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RADIAL AND ANTIPODAL DOMINATION NUMBER OF TREES

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ABSTRACT. Motivated by the existing global domination parameter γ_g, in this paper we make the first move for the study of radial and antipodal domination numbers of a graph. A set $S \subseteq V(G)$ is a radial (an andipodal) dominating set of a graph G if S is a dominating set of both G and $R(G)$ $(A(G))$, where $R(G)$ $(A(G))$ is the radial (andipodal) graph of G. The radial (antipodal) domination number $\gamma_{rad}(G)$ $(\gamma_{diam}(G))$ equals the minimum cardinality of a radial (an antipodal) dominating set of G. For any tree with radius atleast two, we have established the chain $\gamma \leq \gamma_g \leq \gamma_{rad} < \gamma_{diam}$. For any tree T with $rad(T) = 2$, γ_{rad}=γ or $\gamma + 1$. If T is a tree of order n, then γ_{diam}=$\gamma + 1$ if and only if T is either $K_{1,n-1}$ or $K^{'}_{1,n-1}$, where $K^{'}_{1,n}$ is a tree obtained from $K_{1,n-1}$ by adding a new vertex to a pendent. We presented linear programming formulation to identify radial and antipodal dominating sets of a graph. Computational time studied was carried out on randomly generated graphs by use of MATLAB function. The computational time to find radial and antipodal domination was recorded as less than a second for those graphs with fewer than 100 nodes.

Keywords: global, radial, antipodal domination, LP formulation

AMS Subject Classification: 05C12, 05C69

1. Introduction

For graph theoretic notation and terminology, we follow [18]. For a graph $G = (V, E)$, *the distance* $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The *eccentricity* $e(u)$ of a vertex u is the distance to a vertex farthest from u. The *radius* of G is defined by $rad(G) = \min\{e(u) \mid u \in V(G)\}$ and the *diameter* of G is defined by $diam(G) = \max\{e(u) \mid u \in V(G)\}$. A vertex v in G is called a *central vertex* if $e(v) = rad(G)$. A graph G is called *self-centered graph* if all the vertices of G

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have the same eccentricity, so $rad(G) = diam(G)$. A path of G is called the *diametrical path* if the distance between its end vertices is $diam(G)$. The concept of radial graph was introduced and developed by [7]. Also the concept of antipodal graph was initiated by [17] and was further expanded by [1, 2]. The *radial graph* of a graph G, denoted by $R(G)$, is the graph on the same vertex set as in G with two vertices are adjacent in $R(G)$ if $d(u, v) = rad(G)$. If G is disconnected, then two vertices are adjacent in $R(G)$ if and only if they belong to different components of G. The *antipodal graph* of a graph G, denoted by $A(G)$, is the graph on the same vertex set as in G with two vertices are adjacent in $A(G)$ if $d(u, v) = diam(G)$. If G is disconnected, then two vertices are adjacent in $A(G)$ if and only if they belong to different components of G. A graph G is called a radial (an antipodal) graph if $R(H) = G\ (A(H) = G)$, for some graph H.

In this paper, we introduce the new domination parameters called radial, antipodal domination numbers of a graph. A set $S \subseteq V$ is called a dominating set of G if every vertex in $V - S$ has a neighbour in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. The set with cardinality γ is called γ-set of G. A set $S \subseteq V$ is called a global dominating set of G if S is a dominating set of both G and its complement $\overline{G}$. The minimum cardinality of all global dominating sets of G is called the global domination number $\gamma_g(G)$ of G. We define, a set $S \subseteq V$ is called a radial dominating set of G if S is a dominating set of both G and $R(G)$. A set $T \subseteq V$ is called an antipodal dominating set of G if T is a dominating set of both G and $A(G)$. The radial (antipodal) domination number $\gamma_{rad}(G)$ ($\gamma_{diam}(G)$) of G is the minimum cardinality of the radial (antipodal) dominating set of G.

2. Main Results

We start with some basic properties and results of $\gamma_{rad}(G)$ and $\gamma_{diam}(G)$.

Lemma 2.1.

1 . *For any graph G of order n,* $\gamma(G) \leq \min\{\gamma_{rad}(G),\ \gamma_{diam}(G)\}$.
2 . $\gamma_{rad}(G) \geq \max\{\gamma(G), \gamma(R(G))\}$.
3 . $\gamma_{diam}(G) \geq \max\{\gamma(G), \gamma(A(G))\}$.

Proposition 2.2. *For the complete graph $K_n, n \geq 1$, $\gamma_{rad}(K_n) = 1$ and $\gamma_{diam}(K_n) = 1$.*

Proof. Since K_n is a self-centered graph with $rad(K_n)$=$diam(K_n)$= 1, $R(K_n) \cong K_n$ and $A(K_n) \cong K_n$. □

Corollary 2.3. $\gamma_{diam}(G) = 1$ *if and only if G is complete.*

Proposition 2.4. *For the path $P_n,\ n \geq 2$,*

$$\gamma_{rad}(P_n) = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd and } n \neq 5 \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \\ \dfrac{n+1}{2}, & \text{if } n = 5 \end{cases}$$

and $\gamma_{diam}(P_n) = n - 1$.

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of P_n. The radial and antipodal graphs of P_n are

$$R(P_n) \cong \begin{cases} \left(\frac{n}{2}\right) K_2, & \text{if } n \text{ is even} \\ P_3 \cup \left(\frac{n-3}{2}\right) K_2, & \text{if } n \text{ is odd} \end{cases}.$$

and $A(P_n) \cong K_2 \cup ((n-2)K_1)$. When $n = 5$, $\gamma_{rad}(P_5) = 3$. Let n be odd and $n \neq 5$. Since the number of components in $R(P_n)$ is $1 + \left(\frac{n-3}{2}\right)$ and the domination number of any graph is greater than or equal to the number of components, so $\gamma_{rad}(P_n) \geq \left(\frac{n-1}{2}\right)$. Let r be the radius of P_n. Then v_{r+1} is a central vertex of P_n. The set $D = \{v_{r+1}\} \cup \{v_i \ / \ i \equiv 0(mod\ 2) \text{ and } 2 \leq i \leq r\} \cup \{v_i \ / \ i \equiv 1(mod\ 2) \text{ and } r+1 \leq i \leq n-1\}$ is a dominating set of both P_n and $R(P_n)$. Then $|D| = \left(\frac{n-1}{2}\right)$ and so $\gamma_{rad}(P_n) \leq \left(\frac{n-1}{2}\right)$. Hence $\gamma_{rad}(P_n) = \left(\frac{n-1}{2}\right)$. For even cases, it can be easily seen. □

Corollary 2.5. *A graph G with n vertices has a single diametrical path if and only if $\gamma_{diam}(G) = n - 1$.*

Proof. G has a single diametrical path if and only if the graph $A(G)$ have only one edge. So $\gamma(A(G)) = n - 1$. □

Proposition 2.6. *For the cycle C_n, $n \geq 3$,*

$$\gamma_{rad}(C_n) = \gamma_{diam}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1, & \textit{if } n \textit{ is odd and } n \equiv 5(mod\ 6) \\ \left\lceil \frac{n}{3} \right\rceil, & \textit{if } n \textit{ is odd and } n \not\equiv 5(mod\ 6) \\ \frac{n}{2}, & \textit{if } n \textit{ is even} \end{cases}$$

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of C_n. Since C_n is a self-centered graph with $rad(C_n) = diam(C_n)$, $R(C_n) \cong A(C_n) \cong \begin{cases} C_n, & \text{if } n \text{ is odd} \\ \left(\frac{n}{2}\right) K_2, & \text{if } n \text{ is even} \end{cases}$. Case (i): n is odd and $n \equiv 5(mod\ 6)$. Let C_n $(n \geq 11)$. Every γ-set of C_n is obtained by choosing one element for every three consecutive vertices. Then any dominating set of C_n for $n = 3k+2$ is of the form $D' = \{v_{[l]}, v_{[l+3]}, v_{[l+6]}, \ldots, v_{[l+3k]}\}$, where $[l]$ denotes multiplication modulo n and $|D'| = k + 1$. Let $r = rad(C_n)$. Here, $r = \frac{n-1}{2}$ and $r \equiv 2(mod\ 3)$. Since $r + 1 \equiv 0(mod\ 3)$, every γ-set contains at least two vertices with distance $r + 1$ in C_n. $N(v_{[l]}) = \{v_{[l+r]}, v_{[l+r+1]}\}$ and $N(v_{[l+r]}) = \{v_{[l+r+1]}, v_{[l+r+2]}\}$ in $R(C_n)$. So, $v_{[l]}$ dominates $v_{[l+r+1]}$ and $v_{[l+r]}$ dominates $v_{[l+r+2]}$ in $R(C_n)$. There are remaining $k - 1$ vertices in D'. Suppose each of these vertices dominate exactly two vertices in $R(C_n)$, then totally $2(k-1) + 2 = 2k$ vertices are dominated by D' in $R(C_n)$. Therefore, there is a vertex of $R(C_n)$ which is not dominated by D'. Hence, every dominating set of C_n is not a radial dominating set of $R(C_n)$. So, $\gamma_{rad}(C_n) \geq \gamma + 1$. Suppose $x \in V(C_n)$ is not dominated by D'. Then $D' \cup \{x\}$ is a radial dominating set of C_n and $D' = \gamma + 1$. Hence, $\gamma_{rad} = \gamma + 1$. Case (ii): n is odd and $n \not\equiv 5(mod\ 6)$. This case can be seen as two sub-cases such as $n \equiv 0(mod\ 3)$ and $n \equiv 1(mod\ 3)$. The set $D' = \{v_1, v_4, v_7, \ldots, v_{3(k-1)+1}\}$ is a radial dominating set of C_n, where $n = 3k$ and k is odd. Since $|D'| = \gamma(C_n)$ and D' is also a dominating set of $R(C_n)$, $\gamma_{rad}(C_n) = \gamma$. The set $D'' = \{v_1, v_4, v_7, \ldots, v_{\frac{n-5}{2}}, v_{\frac{n-1}{2}}, v_{\frac{n+5}{2}}, \ldots, v_{3(k-1)}\}$ is a radial dominating set of C_n for $n \equiv 1(mod\ 3)$, where $n = 3k + 1$. Since $|D''| = \gamma(C_n)$ and D'' is also a dominating set of $R(C_n)$, $\gamma_{rad}(C_n) = \gamma(C_n)$. For even cases, it can be easily seen. □

Proposition 2.7. *For the star graph $K_{1,n}$, $n \geq 2$, $\gamma_{rad}(K_{1,n}) = 1, \gamma_{diam}(K_{1,n}) = 2$.*

Proof. Since $rad(K_{1,n})$=1 and $diam(K_{1,n})$=2, $R(K_{1,n}) \cong K_{1,n}$ and $A(K_{1,n}) \cong K_1 \cup K_n$. □

Proposition 2.8. *For the complete r-partite graph $K_{n_1,n_2,\cdots,n_r}$, where $n_i \geq 2$ for all $1 \leq i \leq r$, $\gamma_{rad}(K_{n_1,n_2,\cdots,n_r}) = \gamma_{diam}(K_{n_1,n_2,\cdots,n_r}) = r$.*

Proof. Since $K_{n_1,n_2,\cdots,n_r}$ is a self-centered graph with rad=$diam$=2, $R(K_{n_1,n_2,\cdots,n_r}) \cong A(K_{n_1,n_2,\cdots,n_r}) \cong K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$. □

Proposition 2.9. *For the wheel graph $W_{1,n}$, $n \geq 4$, $\gamma_{rad}(W_{1,n}) = 1, \gamma_{diam}(W_{1,n}) = 3$.*

Proof. Since $rad(W_{1,n})$=1 and $diam(W_{1,n})$=2 for a wheel graph, $R(W_{1,n}) \cong W_{1,n}$ and $A(W_{1,n}) \cong K_1 \cup H$, where H is the $(n-3)-$regular graph. □

3. Trees

In this section, we obtain γ_{rad} and γ_{diam} for a tree of order n. Throughout this section, let P denote the set of pendent vertices of a tree T and l denote the number of vertices in P. That is, $|P| = l$. It is easy to verify that, for a tree, other than K_2, the bounds of radial and antipodal domination numbers are $1 \leq \gamma_{rad} \leq n - l$ and $2 \leq \gamma_{diam} \leq n - 1$ respectively. For any tree T, $diam(T) = 2rad(T)$ or $2rad(T) - 1$.

Theorem 3.1. [10] *Let S be the set of support vertices of a tree T. If $diam(T) \in \{2, 3, 4, 5\}$, then $\gamma(T) = |S|$.*

In [14], a vertex v in a graph G is called an eccentric vertex of a center vertex if $d(u, v) = e(u) = rad(G)$ for some vertex u. Let $EC(T)$ be the set of all eccentric vertices of T. A pendent vertex v of G is said to be lonely pendent if v is the one and only pendent of its support. Let $LP(T)$ be the set of all lonely pendent vertices of T.

Theorem 3.2. *For any tree T with order $n \geq 2$, $\gamma_{diam}(T) \geq n - l + 1$. Moreover, $\gamma_{diam}(T) = n - l + 1$ if and only if $P(T) \subseteq EC(T)$ and $LP(T) \neq \phi$.*

Proof. Since every non-pendent vertices of T are isolates in $A(T)$, at least $n-l+1$ vertices are needed to dominate the vertices in T and $A(T)$.

To prove equality, let us assume that T be a tree of order n and $\gamma_{diam}(T) = n - l + 1$. Let $w \in P(T)$ and $deg(w)$ in $A(T)$ be $l-1$. Let $x \in P(T)$ and $x \neq w$. Suppose $x \notin EC(T)$. Then $d(x, u) \leq rad(T) - 2$, for all $u \in C(T)$, where $C(T)$ is the set of all centers of T. Let $xv_1v_2 \cdots v_{diam(T)-1}w$ be a diametrical path in T and v_i be a central vertex. $diam(T) = d(x, v_i) + d(v_i, w) \leq 2rad(T) - 2$, which is a contradiction to the fact that $diam(T) = 2rad(T)$ or $2rad(T) - 1$, for any tree T. Then $x \in EC(T)$ and $P(T) \subseteq EC(T)$. Since every pendent vertex has diametrical path with w in T, w must be a lonely pendent vertex of T. Then $w \in LP(T) \neq \phi$.

Let T be a tree with $P(T) \subseteq EC(T)$ and $LP(T) \neq \phi$. Let $w \in LP(T)$. Suppose T has a unique central vertex and let it be a. Then $diam(T) = 2rad(T)$. Since $P(T) \subseteq EC(T)$, any pendent vertex of T other than w has a diametrical path with w. Then $deg(w)$ in $A(T)$ is $|P| - 1$. The set of all non-pendent of w forms an antipodal dominating set of T. Therefore, $\gamma_{diam}(T) \leq n-l+1$ and since $\gamma_{diam}(T) \geq n-l+1$, we get $\gamma_{diam}(T) = n-l+1$.

Suppose T has two center vertices and let it c_1 and c_2. Let $C_1 = \{x \in P(T) \mid d(x, c_1) = rad(T)\}$ and $C_2 = \{x \in P(T) \mid d(x, c_2) = rad(T)\}$. From the definition of C_1 and C_2, $C_1 \cap C_2 = \phi$. Assume that with out loss of generality, $w \in C_2$. Since $e(c_1) = e(c_2) = rad(T)$, $d(x, c_1) = rad(T) - 1$, for all $x \in C_2$ and $d(x, c_2) = rad(T) - 1$, for all $x \in C_1$. Let $x \in C_1$. Then $d(x, c_1) = rad(T)$ and $d(w, c_1) = rad(T) - 1$. Consider the path $P' : P_{xc_1}P_{c_1w}$. Since the length of P' is a $2rad(T) - 1$, it is a diametrical path of T. Then every vertex of C_1 has a diametrical path with w in T. Let $x \in C_2$. Then $d(x, c_2) = rad(T) - 1$ and $d(w, c_2) = rad(T)$. Again consider the path $P'' : P_{xc_2}P_{c_2w}$ which is a diametrical path of T. Every vertex of C_2 other than w has a diametrical path with w in T. Hence $deg(w)$ in $A(T)$ is $|P| - 1$. The set of all non-pendent vertices and w forms an antipodal dominating set of T. Therefore, $\gamma_{diam}(T) \leq n - l + 1$ and since $\gamma_{diam}(T) \geq n - l + 1$, $\gamma_{diam}(T) = n - l + 1$ □

Sampathkumar [15] and Rall [13] shown that, if T is a tree, then $\gamma(T) \leq \gamma_g(T) \leq \gamma(T) + 1$. In addition Rall proved that the stars and a certain type of tree T of diameter four has $\gamma_g(T) = \gamma(T) + 1$ and other trees has $\gamma_g(T) = \gamma(T)$.

Theorem 3.3. [13] *Let T be a tree with $\gamma_g(T) = \gamma(T) + 1$ if and only if T is a star or T is a tree of diameter four which is constructed from two or more stars, each having at least two leaves, by connecting the centers of these stars to a common vertex.*

T is a tree with $rad(T) = 1$, it is obviously a star. Then the domination number and radial domination number is one whereas the global domination number is two. So here, $\gamma(K_{1,n}) = \gamma_{rad}(K_{1,n}) < \gamma_g(K_{1,n})$. In the following theorem we have established the chain between γ, γ_g, γ_{rad} and γ_{diam} for any tree with radius at least two.

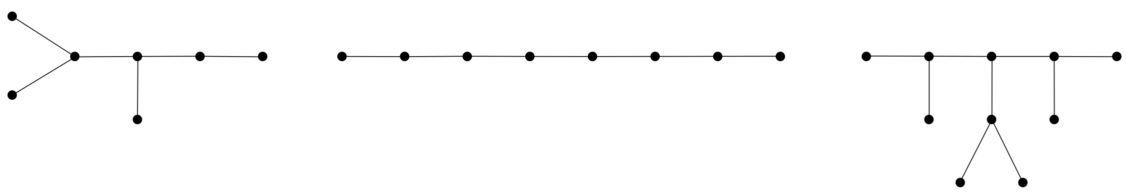
Theorem 3.4. *For any tree T with $rad(T) \geq 2$, $\gamma \leq \gamma_g \leq \gamma_{rad} < \gamma_{diam}$.*

Proof. Since $\gamma_g(T) = \gamma(T)$ or $\gamma(T) + 1$, we have the following cases. When $\gamma_g = \gamma$, $\gamma_g \leq \gamma_{rad}$. Suppose $\gamma_g = \gamma + 1$. T is a tree obtained from the construction described in Theorem 3.3. Here radius of T is two and the common vertex is not a support vertex where as all other non-pendent vertices are support vertices. Hence by Theorem 3.3, $\gamma_{rad}(T) = \gamma(T) + 1 = \gamma_g(T)$.

Let T be a tree of order n. Let $D = V - P$. Since the eccentricity of a pendent vertex in a tree is greater than radius, every pendent vertex in P having a distance r to a non-pendent vertex in T. Then D is a dominating set for both T and $R(T)$. $\gamma_{rad}(T) \leq |D| = n - l$. From Theorem 3.2, $\gamma_{diam}(T) \geq n - l + 1$. So $\gamma_{rad}(T) \leq n - l < n - l + 1 \leq \gamma_{diam}(T)$. Hence $\gamma_{rad}(T) < \gamma_{diam}(T)$. □

Remark 3.5. For a star, $\gamma_{rad} = 1$ and $\gamma_{diam} = 2$. Then $\gamma_{rad} < \gamma_{diam}$ holds for any tree of order $n \geq 2$.

Example 3.6. Trees with a) $\gamma = \gamma_g = \gamma_{rad} = 3 < \gamma_{diam} = 4$, b) $\gamma = \gamma_g = 3 < \gamma_{rad} = 4 < \gamma_{diam} = 7$, c) $\gamma = 3 < \gamma_g = 4 = \gamma_{rad} < \gamma_{diam} = 6$,



a) b) c)

Theorem 3.7. *Let T be a tree with $rad(T) = 2$. Then $\gamma_{rad}(T)$ is either $\gamma(T)$ or $\gamma(T)+1$.*

Proof. Let T be a tree with radius 2. Suppose every non-pendent vertex is a support vertex. Let D be the collection of all non-pendent vertices of T. Every pendent vertex having distance r with at least one non-pendent vertex of T. Hence D is dominating set of $R(T)$. By Theorem 3.1, D is a minimum dominating set of T and $\gamma(T) = |D|$. Therefore, $\gamma_{rad}(T) \leq |D| = \gamma(T)$. We know that, $\gamma_{rad}(T) \leq |D| = \gamma(T)$. Hence, $\gamma_{rad}(T) = \gamma(T)$.

Let T be a tree with radius 2 and having a non-pendent vertex which is not a support vertex. Let $diam(T) = 3$ and $P : v_1v_2v_3v_4$ be a diametrical path of T. Any other vertex of T must be adjacent with either v_2 or v_3. Hence v_2 and v_3 both of them are support vertices which is a contradiction to the fact that T has a non-pendent vertex which is a support vertex. Therefore $diam(T) = 4$. Let $P : v_1v_2v_3v_4v_5$ be a diametrical path of T. Here v_2 and v_4 are support vertices. Let x be a neighbour of v_3. Since $diam(T) = 4$, neighbour of x must be a pendent. Then x is a support vertex of T. Hence v_3 is the only non-pendent vertex which is not a support vertex. It is clearly seen that v_3 is the central vertex of T and every pendent vertex of T is adjacent only with the central vertex in $A(T)$. Hence every γ_{rad}-set of T should contain the central vertex. Let D be the set of all support vertices of T. Since the central vertex is the one and only pendent vertex which is not a support vertex, $|D| = n - l - 1$. By Theorem 3.1, $\gamma(T) = |D|$. Then $\gamma_{rad}(T) \geq |D| + 1 = n - l - 1 + 1 = n - l$. The upper bound for $\gamma_{rad}(T)$ is $n - l$. Hence $\gamma_{rad} = n - l = |D| + 1 = \gamma + 1$. □

Definition 3.8. *A $K'_{1,n-1}$ on n vertices is a tree obtained from $K_{1,n-2}$ by adding a vertex v to a pendent vertex of $K_{1,n-2}$*

Theorem 3.9. *Let T be a tree of order n. Then $\gamma_{diam}(T) = \gamma(T) + 1$ if and only if T is either $K_{1,n-1}$ or $K'_{1,n-1}$.*

Proof. It is obvious that, if T is either $K_{1,n-1}$ or $K'_{1,n-1}$, then $\gamma_{diam}(T) = \gamma(T) + 1$.

On the other hand, let T be a tree with $\gamma_{diam}(T) = \gamma(T) + 1$. Since $V - P$ is the set of non-pendent vertices and it is a dominating set of T, $\gamma(T) \leq n - l$. Suppose $\gamma_{diam}(T) \geq n - l + 2$. $\gamma_{diam}(T) \geq \gamma(T) + 2$, which is a contradiction. Then $\gamma_{diam}(T) < n - l + 2$. Thus $\gamma_{diam}(T) \leq n - l + 1$. By Theorem 3.2, $\gamma_{diam}(T) \geq n - l + 1$. Hence $\gamma_{diam}(T) = n - l + 1$. Since $\gamma(T) = n - l$, every non-pendent vertex is a support vertex in T. Since $\gamma_{diam}(T) = n - l + 1$ and $A(T)$ has $n - l$ isolates, there is a vertex u in $A(T)$ with degree $n - l - 1$.

Suppose T has more than 2 support vertices. Then $diam(T) \geq 4$. Let $P : u_1u_2 \cdots u_{n-1}u_n(= u)$ be a diametrical path in T and $x \in T$ be a pendent vertex adjacent to u_3. Since $d(x, u_n) < diam(T)$, x is not adjacent to u_n in $A(T)$, which is a contradiction. Hence T has at most 2 support vertices and $diam(T) \leq 3$. Let $diam(T) = 3$ and $P' : v_1v_2v_3v_4$ be a diametrical path of T. Without loss of generality, assume $v_3 \in LP(T)$. If $v_2 \in LP(T)$, then $T \cong K'_{1,2}$. Suppose $v_2 \notin LP(T)$. Then v_2 have at least one neighbour other than v_1 and all the neighbours of v_2 are pendent vertices. $< V(T) - \{v_4\} > \cong K_{1,n-1}$ and v_4 is a vertex attached to the pendent v_3 of $K_{1,n-1}$. Hence, $T \cong K'_{1,n-1}$. A tree of $diam(T) = 2$ is a star. □

Let $d_p(v)$ be the number of pendent vertices adjacent to a vertex v in T.

Proposition 3.10. *Let T be a tree with $d_p(v) \geq 1$, for all $v \in V - P$. Then $\gamma(T) = n - l = \gamma_{rad}(T)$.*

Proof. Suppose T is a tree with $d_p(v) \geq 1$, for all $v \in V - P$. Since $V - P$ is a minimum dominating set of both of T and $R(T)$, the result follows. □

Proposition 3.11. *For any two positive integers k and n with $2 \leq k \leq n-1$, there exists a tree T of order n with $d_p(v) \geq 1$, for all $v \in V-P$ and $\gamma_{diam}(T) = k$.*

Proof. Up to $n = 8$, it is easily check it out.

Let $n \geq 9$. Then the following trees (i) star with n vertices having $\gamma_{diam}(T) = 2$, (ii) $\gamma_{diam}\left(K'_{1,n-1}\right) = 3$ and $\gamma_{diam}(P_n) = n-1$ satisfy the result for $k = 2$, 3 and $n-1$ respectively. It is enough to show for $4 \leq k \leq n-2$.

Case (1): $k = 4$ Let T be a tree obtained from $P_5 : v_1v_2v_3v_4$ by adding $n-5$ vertices to v_2 and one vertex w to v_3. Then $\{v_2, v_3, v_4, w\}$ is a minimum dominating set of both T and $A(T)$. Hence $\gamma_{diam}(T) = 4$.

Case (2): $5 \leq k \leq n-2$ Let T be a tree obtained from $P_5 : v_1v_2v_3v_4v_5$ by adding $(n-k-1)$ vertices to v_2 and $k-4$ vertices $u_1, u_2, \cdots, u_{k-4}$ to v_3. Then $\{v_2, v_3, v_4, v_5, u_1, u_2, \cdots, u_{k-4}\}$ is a dominating set for both T and $A(T)$ and $|D| = k$. Hence $\gamma_{diam}(T) = k$. □

Theorem 3.12. *Let T be a tree of order n. Then $\gamma_{diam}(T) = \gamma_{rad}(T) + 1$ and $\gamma_{rad}(T) = \gamma(T) + 1$ if and only if T is a tree of diameter four which is constructed from two or more stars by connecting the centers to a pendent of P_3.*

Proof. Let T be a tree described in the Theorem. Since $diam(T) = 4$ and by Theorem 3.1 the set of all non-pendent vertices except the common vertex form a minimum dominating set of T, $\gamma(T) = |V-P| - 1 = n-l-1$. Since the set of all non-pendent vertices form a minimum radial dominating set of T, we have $\gamma_{rad}(T) = |V-P|$. Also the set of non-pendent vertices and pendent of P_2, described in the Theorem, form a minimum antipodal dominating set of T. Then $\gamma_{diam}(T) = |V-P| + 1$. Hence $\gamma_{rad}(T) = \gamma(T) + 1$ and $\gamma_{diam}(T) = \gamma_{rad}(T) + 1$.

Let T be a tree of order n with $\gamma_{diam}(T) = \gamma_{rad}(T) + 1$ and $\gamma_{rad}(T) = \gamma(T) + 1$. By Theorem 3.2, $\gamma_{diam}(T) \geq |V-P|+1$. Then, $\gamma_{rad}(T)+1 \geq |V-P|+1$. $\gamma_{rad}(T) \geq |V-P|$. $\gamma(T) + 1 \geq |V-P|$. So $\gamma(T) \geq |V-P| - 1$. $V-P$ is a dominating set of T implies that $\gamma(T) \leq |V-P|$. Hence, $|V-P| - 1 \leq \gamma(T) \leq |V-P|$. Suppose $\gamma(T) = |V-P|$. Since the set of all non-pendent vertices forms a radial dominating set of T, $\gamma_{rad}(T) \leq |V-P|$. Hence $\gamma(T) \leq \gamma_{rad}(T) \leq |V-P|$ and so $\gamma_{rad}(T) = |V-P|$, which is a contradiction to $\gamma_{rad}(T) = \gamma(T) + 1$. Thus, $\gamma(T) = |V-P| - 1$, $\gamma_{rad}(T) = |V-P|$ and $\gamma_{diam}(T) = |V-P| + 1$. That is, $\gamma(T) = n-l-1$, $\gamma_{rad}(T) = n-l$ and $\gamma_{diam}(T) = n-l+1$. Let $diam(T) = 3$ and $v_1v_2v_3v_4$ be a diametrical path of T. Let u be a neighbour of v_2 in T. Since $diam(T) = 3$, u must be a pendent. Every neighbour of v_2 and v_3 is a pendent vertex. Then the set $\{v_2, v_3\}$ is a minimum dominating set of T and $\gamma(T) = 2 = |V-P|$, which is a contradiction to $\gamma(T) = |V-P| - 1$. Therefore, $diam(T) \geq 4$. Suppose $diam(T) \geq 5$. Since $\gamma_{diam}(T) = n-l+1$, $\langle P \rangle$ in $A(T)$ is a star. Let $u \in P$ and $deg(u) = |P-1|$ in $A(T)$. Let $v_1, v_2, \cdots, v_{diam(T)}(= u)$ be a diametrical path in T. Then every neighbour of v_2 and $v_{diam(T)-1}$ are pendent vertices. Let x be a pendent neighbour of v_3. Then $d(x, u) = diam(T) - 1$, which a contradiction to $d(x, u) = diam(T)$, for all $x \in P$. Hence $diam(T) = 4$. Let $v_1v_2v_3v_4v_5(= u)$ be a diametrical path in T. If v_3 has a neighbour x, with $deg(x) = 1$, then x is the isolate vertex in $A(T)$, which is a contradiction. Then $deg(x) \geq 2$. Since $\gamma(T) = |V-P| - 1$, T contains only one non-pendent vertex. So v_3 is the non-pendent vertex and is not a support. Since $diam(T) = 4$, neighbour of x other than v_3 must be pendent. Then T is a tree of diameter four which is constructed from two or more stars by connecting the centers to a common vertex. Since degree of u is $|P|-1$ in $A(P)$, u is a lonely pendent vertex of T. Therefore, one of those star should be P_2. Hence T has the structure as claimed in the statement of the Theorem. □

4. Linear programming formulation

Recently, many variety of domination parameters are computed by formulating the domination problem into a linear programming problem. Kartica et. al., presented linear programming formulation for convex and weakly convex domination set problem [8]. Weighted total domination problem and double roman domination problem are formulated as integer linear programming problem in [4, 9]. We have provided a linear programming formulation for both radial and antipodal domination number of a graph. Further, we have developed MATLAB programming function for these domination parameters. Computational time have been calculated for randomly generated graphs. As described in the earlier work [11, 16], linear programming formulation problem to identify restrained domination, strong domination and strong restrained domination. Also we developed linear programming formulation for generalized domination parameters such as (k, k', k'')-domination, $[j, k]$-domination, efficient (j, k)-domination, factor domination and R-domination [12]. By use of this formulation, MATLAB functions were coded, and further simulation study was carried out to understand the chain relationship among these domination parameter under four different types of network topologies. So, these type of formulation will be supportive for the future studies on these domination parameters.

Let G be a graph with the vertex set $v_1, v_2, \cdots, v_n$. Let A be the adjacency matrix of G. The graph G^k is obtained from G with same vertex set and two vertices v_1 and v_2 are adjacent if $d(v_1, v_2) = k$. Let $rad = r$ and $diam = d$. Then G^r and G^d are the radial and antipodal graph of G, respectively. Then $A^k = [a^k_{ij}]$ denotes the adjacency matrix of G^k. Let $C = [c_{ij}] = A + I_{n\times n}$, $D = [d_{ij}] = A^r + I_{n\times n}$ and $F = [f_{ij}] = A^d + I_{n\times n}$. Then the linear programming formulation for radial domination number is defined as follows:
Minimum $x_1 + x_2 + \cdots + x_n$

$$\text{Subject to } \sum_{j=1}^{n} c_{ij}x_j \geq 1 \text{ for } i = 1, 2, \ldots, n \tag{4.1}$$

$$\sum_{j=1}^{n} d_{ij}x_j \geq 1 \text{ for } i = 1, 2, \ldots, n \tag{4.2}$$

$$x_i \in \{0, 1\} \text{ for } i = 1, 2, \ldots, n \tag{4.3}$$

Let $RD = \{v_i | x_i = 1 \text{ for all } i = 1, 2, \ldots, n\}$, where x_i is the output yield by the BLPP. Then the binary linear programming problem is:
Minimum $x_1 + x_2 + \cdots + x_n$

$$\text{Subject to } \sum_{j=1}^{n} c_{ij}x_j \geq 1 \text{ for } i = 1, 2, \ldots, n \tag{4.4}$$

$$\sum_{j=1}^{n} f_{ij}x_j \geq 1 \text{ for } i = 1, 2, \ldots, n \tag{4.5}$$

$$x_i \in \{0, 1\} \text{ for } i = 1, 2, \ldots, n \tag{4.6}$$

The set $AD = \{v_i | x_i = 1 \text{ for all } i = 1, 2, \ldots, n\}$ is obtained by the decision variables x_i in the above BLPP. We define, ${deg_G}^D(v)$ is the cardinality of $N_G(v) \cap D$, where $D \subseteq V(G)$. The following theorems are ensuring that the optimality of the BLPP's.

Theorem 4.1. *The set RD is a minimum radial dominating set of G.*

Proof. Since the BLPP is the minimization problem, it is enough to show that RD is a radial dominating set of G. For any vertex $v_i \in V$, the sum of the ith row of (4.1) and

(4.2) are

$$\sum_{j=1}^{n} c_{ij}x_j = \begin{cases} deg_G^{RD}\, v_i + 1 & \text{if } v_i \in RD \\ deg_G^{RD}\, v_i & \text{if } v_i \notin RD \end{cases}$$

and

$$\sum_{j=1}^{n} d_{ij}x_j = \begin{cases} deg_{G^r}^{RD}\, v_i + 1 & \text{if } v_i \in RD \\ deg_{G^r}^{RD}\, v_i & \text{if } v_i \notin RD \end{cases}$$

respectively. Since RD satisfies (4.1) and (4.2) constraints, $deg_G^{RD}\, v_j$ and $deg_{G^r}^{RD}\, v_j$ is greater than are equal to 1 for any vertex $v_j \notin RD$. This implies that, v_j is adjacent to at least one vertex of RD in both G and G^r. Hence, RD is a radial dominating set of G. □

Theorem 4.2. *The set AD is a minimum antipodal dominating set of G.*

Proof. Since the BLPP is the minimization problem, it is enough to show that AD is an antipodal dominating set of G. For any vertex $v_i \in V$, the sum of the ith row of (4.4) and (4.5) are

$$\sum_{j=1}^{n} c_{ij}x_j = \begin{cases} deg_G^{AD}\, v_i + 1 & \text{if } v_i \in AD \\ deg_{AD}^{G}\, v_i & \text{if } v_i \notin AD \end{cases}$$

and

$$\sum_{j=1}^{n} f_{ij}x_j = \begin{cases} deg_{G^d}^{AD}\, v_i + 1 & \text{if } v_i \in AD \\ deg_{G^d}^{AD}\, v_i & \text{if } v_i \notin AD \end{cases}$$

respectively. Since AD satisfies (4.4) and (4.5) constraints, $deg_G^{AD}\, v_j$ and $deg_{G^d}^{AD}\, v_j$ is greater than are equal to 1 for any vertex $v_j \notin AD$. This implies that, v_j is adjacent to at least one vertex of AD in both G and G^d. Hence, AD is an antipodal dominating set of G. □

Experiments were carried out with a computer having the specification as Intel(R) Core(TM) $i5-3470S$ CPU @ 2.90 GHz 2.90 GHz, 4.00GB RAM and windows 7 operating system. To solve linear programming, we have used CPLEX optimization solver. The computational time for finding radial and antipodal domination number was studied with randomly generated graph Erdos-Renyi graphs. These graphs are generated by use of the MATLAB code `step1_randomgraph`. 12 random graphs are generated with the probability value of 0.25, 0.5 and 0.75. Random graphs with fewer than 100 nodes are recorded as less than a second as computational time (Table 1). Further study may be carried out to reduce the computational time by reducing number of constraints or choosing initial point of approximation.

5. Conclusions

In this paper we introduced the concept of radial and antipodal domination, initiated a study of the corresponding parameters. For any tree T with radius at least two, we have established the chain between global, radial and andipodal domination parameters. We have provided the lower bound of andipodal domination interms of number of vertices and pendent vertices. Also, We have characterized trees of diameter four with $\gamma_{diam} = \gamma_{rad} + 1$ and $\gamma_{rad} = \gamma + 1$. In addition to this, we have presented linear programming formulation for radial and antipodal domination and further MATALB coded are generated to understand the computational time. It is noticed that, radial and antipodal domination number of

MATLAB code for finding radial and antipodal domination number is as follows:

Algorithm 1 Finding radial (γ_{rad}) and antipodal (γ_{diam}) domination number of a graph

Input: Adjacency matrix(adj) of the graph G

Output: Radial and antipodal domination numbers

1: $n = size(adj, 1); Dist = johnson\_all\_sp(adj); Dist(isnotfinite(Dist)) = 0;$

2: $r = min(Dist(:)); A^r = Dist;$

3: $A^r(A^r \neq r) = 0; A^r(A^r == r) = 1;$ //Generating matrix A^r //

4: $d = max(Dist(:)); A^d = Dist;$

5: $A^d(A^d \neq d) = 0; A^d(A^d == d) = 1;$ //Generating matrix A^d //

6: $C = adj + eye(n); D = A^r + eye(n); F = A^d + eye(n);$

7: $f = ones(n, 1); b = ones(2 * n, 1); X1 = [C; D]; X2 = [C; F];$

8: Find optimum value of the BLPP;(Radial domination)
   Minimize $f * x$;
   Subject to $X1 * x \geq b$;
   $x_i = 0$ or 1 for each $i = 1$ to n;

9: Find optimum value of the BLPP;(Antipodal domination)
   Minimize $f * y$;
   Subject to $X2 * y \geq b$;
   $y_i = 0$ or 1 for each $i = 1$ to n;

10: $RD = \{v_i | x_i = 1\}$; & $AD = \{v_i | y_i = 1\}$; //Radial and Antipodal dominating set//

11: $\gamma_{rad} = |RD|$; & $\gamma_{diam} = |AD|$; //Radial and Antipodal domination number//

TABLE 1. Computational time for radial and antipodal domination.

p	n	m	d	r	γ_{diam}	Time (sec)	γ_{rad}	Time (sec)
0.25	50	323	3	2	31	0.0123	2	0.0712
0.25	100	1169	3	2	90	0.0164	7	0.8636
0.25	150	2827	2	2	7	9.8699	7	9.7843
0.25	200	5063	2	2	7	84.9792	7	84.5816
0.5	50	632	2	2	4	0.0620	4	0.0861
0.5	100	2500	2	2	4	0.4068	4	0.3369
0.5	150	5544	2	2	5	20.0210	5	20.0148
0.5	200	9987	2	2	5	127.1630	5	154.4178
0.75	50	927	2	2	5	0.1100	5	0.1123
0.75	100	3754	2	2	6	0.5197	6	0.4336
0.75	150	8371	2	2	7	9.7175	7	9.3154
0.75	200	14827	2	2	7	44.2700	7	44.0339

graphs with less than 200 nodes is computed in at most of 3 seconds. This formulation is useful for future studies on this domination parameters.

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