TWMS J. App. and Eng. Math. V.15, N.3, 2025, pp. 703-715

ON THE CONVERGENCE OF SEQUENCES IN FUZZY TOPOLOGICAL SPACES

J. K. MOHAN[1], T. K. SHEEJA[2*], A. S. KURIAKOSE[3], §

Abstract. Analogous to classical topology, the concept of fuzzy nets, in particular, fuzzy sequences and its convergence play a fundamental role in fuzzy topology. There are several different ways of defining fuzzy convergence. The present paper is based on the definition of convergence of fuzzy sequences in terms of quasi-coincidence and Q-neighbourhoods. The study aims at investigating the convergence of fuzzy sequences in various fuzzy topological spaces. The nature of convergent sequences of fuzzy points in certain fuzzy topological spaces such as fuzzy indiscrete, fuzzy discrete, fuzzy co-finite etc. are studied. Characterization theorems for the convergence of fuzzy sequences in each space are obtained. Also, characterizations of fuzzy indiscrete topological space using convergence of fuzzy sequences are provided. The concepts of maximal limit, l-set, l_m-set and fuzzy l-set of fuzzy sequences are introduced and their properties are explored.

Keywords: Fuzzy topology, Fuzzy sequence, Q- neighbourhood, Fuzzy convergence, Maximal limit

AMS Subject Classification: 03B52, 03E72, 54A40

1. Introduction

Fuzzy mathematics has incontrovertibly demonstrated its effectiveness in handling uncertainty and vagueness. The idea of a fuzzy set, initiated by L. A. Zadeh [1] in 1965, revolutionized the world of Mathematics by replacing the two way nature of membership and non-membership by the concept of partial membership. The perception of fuzzy set provides a wider structure, compared to classical set theory, which generalizes various concepts on all branches of Mathematics, especially on topology. C. L. Chang [2] proposed the concept of fuzzy topology in 1968 by incorporating fuzzy set theoretical approach to

[1] Department of Mathematics, Maharaja's College, Ernakulam, Affiliated to M.G. University, Kottayam, India.
e-mail: jyothiskmohan@gmail.com; ORCID: https://orcid.org/0000-0002-2644-3059.
[2] Department of Mathematics, T.M. Jacob Memorial Govt. College, Manimalakunnu, India.
e-mail: sheejakannolil@gmail.com; ORCID: https://orcid.org/0000-0002-6638-7688.
[3] Prinicipal, St. Kuriakose College of Management and Science, Kuruppampady, India.
e-mail: asunnyk@gmail.com; ORCID: https://orcid.org/0009-0002-3063-6459.
[*] Corresponding Author
§ Manuscript received: September 3, 2023; accepted: October 26, 2023.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.3; © Işık University, Department of Mathematics, 2025; all rights reserved.
The first author is supported by Junior Research Fellowship by CSIR, India (File No: 08/0724(15474)/2022-EMR-I).

.

topological concepts. In [2], Chang extended some basic definitions and results in general topology into fuzzy topology. Later, in 1976, R. Lowen [3] proposed a modification to Chang's definition of fuzzy topology. In the present paper, we follow Chang's definition of fuzzy topology.

Chang defined neighbourhood of a fuzzy set and introduced the concept of sequence of fuzzy sets in his paper [2]. It should be noted that in ordinary topology neighbourhood of a *point* is defined whereas Chang defined neighbourhood of a *fuzzy set*. In order to make the study of convergence meaningful, in 1974, C. K. Wong [4] proposed the concept of a fuzzy point. He defined fuzzy point in such a way that the membership value of the support of a fuzzy point lies in $(0, 1)$. That is, the support of a fuzzy point will not take the value 1 as a membership value. Also, according to him, a fuzzy point p belongs to a fuzzy set A if and only if $\mu_p(x) < \mu_A(x)$, $\forall x$, where μ denotes the membership function. But, this definition implies that a fuzzy point is contained in a fuzzy set iff its membership function takes non-zero values everywhere, which appears to be somewhat unnatural.

So, in order to rectify this, in [5], R. Srivastava, S. N. Lal, and A. K. Srivastava modified the definition of 'belongs to' given by Wong. According to them, a fuzzy point p belongs to a fuzzy set A if and only if the membersip value of the support of p is strictly less than that in A, and for all other points the membership value in p is less than or equal to the corresponding values in A. Also, the authors considered the definition of fuzzy point given by Wong in [4].

In 1980, P. P. Ming and L. Y. Ming redefined fuzzy points so that a crisp singleton, equivalently, an ordinary point, is a special case of a fuzzy point [6]. In [6], the authors introduced the concept of quasi-coincidence and replaced 'belongs to' by 'quasi-coincidence' in the definition of neighbourhood and called it as a Q-neighbourhood. They also introduced nets of fuzzy points and its convergence in terms of quasi-coincidence and Q-neighbourhoods of fuzzy points. Also, they defined fuzzy Hausdorff space (T_2 space) by using the concept of Q-neighbourhoods. With this new approach, the authors were able to generalize almost all the theorems concerning the neighbourhood structure of a fuzzy point and the theory of convergence in general topology to fuzzy topology [6].

In [7], M. Guloglu and D. Coker combined fuzzy Q-neighbourhood system of fuzzy points with the convergence structures using L-filter approach. In 1992, M. M. Stadler and M. A. de P. Vicente proposed N-convergence of fuzzy nets in a fuzzy topological space [8]. The authors defined N-convergence in terms of neighborhoods of fuzzy points and explored its properties. D. N. Georgiou and B. K. Papadopoulos introduced the notion of fuzzy continuous convergence on the set of fuzzy continuous functions in 1999 [9]. In the same year, V. Gregori and A. Vidal [10] introduced gradation of openness for the open sets of Chang fuzzy topological space by means of a map from $I^X \to I$, where $I = [0, 1]$. In 2006, F. G. Lupianez defined the notions of nets and filters in the intuitionistic fuzzy sense and obtained some results on convergence [11]. Further studies on convergence of fuzzy nets in fuzzy topological spaces can be seen in [12], [13], [14], [15], [16], [17], [18], [21] etc.

Throughout the present paper, we use the definitions of fuzzy points, nets of fuzzy points and its convergence, and Hausdorff spaces proposed by P. P. Ming and L. Y. Ming [6]. They defined the convergence of fuzzy nets in terms of quasi-coinceidence and Q-neighbourhoods. The advantage of using Q-neighbourhoods instead of neighbourhoods is that, unlike in the case of neighbourhoods, the complement of a Q-neighbourhood of a fuzzy point will never contain the fuzzy point. Moreover, in this paper, we consider sequences of fuzzy points only. Also, by fuzzy sequence we mean sequence of fuzzy points.

In section 2, some basic definitions and results in fuzzy set theory and fuzzy topology that are used in the subsequent sections are recalled. A characterization of fuzzy indiscrete spaces in terms of convergence of sequences of fuzzy points is presented in section 3. Also, characterizations of fuzzy convergent sequences in fuzzy co-finite and finer topological spaces including fuzzy co-countable and fuzzy discrete spaces are obtained. In section 4, the concepts of maximal limits, l-sets, l_m-sets and fuzzy l-sets of sequences of fuzzy points are proposed and their properties are investigated.

2. Preliminaries

This section is devoted to a few existing definitions and results which are valid in fuzzy set theory and fuzzy spaces, that serve as a base for the sequel. For uniformity, we have altered certain notations in some of the definitions.

Definition 2.1. [1]*Let X be a space of points, with a generic element of X denoted by x. A fuzzy set A in X is characterized by a membership function μ_A which associate with each point in X a real number in the interval $[0, 1]$, with the value of μ_A at x representing the grade of membership of x in A.*

The grade of membership of x in the fuzzy set A can also be represented as $A(x)$.

Definition 2.2. [2]*Let A and B be fuzzy sets in a space X, with the grades of membership of $x \in X$ in A and B denoted by $\mu_A(x)$ and $\mu_B(x)$, respectively. Then,*

$$\begin{aligned} A = B &\iff \mu_A(x) = \mu_B(x) \ for\ all\ x \in X. \\ A \subseteq B &\iff \mu_A(x) \leq \mu_B(x) for\ all\ x \in X. \\ C = A \cup B &\iff \mu_C(x) = Max[\mu_A(x), \mu_B(x)] \ for\ all\ x \in X. \\ D = A \cap B &\iff \mu_D(x) = Min[\mu_A(x), \mu_B(x)] \ for\ x \in X. \\ E = A' \ or\ A^c &\iff \mu_E(x) = 1 - \mu_A(x) \ for\ all\ x \in X. \end{aligned}$$

Definition 2.3. [19]*The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A.*

The support of a fuzzy set A is usually denoted by $S(A)$ or $Supp(A)$.

Definition 2.4. [1] *A fuzzy set A is said to be empty, finite or countable fuzzy set if and only if the support of A is empty, finite or countable respectively.*

Definition 2.5. [1]*Two fuzzy sets A and B are disjoint if $A \cap B$ is empty.*

It is evident that the fuzzy sets A and B are disjoint if and only if their supports are disjoint.

Definition 2.6. [2]*A fuzzy topology is a family δ of fuzzy sets in X which satisfies the following conditions:*

(1) $\phi, X \in \delta$
(2) *If $A, B \in \delta$, then $A \cap B \in \delta$*
(3) *If $A_i \in \delta$ for each $i \in I$, then $\cup_I A_i \in \delta$.*

δ is called a fuzzy toplogy for X, and the pair (X, δ) is a fuzzy topological space, or fts for short. Every member of δ is called a δ-open fuzzy set (or simply an open set). A fuzzy set is δ- closed (or simply closed) if and only if its complement is δ-open.

Here, the symbol ϕ is used to denote the empty fuzzy set. That is, if μ is the membership function, then $\mu_\phi(x) = 0$ for all $x \in X$. Also, X denote the fuzzy set for which $\mu_X(x) = 1$ for all $x \in X$.

Definition 2.7. [2] *The fuzzy topology that contains only the fuzzy sets X and ϕ is called the indiscrete fuzzy topology on X.*

Definition 2.8. [2] *The fuzzy topology that contains all the fuzzy subsets of X is called the discrete fuzzy topology on X.*

Definition 2.9. *A fuzzy topological space (X, δ) is said to be fuzzy co-finite if*
$$\delta = \phi \cup \{A \in I^X : Supp(A^c) \text{ is finite}\},\ I = [0, 1].$$

Definition 2.10. *A fuzzy topological space (X, δ) is said to be fuzzy co-countable if*
$$\delta = \phi \cup \{A \in I^X : Supp(A^c) \text{ is countable}\},\ I = [0, 1].$$

Definition 2.11. [6] *A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one point, say $x \in X$. If its value at x is $\lambda, 0 < \lambda \leq 1$, we denote this fuzzy point by x_λ, where the point x is called its support.*

Definition 2.12. [6] *The fuzzy point x_λ is said to be contained in a fuzzy set A, denoted by $x_\lambda \in A$, if and only if $\lambda \leq A(x)$. Evidently every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A.*

Definition 2.13. [6] *Two fuzzy sets A, B in X are said to be intersecting if and only if there exists a point $x \in X$ such that $(A \cap B)(x) \neq 0$. For such a case, we say that A and B intersect at x.*

Definition 2.14. [5] *Two fuzzy points in X are said to be distinct iff their supports are distinct.*

Definition 2.15. [6] *A fuzzy point x_λ is said to be quasi-coincident with the fuzzy set A, denoted by $x_\lambda q A$, if and only if $\lambda > A^c(x)$, or $\lambda + A(x) > 1$.*

Definition 2.16. [6] *A fuzzy set A in (X, δ) is said to be quasi-coincident with B, denoted by AqB, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$, or $A(x) + B(x) > 1$. If this is true, we also say that A and B are quasi-coincident (with each other) at x.*

Definition 2.17. [6] *A fuzzy set A in (X, δ) is called a Q-neighbourhood of x_λ if and only if there exists a $B \in \delta$ such that $x_\lambda q B \leq A$. The family consisting of all the Q-neighbourhoods of x_λ is called the system of Q-neighbourhoods of x_λ. A Q-neighbourhood of a fuzzy point generally does not contain the point itself.*

Definition 2.18. [6] *A fuzzy topological space (X, δ) is fuzzy T_2 (Hausdorff) space if and only if for any two fuzzy points x_λ and y_γ satisfying $x \neq y$, there exist Q-neighbourhoods B and C of x_λ and y_γ, respectively, such that $B \cap C = \phi$.*

Definition 2.19. [6] *Let $(D, \geq)$ be a directed set. Let X be an ordinary set. Let χ be the collection of all the fuzzy points in X. The function $S : D \to \chi$ is called a fuzzy net in X. In other words, a fuzzy net is a pair $(S, \geq)$ such that S is a function from $D \to \chi$ and $\geq$ directs the domain of S. For $n \in D, S(n)$ is often denoted by S_n and hence a net S is often denoted by $\{S_n, n \in D\}$. $S : D \to \chi$ is called a fuzzy sequence if D is N, the set of all natural numbers, and $\geq$ is the usual greater than or equal to.*

Definition 2.20. [6] *Let $S = \{S_n, n \in D\}$ be a fuzzy net in X. S is said to be quasi-coincident with the fuzzy set A if and only if, for each $n \in D, S_n$ is quasi-coincident with A. S is said to be eventually quasi-coincident with A if and only if there is an element m of D such that, if $n \in D$ and $n \geq m$, then S_n is quasi-coincident with A.*

Definition 2.21. [6]*A net S in a fuzzy topological space (X, δ) is said to converge to a point $e = x_\lambda$ in X relative to δ if and only if S is eventually quasi-coincident with each Q-neighbourhood of e.*

Theorem 2.1. [6]*In a fuzzy topological space (X, δ), if a fuzzy net S converges to a fuzzy point x_λ, then for every $\mu \in (0, \lambda]$, S converges also to x_μ.*

Theorem 2.2. [6]*In a fuzzy topological space (X, δ), every fuzzy net does not converge to two fuzzy points with different supports if and only if (X, δ) is a fuzzy T_2 space.*

3. Convergence of Sequences in Different Fuzzy Topological Spaces

Fuzzy convergence is an inevitable and vital concept in fuzzy topology. In this section, we study the nature of convergent sequences in different fuzzy topological spaces such as fuzzy indiscrete, fuzzy co-finite, fuzzy co-countable and fuzzy discrete spaces. We begin the section with the following proposition.

Proposition 3.1. *In a fuzzy topological space (X, δ), each fuzzy point is quasi-coincident with all of its Q-neighbourhoods.*

Proof. Let x_λ be a fuzzy point in the fuzzy topological space (X, δ) and A be a Q-neighbourhood of x_λ. Then, by definition of Q-neighbourhood, there exists $B \in \delta$ such that $x_\lambda q B$ and $B \leq A$.

$\implies \lambda > 1 - B(x)$ and $B(x) \leq A(x)$.

$\implies \lambda > 1 - B(x)$ and $1 - B(x) \geq 1 - A(x)$.

$\implies \lambda > 1 - A(x)$.

Therefore, x_λ is quasi-coincident with A. Hence, in general, each fuzzy point in (X, δ) is quasi-coincident with all of its Q-neighourhoods. □

Remark 3.1. *It should be noted that if a fuzzy point x_λ in a fuzzy topological space (X, δ) is quasi-coincident with a fuzzy set A, then A need not be a Q-neighbourhood of x_λ. For, let X be a non-empty set and (X, δ) be fuzzy indiscrete. Consider the fuzzy point x_λ, where $\lambda = 0.3$, $x \in X$. Let A be a fuzzy set for which $A(x) = 0.9$. Then, evidently, x_λ is quasi-coincident with A. But, there does not exist $B \in \delta$ such that $x_\lambda q B$ and $B \leq A$. Hence, A is not a Q-neighbourhood of x_λ. Thus, we can conclude that a fuzzy point can be quasi-coincident with a fuzzy set which is not a Q-neighbourhood of it.*

For convenience, we define the following:

Definition 3.1. *The set of all supports of the terms of a fuzzy sequence in a fuzzy topological space is called the range set of the fuzzy sequence.*

Note 3.1. *Since, the terms of a fuzzy sequence are fuzzy points the range set of any fuzzy sequence in a fuzzy topological space (X, δ) is a crisp subset of X.*

The following theorem provides a characterization of fuzzy indiscrete topological space in terms of convergence of fuzzy sequences.

Theorem 3.1. *A fuzzy topological space (X, δ) is fuzzy indiscrete if and only if every fuzzy sequence in (X, δ) converge to each and every fuzzy points in X.*

Proof. Let $S = \{S_n, n \in N\}$ be a fuzzy sequence in the fuzzy indiscrete topological space (X, δ). Let x_λ be a fuzzy point in (X, δ). Then X is the only Q-neighbourhood of x_λ. Evidently, any fuzzy point is quasi-coincident with X. So, in particular, each and every terms of sequence S are quasi coincident with X. Hence, sequence S converges to x_λ. Since, sequence S and the fuzzy point x_λ were arbitrary, in a fuzzy indiscrete topological space every fuzzy sequence converges to every fuzzy point.

Conversely, assume that every fuzzy sequence in (X, δ) converges to each and every fuzzy points in X. Let e be a fuzzy point in X and Q be a Q-neighbourhood of e. We prove that $Q = X$. For this, let $x \in X$. Define a sequence $S = \{x_{1/n} : n \in N\}$ where $x \in X$. Then, by our assumption, sequence S converges to e. Therefore, by definition of convergence, S is eventually quasi-coincident with every Q-neighbourhoods of e and hence, in particular, S is eventully quasi-coincident with Q.
Therefore, there exists $m \in N$ such that $S_n q Q, \ \forall n \geq m$.

$$\implies x_{1/n} q Q, \ \forall n \geq m$$

$$\implies \frac{1}{n} > 1 - Q(x), \ \forall n \geq m.$$

That is, $Q(x) > 1 - \frac{1}{n}, \ \forall n \geq m$. This is possible only if $Q(x) = 1$. Since, x was arbitrary, $Q(x) = 1, \ \forall x \in X$. Therefore, $Q = X$. Since, e and Q were arbitrary, X is the only Q-neighbourhood of any fuzzy point in X.

Now, we claim that X is the only non-empty open fuzzy set in (X, δ). If possible, suppose there exists a non-empty open fuzzy set A distinct from X in (X, δ). Then, there exists $x \in X$ such that $0 < A(x) < 1$. Now, choose $\lambda \in (1 - A(x), 1]$ so that $A(x) + \lambda > 1$. Then, A is a Q- neighbourhood of the fuzzy point x_λ. This contradicts the fact that X is the only Q- neighbourhood of every fuzzy point in (X, δ). So, X and ϕ are the only fuzzy open sets in (X, δ). Hence, (X, δ) is fuzzy indiscrete. □

Now, we are in a stage to characterize convergent sequences in fuzzy co-finite topological spaces. The next theorem serves the purpose for a fuzzy co-finite topological space.

Theorem 3.2. *In a fuzzy co-finite topological space (X, δ), a sequence of fuzzy points converges if and only if there exist $x \in X$ and $\lambda \in (0, 1]$ such that the terms of the sequence are eventually fuzzy points with support x and membership value at least λ.*

Proof. Let (X, δ) be a fuzzy co-finite topological space and let $S = \{S_n, n \in N\}$ be a sequence of fuzzy points in (X, δ). Suppose that S converges to the fuzzy point x_λ. First, we prove that the terms of the sequence are eventually fuzzy points with support x.

We claim that no fuzzy points with support different from x can quasi-coincident with all the Q-neighbourhoods of x_λ. For, let y_γ be a fuzzy point such that $x \neq y$. Choose $\alpha \in (1-\lambda, 1]$ so that $\lambda > 1-\alpha$. Now, consider the fuzzy set A for which $A(x) = \alpha, A(y) = 0$ and $A(z) = 1$ for all $z \in X - \{x, y\}$. Then, A is a fuzzy open set which is quasi-coincident with x_λ. So, A is a Q-neighbourhood of x_λ. But y_γ is not quasi-coincident with A. That is, there is a Q-neighbourhood of x_λ, which is not quasi-coincident with y_γ. Hence, by definition of convergence, the terms of S are eventually fuzzy points with support x. That is, there exists $n_1 \in N$ such that S_n has support x for all $n \geq n_1$.

Next, we prove that, eventually, the terms of the sequence have membership value at least λ. By our previous assertion, there is no need to consider fuzzy points with support different from x. Let $\gamma \in (0, 1]$ such that $\gamma < \lambda$. Besides, choose $\alpha = 1 - \gamma$ so that $\lambda + \alpha > 1$, but $\gamma + \alpha = 1$. Now, consider the fuzzy set B such that $B(x) = \alpha$ and $B(y) = 1, \forall y \in X - \{x\}$. Then, B is a fuzzy open set such that $x_\lambda \ qB$. So, B is a Q-neighbourhood of x_λ. But, x_γ is not quasi coincident with B, since $\gamma = 1 - \alpha$. Thus, for each $\gamma < \lambda$, we are able to find a Q-neighbourhood B of x_λ such that x_γ is not quasi-coincident with B. Therefore, no fuzzy point with support x and membership value less than λ can be quasi-coincident with every Q-neighbourhoods of x_λ. Hence, eventually the terms of S must have membership value at least λ. That is, there exists $n_2 \in N$ such that S_n has membership value greater than or equal to $\lambda, \forall n \geq n_2$. Now, let $m = max(n_1, n_2)$. Then, it follows that S_n has support x and membership value at least $\lambda, \forall n \geq m$. Thus, there exists $x \in X$ and $\lambda \in (0,1]$ such that eventually the terms of the sequence are fuzzy points with support x and membership value at least λ. This proves the necessary part.

Conversely, suppose that there exists $x \in X$ and $\lambda \in (0,1]$ such that the terms of sequence S are eventually fuzzy points with support x and membership value at least λ. That is, there exists $n_0 \in N$ such that S_n has support x and membership value greater than or equal to $\lambda, \forall n \geq n_0$. Now, consider the fuzzy point x_λ. Let A be a Q-neighbourhood of x_λ. Then, by proposition 3.1, x_λ is quasi-coincident with A. So, by our assumption, S_n is quasi-coincident with A, $\forall n \geq n_0$. Thus, sequence S converges to the fuzzy point x_λ. Hence, S is convergent. □

Now, we prove that in any fuzzy topological space finer than fuzzy co-finite space on a given set X, convergent sequences behave in the same way as in the case of fuzzy-cofinite space.

Corollary 3.1. *In a fuzzy topological space (X, δ), finer than fuzzy co-finite space, a sequence of fuzzy points converges if and only if there exist $x \in X$ and $\lambda \in (0,1]$ such that the terms of the sequence are eventually fuzzy points with support x and membership value at least λ.*

Proof. Let (X, δ) be a fuzzy topological space which is finer than fuzzy-cofinite space on X. Now, the fuzzy sets A and B that we have defined in the proof of theorem 3.2 are fuzzy co-finite and so are open in (X, δ). Hence, the corollary follows from the proof of theorem 3.2. □

Since, each fuzzy co-finite subset of X is indeed fuzzy co-countable, fuzzy co-countable topology is finer than fuzzy co-finite topology. Also, the fuzzy discrete topology is the finest fuzzy topology on any given set X. Hence, by corollary 3.1, in fuzzy co-countable and discrete topological spaces, a sequence of fuzzy points converges if and only if there exist $x \in X$ and $\lambda \in (0,1]$ such that the terms of the sequence are eventually fuzzy points with support x and membership value at least λ.

Theorem 3.3. *The range set of a convergent fuzzy sequence in a fuzzy co-finite (or finer) topological space is finite.*

Proof. Let (X, δ) be a fuzzy co-finite (or finer) topological space and let $S = \{S_n, n \in N\}$ be a convergent fuzzy sequence in (X, δ). Suppose that the fuzzy sequence S coverges to the fuzzy point x_λ. Then, by Theorem 3.2, there exists $m \in N$ such that S_n has support x, $\forall n \geq m$. Now, let A be the range set of S. Then, $|A| \leq m$, which is a finite value, where $|A|$ denotes the cardinality of the set A. Therefore, A is a finite set. Since, the fuzzy sequence S was arbitrary, the range set of any convergent fuzzy sequence in a fuzzy co-finite (or finer) topological space is a finite set. □

Since, theorem 3.3 is true for any space finer than fuzzy co-finite topological space, it follows that the range sets of convergent fuzzy sequences in both fuzzy co-countable and fuzzy discrete topological spaces are finite.

Now, theorem 3.3 implies that if a fuzzy sequence S in a fuzzy co-finite (or finer) topological space converges, then the range set of S is finite. But, the converse need not be true. That is, if the range set of a fuzzy sequence S in a fuzzy co-finite (or finer) topological space is finite, then it is not necessarily true that S is convergent. For, let X be a set with more than one elements and let (X, δ) be the fuzzy co-finite topological space. Now, for distinct points x and y in X, consider the fuzzy sequence $S = \{S_n, n \in N\}$ in (X, δ) such that

$$S_n = \begin{cases} x_{1/n}, & if\ n\ is\ odd \\ y_{1/2n}, & if\ n\ is\ even. \end{cases}$$

Then, evidently, range set of S is $\{x, y\}$, which is a finite set. But, clearly, there does not exist $z \in X$ and $m \in N$ such that S_n has support $z, \forall n \geq m$. So, by theorem 3.2, S is not convergent. That is, there exists a non-convergent sequence with finite range set in the fuzzy co-finite topological space (X, δ). Thus, the converse of theorem 3.3 does not hold in general.

Next, we investigate the consequences of the characerization theorems, that we have discussed above, for convergent fuzzy sequences in fuzzy co-finite and finer topological spaces. In the following remark, we examine whether fuzziness of the terms has an influence on the convergence of fuzzy sequences in fuzzy topological spaces compared to that in general topology.

Remark 3.2. *In general topology, a sequence in a co-finite topological space converges if and only if at most one term of the sequence repeats infinitely many times as the terms of the sequence* [20]. *But, in a co-countable or discrete topological space, a sequence converges if and only if it is eventually constant* [20]. *So, if X is a given set, then the collection of all convergent sequences in X with co-finite topology is larger than that in the case of X with co-countable or discrete topology. Also, the theorems we have proved above asserts that, if X is a given set, then the collection of all convergent fuzzy sequences in X with fuzzy co-finite, fuzzy co-countable and fuzzy discrete topology are the same. So, we can conclude that, fuzziness of the terms has an influence on the convergence of sequences in a fuzzy topological space compared to that in general topological space.*

Theorem 3.4. *In a fuzzy co-finite topological space (X, δ), if a sequence of fuzzy points converge to two fuzzy points, then their supports are the same.*

Proof. Let (X, δ) be a fuzzy co-finite space. Let $S = \{S_n, n \in D\}$ be a sequence of fuzzy points in X which converges to both x_λ and y_γ. Now, since S converges to x_λ, by theorem 3.2, eventually the terms of seuence S has support x. That is, there exists $n_1 \in N$ such that S_n has support x for all $n \geq n_1$. Similarly, since S converges to y_γ, eventually the terms of S has support y. That is, there exists $n_2 \in N$ such that S_n has support y for all $n \geq n_2$. Let $m = max(n_1, n_2)$. Then, as $n \geq m$, S_n has both x and y as its support. This implies that $x = y$. Thus, in a fuzzy co-finite topological space if a sequence of fuzzy points converge to two fuzzy points, then their supports are the same. □

By corollary 3.1, it is evident that theorem 3.4 is valid in the case of any fuzzy topological space (X, δ) finer than fuzzy co-finite topology on X. So, in particular, theorem 3.4 remains valid in the case of fuzzy co-countable and discrete topological spaces.

The next theorem along with theorem 3.4 can be used as a tool to discover the significance of fuzzy nets over the particular case, fuzzy sequences, in characterizing T_2-spaces.

Theorem 3.5. *The co-finite fuzzy topological space (X, δ) is not T_2 whenever X is an infinite set.*

Proof. Let X be an infinite set and let δ be the co-finite fuzzy topology defined on X. We prove, (X, δ) is not T_2. If possible, suppose (X, δ) is T_2. Let x_λ and y_γ be two distinct fuzzy points in X. Then, by our assumption, there exists disjoint fuzzy open sets U and V such that $x_\lambda q U$ and $y_\gamma q V$. Then, evidently, $U \neq \phi$ and $V \neq \phi$. That is, U and V are two non-empty fuzzy open sets in the fuzzy co-finite topological space (X, δ). This implies, both support of U^c and support of V^c are finite subsets of X. Now,

$$Supp(U^c \cup V^c) = Supp(U^c) \cup Supp(V^c).$$

That is,

$$Supp((U \cap V)^c) = Supp(U^c) \cup Supp(V^c).$$

Clearly, the set in the R.H.S of the above equation is a finite subset of X. Therefore, $Supp((U \cap V)^c)$ is also finite.

$$\implies U \cap V \neq \phi.$$

This contradicts the fact that U and V are disjoint fuzzy sets. Hence, (X, δ) is not T_2. □

Remark 3.3. *With the aid of theorems 3.4 and 3.5, we can show that the necessary part of theorem 2.2, the characerization of fuzzy T_2- spaces by nets, does not hold, in general, if we replace fuzzy nets by fuzzy sequences. For, let X be an infinite set and let δ be the fuzzy co-finite topology defined on X. Then, by theorem 3.5, (X, δ) is not T_2. But, by theorem 3.4, in (X, δ), if a sequence of fuzzy points converge to two fuzzy points then their supports are the same. That is, (X, δ) is a non-Hausdorff space in which every fuzzy sequence does not converge to two fuzzy points with different supports.*

4. Maximal limits, l-sets, l_m-sets and fuzzy l-sets of sequences of fuzzy points

In this section, we define maximal limits, l-sets, l_m-sets and fuzzy l-sets of sequences of fuzzy points. Moreover, some results based on these new terminologies are also included.

We begin with the definition of maximal limit of a fuzzy sequence.

Definition 4.1. *A fuzzy point x_λ is called a maximal limit of a sequence of fuzzy points S in a fuzzy topological space (X, δ) if and only if*

$$\lambda = max\{\mu \in (0, 1] : S \text{ converges to } x_\mu\}.$$

It is evident that if x_λ is a maximal limit of a sequence of fuzzy points, then no fuzzy points containing x_λ can be a limit of the fuzzy sequence. This justifies the term *maximal limit*.

By theorem 2.1, we immediately have the following theorem.

Theorem 4.1. *If x_λ is a maximal limt of a fuzzy sequence S, then x_μ is a limit of S if and only if $\mu \leq \lambda$.*

Now, analogous to the definition of derived set of an ordinary set, we define two types of sets namely l-set and l_m-set for sequence of fuzzy points, which will help us to characterize fuzzy indiscrete space.

Definition 4.2. *The set of all limits of a sequence of fuzzy points S, denoted by S_l, is called the l- set of the fuzzy sequence. That is,*

$$S_l = \{x_\lambda : S \text{ converges to } x_\lambda\}.$$

Definition 4.3. *The set of all maximal limits of a sequence of fuzzy points S, denoted by S_{l_m}, is called the l_m-set of the fuzzy sequence. That is,*

$$S_{l_m} = \{x_\lambda : x_\lambda \text{ is a maximal limit of } S\}.$$

It is clear from the definitions that both l-set and l_m-set of a sequence of fuzzy points are crisp sets of fuzzy points.

Result 4.1. *For any sequence $S = \{S_n, n \in N\}$ of fuzzy points, $S_{l_m} \subseteq S_l$ and $S_{l_m} = S_l$ if and only if $S_l = \phi$.*

Proof. Clearly, every maximal limit is a limit of the fuzzy sequence. Hence, the inclusion follows.

Now, suppose $S_{l_m} = S_l$. If $S_l \neq \phi$, then there exists $x_\lambda \in S_l$ for some $x \in X$ and $\lambda \in (0, 1]$. Then, by theorem 2.1, $x_\mu \in S_l$ for all $\mu \in (0, \lambda]$. That is, there exists $\mu < \lambda$ such that $x_\mu \in S_l$. But, this $x_\mu \notin S_{l_m}$, by definition. Therefore, $S_{l_m} \neq S_l$, which is a contradiction. Hence, $S_l = \phi$.
The converse is trivial in view of the fact that if a sequence of fuzzy points has no limits, then it obviously has no maximal limits. □

Theorem 4.2. *A fuzzy topological space (X, δ) is fuzzy indiscrete if and only if the l_m-set of any sequence of fuzzy points in (X, δ) is the fuzzy set X.*

Proof. By theorem 3.1, we have, a fuzzy topological space (X, δ) is fuzzy indiscrete if and only if every fuzzy sequences in (X, δ) converge to every fuzzy points. So, in a fuzzy indiscrete space, for each $x \in X$, x_1 is a maximal limit of any fuzzy sequence. Hence, S_{l_m} is the crisp set X. Also, if $S_{l_m} = X$ for any fuzzy sequence, then the converse of theorem 3.1 guarantees that (X, δ) is fuzzy indiscrete. □

Theorem 4.3. *In a fuzzy co-finite (or finer) topological space the l_m-set of any sequence of fuzzy points contains at most one element.*

Proof. By theorem 3.4, we have, in a fuzzy co-finite (or finer) topological space, if a fuzzy sequence converge to two fuzzy points then their supports are the same. So, in a fuzzy co-finite (or finer) topological space, the l_m-set of a sequence of fuzzy points is either empty or contain exactly one element.This proves the theorem. □

Theorem 4.4. *If $\delta_1, \delta_2, ...$ are topologies on a non-empty set X such that $\delta_1 \subseteq \delta_2 \subseteq ...$, then $S_l^1 \supseteq S_l^2 \supseteq ...$ for any sequence S, where S_l^i denotes the l-set of sequence S in (X, δ_i) for each i.*

Proof. As the topology on a set X get finer, the collection of Q-neighbourhoods of a fuzzy point x_λ get larger and so the possibility of x_λ to become the limit of a fuzzy sequence decreases. Hence, if $\delta_1, \delta_2, ...$ are topologies on a non-empty set X such that $\delta_1 \subseteq \delta_2 \subseteq ...$, then $S_l^1 \supseteq S_l^2 \supseteq ...$ for any fuzzy sequence S. □

Now we define fuzzy l-set of a sequence of fuzzy points.

Definition 4.4. *The smallest fuzzy set containing all the limits of a sequence of fuzzy points in a fuzzy topological space is called the fuzzy l-set of the sequence. In otherwords, the fuzzy l-set of a fuzzy sequence S in $(X, \delta)$is the set given by,*

$$\cap\{A \in I^X : A \text{ contains all the limits of } S\}.$$

Obviously, every maximal limit of a fuzzy sequence S is indeed a limit of S. So, the fuzzy l-set of S will contain all the maximal limits of S. Moreover, if a fuzzy set contain all the maximal limits of a fuzzy sequence, then it will definitely contain all the limits of the fuzzy sequence. Hence the fuzzy l-set of a fuzzy sequence can also be defined as the smallest fuzzy set containing all the *maximal limits* of the fuzzy sequence.

Theorem 4.5. *A fuzzy topological space (X, δ) is fuzzy indiscrete if and only if the fuzzy l-set of any sequence in (X, δ) is the fuzzy set X.*

Proof. Let (X, δ) be a fuzzy indiscrete topological space. Let $S = \{S_n, n \in N\}$ be a fuzzy sequence in (X, δ). Then, by theorem 3.1, every fuzzy point is a limit of sequence S. Hence, the fuzzy l-set of sequence S is the fuzzy set X. Since, sequence S was arbitrary, the necessary part of the theorem follows.

Conversely, suppose that the fuzzy l-set of any fuzzy sequence in (X, δ) is X. Consider the fuzzy sequence, $S = \{x_{1/n}, n \in N\}$. Now, since the fuzzy l-set of S is X, there does not exist $\lambda \in (0, 1)$ such that x_γ is not a limit of the fuzzy sequence S for every $\gamma \in (\lambda, 1)$. This implies that x_λ is a limit of $S, \forall \lambda \in (0, 1)$. Then, since the membership values of the terms of the fuzzy sequence S decreases to zero, as $n \to \infty$, and the support of the terms remain as x, it follows from the definition of convergence and Archimedean property that every Q-neighbourhood of x_λ has membership value 1 at x for any λ. Since, $x \in X$ was arbitrary, X is the only Q-neighbourhood of any fuzzy point in (X, δ). Then, as in the proof of the converse part of theorem 3.1, X is the only non-empty open fuzzy set in (X, δ). Hence, (X, δ) is fuzzy indiscrete. □

Theorem 4.6. *The support of the fuzzy l-set of any convergent sequence in a fuzzy co-finite (or finer) topological space (X, δ) is a singleton set.*

Proof. By theorem 3.4, in a fuzzy co-finite (or finer) topological space (X, δ), if a sequence of fuzzy points converge to two fuzzy points then their supports are the same. That is, all the limits of a convergent fuzzy sequence in a fuzzy co-finite (or finer) topological space have the same support. Hence, the support of the fuzzy l-set of any convergent sequence in a fuzzy co-finite (or finer) topological space (X, δ) is a singleton set. □

Corollary 4.1. *The fuzzy l-set of any sequence in a fuzzy co-finite (or finer) topological space (X, δ) is closed.*

Proof. Let (X, δ) be a fuzzy co-finite (or finer) topological space. We prove that the fuzzy l-set of any sequence in (X, δ) is closed. For, let S be a fuzzy sequence in (X, δ). First, suppose S is convergent. Then, by theorem 4.6, the support of the fuzzy l-set of S is a singleton set. So, the fuzzy complement of the fuzzy l-set of S is co-finite and hence is open. Thus, the fuzzy l-set of S is closed. Now, suppose S is not convergent. Then, obviously, the fuzzy l-set of S is ϕ and hence is closed. So, in either case, the fuzzy l-set of S is closed. Since S was arbitrary, the fuzzy l-set of any sequence in (X, δ) is closed, and so the corollary follows. □

5. Conclusion

In any area of analysis, sequences and their convergence play a vital role. In order to understand mathematical concepts more clearly, we can make use of examples and non-examples as powerful tools. In the present paper, we studied the concept of convergence of sequences of fuzzy points in some special fuzzy topological spaces such as fuzzy indiscrete, fuzzy co-finite, fuzzy co-countable, fuzzy discrete etc., in detail. Through this, we were able to characterize fuzzy indiscrete topological space by using fuzzy sequences and its convergence. Also, we obtained that convergent sequences in fuzzy co-finite and finer topological spaces behave identically. The study also revealed the fact that we cannot replace fuzzy nets by fuzzy sequences in the characerization theorem for fuzzy Hausdorff space by using fuzzy nets and its convergence. This helped us to understand the significance of fuzzy nets and its convergence in fuzzy topology.

References

[1] L. A. Zadeh, (1965), Fuzzy sets, Information and control, 8, 338-353.

[2] C. L. Chang, (1968), Fuzzy Toplogical Spaces, Journal of mathematical analysis and applications, 24, 182-190.

[3] R. Lowen, (1976), Fuzzy Topological Spaces and Fuzzy Compactness, Journal of Mathematical Analysis and Applications, 56, 621-633.

[4] C. K. Wong, (1974), Fuzzy Points and Local Properties of Fuzzy Topology, Journal of Mathematical Analysis and Applications, 46, 316-328.

[5] R. Srivastava, S.N. Lal and A. K. Srivastava, (1981), Fuzzy Hausdorff Topological Spaces, Journal of Mathematical Analysis and applications, 81, 497-506.

[6] P. P. Ming and L. Y. Ming, (1980), Fuzzy Topology. I. Neighborhood Structure of a Fuzzy point and Moore-Smith Convergence, Journal of Mathematical Analysis and Applications, 76, 571-599.

[7] M. Guloglu and D. Coker, (2005), Convergence in I-fuzzy topological spaces, Fuzzy Sets and Systems, 151, 615-623.

[8] M. M. Stadler and M. A. de P. Vicente, (1992), On N-convergence of fuzzy nets, Fuzzy Sets and Systems, 51, 203-217.

[9] D. N. Georgiou and B. K. Papadopoulos, (1999), Convergence in fuzzy topological spaces, Fuzzy Sets and Systems, 101, 495-504.

[10] V. Gregori and A. Vidal, (1999), Fuzziness in Chang's Fuzzy Topological Spaces, Rendiconti dell'Istituto di Matematica dell'Universita di Trieste, 111-121.

[11] F. G. Lupianez, (2006), Nets and filters in intuitionistic fuzzy topological spaces, Information Sciences, 176, 2396-2404.

[12] M. K. Luo, Y. M. Liu and J. H. Liang, (1998), On Convergence Classes in L-fuzzy Topological Spaces, Journal of Mathematical Analysis and Applications, 224, 316-331.

[13] A. A. Nouh, (2005), On Convergence Theory in Fuzzy Topological Spaces and its Applications, Czechoslovak Mathematical Journal, 55, 295-316.

[14] K. K. Mondal and S. K. Samanta, (2005), Fuzzy Convergence Theory-I, Journal of the Korean Society of Mathematical Education Series B: The Pure and Applied Mathematics, 12(1), 75-91.

[15] K. K. Mondal and S. K. Samanta, (2006), Convergence of Generalized Nets and Filters in Graded L-Fuzzy Topological Spaces, Journal of the Korean Society of Mathematical Education Series B: The Pure and Applied Mathematics, 13(1), 47-69.

[16] B. Chen, (2010), α-Generalized Convergence Theory of L- Fuzzy Nets and its Applications, Italian Journal of Pure and Applied Mathematics, 27, 167-178.

[17] W. Yao and L. X. Lu, (2011), Moore-Smith Convergence in L-Fuzzifying Topologial Spaces, Journal of Mathematical Research and Exposition, 31(5), 770-780.

[18] Z. Y. Xiu, Q. H. Li and B. Pang, (2020), Fuzzy Convergence Structures in the Framework of L-convex Spaces, Iranian Journal of Fuzzy Systems, 17(4), 139-150.

[19] G. J. Klir and B. Yuan, (1995), Fuzzy Sets and Fuzzy Logic Theory and Applications, Prentice Hall P T R, Upper Saddle River, New Jersey 07458.

[20] K. D. Joshi, (2006), Introduction to general topology, Second edition, New Age International (P) Limited, Publishers.

[21] Gunduz, C., Bayramov, S., Zeynalov, C., (2024), Separation Axioms in Bipolar Soft Topological Spaces, TWMS J. Pure Appl. Math., V.15, N.1, pp.15-25

Jyothis K. Mohan is graduated from K. E. College, Mannanam, Kerala, India in 2013 and has completed M.Sc. from B. C. M. College, Kottayam, Kerala in 2015. She is awarded with Junior Research Fellowship by CSIR, India in the year 2019. Currently she is a research scholar in Maharaja's College, Ernakulam, Kerala under the guidance of Dr. Sheeja T. K. Her research area is Fuzzy topology.

Dr. Sheeja T. K. is currently working as an Associate Professor of Mathematics at T. M. Jacob Memorial Govt. College, Manimalakunnu, Koothattukulam, Kerala, India. She completed Ph.D from M. G. University, Kottayam, Kerala under the guidance of Dr. Sunny Kuriakose A. She has 19 years of teaching experience in different Govt. Colleges under the Collegiate Education Department, Govt. of Kerala. Her areas of research interest include Fuzzy Set theory, Rough Set Theory and Graph theory. She has published 13 journal papers, 2 book chapters and 3 conference papers.

Dr. Sunny Kuriakose A. is currently the principal at St. Kuriakose College of Management and Science, Kuruppampady, Ernakulam, Kerala, India. He completed his doctoral studies at Cochin University of Sience and Technology under the guidance of Dr. T. Thrivikraman and was awarded the Ph.D degree in 1995. He has supervised 16 research scholars in Mathematics leading to Ph.D from Mahatma Gandhi University, Kottayam, Kerala. His areas of research include Fuzzy Mathematics, Fuzzy Economics and Graph Theory. He has published 67 journal papers and 11 conference papers and has authored/edited 12 books.