-1}(F(x))}\int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))}d\eta = +\infty.$$

So, this system fails to have property (Z_1^+).

Notice that since F is even and g is odd, by Remark 3.1 this system has a homoclinic orbit in the upper half- plane in case (i).

Example 3.2. *Consider system* (1) *with*

$$F(x) = \sqrt[3]{x^2}, \quad H(y) = y^5, \quad a(x) = \frac{1}{\sqrt{1+|\, x\,|}}, \quad g(x) = mx(1+|\, x\,|),$$

$$K(u) = u^3 \quad \textit{with} \quad m, n > 0 \quad \textit{and} \quad 0 < m < \frac{1}{128}.$$

We have

$$L_a(K) := \lim_{u\to 0^-} \frac{K(au)}{K(u)} = \lim_{u\to 0^-} \frac{a^3u^3}{u^3} = a^3.$$

Thus,

$$\lambda = \sup_{a\in(0,1)} (1-a)L_a(K) = \sup_{a\in(0,1)} (1-a)a^3 = \frac{27}{256}.$$

Now, from (25) *it can be written*

$$\lim_{x\to 0^+} \frac{1}{H^{-1}(F(x))} \int_0^x \frac{-a^2(\eta)g(\eta)}{K(-F(\eta))} d\eta = \lim_{x\to 0^+} \frac{m}{x^{\frac{1}{8}}} \int_0^x \eta^{\frac{-7}{8}} d\eta = 8m < \frac{27}{256}.$$

Therefore, by Theorem 2.6 this system has property (Z_1^+). *Notice that since* F *is even and* g *is odd, by Remark 3.1 this system has a homoclinic orbit in the upper half- plane.*

Turning our attention to the lower half-plane, all given results can be formulated about properties (Z_3^+) and (Z_4^-) and finally about the existence of homoclinic orbit in lower half-plane.

4. Conclusions

In this paper the existence of homoclinic orbits in the Liénard-type system has been investigated. Some new and sharp necessary and sufficient conditions are presented about the existence of homoclinic orbits in the upper or lower half-plane. Some examples have also been provided to illustrate the results.

Acknowledgement. The authors would like to thank anonymous referees for their carefully reading the manuscript and such valuable comments which have improved the manuscript significantly.

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