TWMS J. App. and Eng. Math. V.15, N.3, 2025, pp. 728-747

BALANCE SPHERICAL FUZZY GRAPH AND THEIR APPLICATIONS

BISWAJIT SOME[1*], PARIKSHIT DAS[1], ANITA PAL[1], §

Abstract. Spherical fuzzy graphs (SFGs) can be considered an advancement beyond the original idea of picture fuzzy graphs (PFGs). The balanced spherical fuzzy graph constitutes a distinct category within the realm of spherical fuzzy graphs. We propose the idea of balanced spherical fuzzy graphs based on density functions in this study and look into some of its characteristics. This article explores the essential and comprehensive criteria required to determine balanced spherical fuzzy graphs. Additionally we have developed an approach to evaluate the spherical fuzzy graph (SFGs) is balanced or not. In summary, this article provides an exemplification of how the utilization of balanced spherical fuzzy graphs (SFGs) can effectively portray the interconnections among neighboring nations.

Keywords: Spherical fuzzy graph, Balanced spherical fuzzy graph, Average spherical fuzzy graph, Density of a spherical fuzzy graph

AMS Subject Classification: 05C35, 05C92.

1. Introduction

A development of classical set theory is the fuzzy set theory proposed by Zadeh [40] in 1965. Various sectors, such as the chemical industry, telecommunications, decision theory, networking, computer science, and management science, have recognized the far-reaching significance of Zadeh's pioneering concepts. As an extension of fuzzy set (FS) theory, Atanassov [1] proposed intuitionistic fuzzy set (IFS) theory. He expanded the concept of fuzzy set by distinguishing the membership degree α and the non-membership degree β, combined with the criterion $0 \leq \alpha+\beta \leq 1$. Pythagorean fuzzy sets (PYFS) were introduced by Yager [39] by adding a new restrictions, $0 \leq \alpha^2+\beta^2 \leq 1$ which increased the space of membership value. The concept of a picture fuzzy set (PFS) was initially explored by Cuong and Kreinovich in their work cited as [10]. It serves as a direct extension of intuitionistic fuzzy sets and finds applicability in situations where individuals express their opinions in terms of categories such as yes, abstain, no, and rejection. Three degrees are assigned by a picture fuzzy set to the elements membership degree $\alpha : X \rightarrow [0, 1]$, neutral

[1] Department of Mathematics, National Institute of Technology, Durgapur, West Bengal-713209, India.
e-mail: biswajitsome.6@gmail.com, ORCID ID 0009-0004-0677-0548.
e-mail: parikshitdas1010@gmail.com, ORCID ID 0009-0001-1404-4579.
e-mail: anita.pal@maths.nitdgp.ac.in, ORCID ID 0000-0002-2514-5463.
∗ Corresponding author.
§ Manuscript received: October 11, 2023; accepted: March 27, 2024.

membership degree $\beta : X \to [0,1]$, and non-membership degree $\gamma : X \to [0,1]$ under the condition $0 \le \alpha + \beta + \gamma \le 1$, where $\pi = 1 - (\alpha + \beta + \gamma)$ denotes the degree of refusal or rejection. The spherical fuzzy set (SFS), created by Gündogdu and Kahraman [12] is an extension of the picture fuzzy set (PFS) since it increases the space of membership degree α, neutral-membership degree β, and non-membership degree γ in the interval $[0,1]$ with a condition $0 \le \alpha^2+\beta^2+\gamma^2 \le 1$. When contrasted with picture fuzzy models, spherical fuzzy models offer the system increased precision, flexibility, and compatibility. This superiority arises from the fact that the membership degree space of SFGs is more extensive than that of PFGs. Consequently, SFGs can be regarded as a more comprehensive and general framework compared to PFGs. As an illustration, consider a scenario in which a decision-maker provides evaluation information characterized by a membership degree of 0.7, a neutral membership degree of 0.4, and a non-membership degree of 0.3, it can be known that the picture fuzzy number fails to address this because $0.7 + 0.4 + 0.3 > 1$ however $(0.7)^2 + (0.4)^2 + (0.3)^2 \le 1$. Consequently, the utilization of the spherical fuzzy model offers the system enhanced precision, adaptability, and versatility.

Review of Literature. In 1965, Zadeh introduced the notion of fuzzy subsets as a tool to portray and address uncertainty, which is well-documented in his work cited in [41]. Since its inception, the theory of fuzzy sets has attracted substantial interest and stimulated active research in a multitude of academic fields and domains. These fields encompass a wide range of disciplines, including but not limited to the life and medical sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multi-agent systems, pattern recognition, robotics, computer networks, expert systems, decision-making, and automata theory, among various others. Rosenfeld introduced a fuzzy graph in 1975, which exhibited a structural resemblance to concepts found in graph theory, as documented in his work [29]. Over time, an array of fuzzy sets and relations have been employed to develop diverse extensions of fuzzy graphs, which have been documented in various publications within the literature. Santhimaheswari et al.[31] investigated strongly edged regular and totally regular FGs . Al-Hawary [15, 16, 17, 18, 46] thought about these kind of ideas related to fuzzy graphs. The notion of FGs was expanded by Parvathi and Karunambigai [22] to include intuitionistic fuzzy graphs (IFGs). After that, Akram and Davvaz [2] examined strong IFGs. Pythagorean fuzzy graphs (PYFGs) were initially proposed by Naz et al. in their work [27] as an evolution of intuitionistic fuzzy graphs(IFGs), and their study encompassed applications in the realm of decision-making. Karunambigai et al. [23] defined the concepts of density of intuitionistic fuzzy graphs, balanced intuitionistic fuzzy graphs, and direct products of intuitionistic fuzzy graphs. In 2014, Samanta et al. introduced the concept of the fuzzy planner graph, as documented in their work [32]. Furthermore, in their subsequent publication [33], Samanta and Pal extensively elaborated on the notion of two distinct types of edges within this framework : effective edges and considerable edges. Also, in 2013, Pal et al.[28] have studied about fuzzy k-competition graphs. Akram et al.[3] gives some Specific types of Pythagorean fyzzy graphs and application to decision-making. The idea of intuitionistic fuzzy competition graphs was covered by Sahoo and Pal [34]. The notion of a neutrality degree was first introduced by Cuong and Kreinovich [8, 9] in their work when they extended the concept of picture fuzzy sets (PFSs) to picture fuzzy graphs (PFGs), also in their article, they presented various categories of picture fuzzy graphs, which encompass strong PFGs, regular PFGs, complete PFGs, and the complement of PFGs. In their article, Zuo and colleagues, as detailed in [38], provide clear and comprehensive definitions and explanations for various concepts related to picture fuzzy graphs

(PFGs). These concepts include isomorphism of PFGs, the cartesian product, composition, join, direct product, lexicographic product, and strong product within the context of PFGs. Following this, the article by Xiao et al. [37], proceeds to introduce the notion of picture fuzzy multi-graphs (PFMGs). Subsequently, it delves into a thorough examination of regular picture fuzzy graphs (RPFGs) and their real-world utility in the realm of networking communications. The concept of PFG dominance, including its particular sequence and size, was pioneered by Ismayil and Bosley in [19]. Additionally, edge domination within PFGs has been the subject of investigation by Ismayil et al., as detailed in their research documented in [20]. In 2019, Akram and Habib introduced the concept of q-rung picture fuzzy line graphs in their work [4]. They also established a necessary condition associated with this particular graph. Moreover, Dombi operators have been recently employed in the context of picture fuzzy graphs, leading to the development of picture Dombi fuzzy graphs, as demonstrated by Mohanta et al. in [25]. In 2019 Jan et al. [44] introduce a new notion of interval-valued q-rung ortho pair fuzzy graph (IVQ-ROPFG) and to study the related graphical terms such as subgraph, complement, degree of vertices and path etc. Each of the graphical concept is demonstrated with an example. In recent time Koczy et al.[43] analysis social network and a wi-fi network using the concept of picture fuzzy graph (PFG). In this article Jan et al.[45] proposed methodology combines complex intuitionistic fuzzy sets (CIFS) and soft sets (SS) and evaluates the significance of the operating system using complex intuitionistic fuzzy soft relations (CIFSRs), which involve merging CIFSSs through Cartesian product . Spherical fuzzy graph is an extension of picture fuzzy graphs. The spherical fuzzy set (SFS), proposed by Gündogdu and Kahraman [12] , is an extension of PFS, as it provides enlargement of the space of degrees of truthness α, abstinence β , and falseness γ in the interval $[0, 1]$ with a condition $0 \leq \alpha^2 + \beta^2 + \gamma^2 \leq 1$. In order to address vagueness while taking into account membership degree α, neutral membership degree β, and non membership degree γ, and satisfying the requirement $0 \leq \alpha^2 + \beta^2 + \gamma^2 \leq 1$ a new tool known as spherical fuzzy graph was established. Akram et al.[5] introduced an application for decision-making within the context of spherical fuzzy graphs. In 2020, Zadam et al.[42] presented an innovative approach for addressing decision-making and shortest path problems based on T-spherical fuzzy information. Jan et al.[21] gives an examination of double domination, employing the principles of spherical fuzzy information, was conducted, and this analysis included practical applications. Guleria et al.[14] explored the realm of T-spherical fuzzy graphs, delving into operations and their applications within diverse selection processes. Shoaib and collaborators [36] introduced the notion of a complex spherical fuzzy graph and elucidated its practical applications, thereby offering valuable insights into this innovative concept. Energy of spherical fuzzy graphs researched by Mohamed et al. [26]. Balanced fuzzy graphs hold a prominent position across a multitude of domains and fields, exerting a substantial influence on various disciplines. A new concept of balanced intuitionistic fuzzy graph introduced by Karunambigai et al. [**?**]. Following this, the article by Rashmanlou and Pal [30], proceeds to introduce the notion of balanced inter-valued fuzzy graphs. Following this Ghorai and Pal [13] introduced on some operation and density of m-polar fuzzy graph and balanced bipolar fuzzy graph represent by Sankar and Ezhilmaran [6]. Various articles have explored in different types of balanced fuzzy graphs balanced picture fuzzy graph [7] and graph based soft-balanced fuzzy clusterating [24] and balanced neutrosophic graph [35].

Motivation. Spherical fuzzy graphs are further developed and extended by picture fuzzy graphs. In the real world, spherical fuzzy graphs are widely used because they may be used to depict a variety of decision-making issues in an uncertain situation. In comparison to picture fuzzy models, spherical fuzzy models offer greater precision, flexibility, and compatibility to the system. For instance, the picture fuzzy model is unable to solve this if a decision-maker provides assessment information with a membership degree of 0.5, a neutral membership degree of 0.3, and a non-membership degree of 0.6, because $0.5 + 0.3 + 0.6 > 1$, but $(0.5)^2 + (0.3)^2 + (0.6)^2 \leq 1$. Therefore the spherical fuzzy model is more adaptable because its membership degree space is greater than the PFG's membership degree space and it deals with the ambiguities in real occurrences in a broad sense. Taking inspiration from this viewpoint, our research study delves into the realm of balanced spherical fuzzy graphs. In doing so, we have introduced a range of definitions, properties, theorems, and methodologies aimed at determining the balanced nature of a spherical fuzzy graph.

Frame work of the article. The article's structure can be outlined as follows : In Section 2, fundamental definitions are presented, laying the necessary groundwork for the development of our primary findings. In section 3, balance spherical fuzzy graphs and strictly balanced fuzzy graph are defined with an example and some related properties are discussed. In Section 4, gives methods to determine whether or not the SFGs are balanced, uses an appropriate illustration, and discusses some significant findings and observations. In Section 5, of the document, an application employing balanced SFG is showcased, while Section 6, serves as a concise segment dedicated to discussion and concluding remarks.

2. Preliminaries

Spherical fuzzy sets represent a broader and more generalized concept compared to picture fuzzy sets. In below, we present the definition of a spherical fuzzy graph.

Definition 2.1. [5] *A spherical fuzzy graph (SFG) is of the form $G = (V, E)$,*

(1) $V = \{v = v_1, v_2, v_3, \cdots, v_n\}$ *such that* $\alpha_V : V \to [0,1]$ *and* $\beta_V : V \to [0,1]$, $\gamma_V : V \to [0,1]$ *denotes the degree of membership and non-membership of the element* $v_i \in V$ *respectively and* $0 \leq \alpha_V^2(v_i) + \beta_V^2(v_i) + \gamma_V^2(v_i) \leq 1\ \forall v_i \in V$.

(2) $E \subseteq V \times V$ *where* $\alpha_E : V \times V \to [0,1]$, $\beta_E : V \times V \to [0,1]$ *and* $\gamma_E : V \times V \to [0,1]$ *are such that*

$$\alpha_E(v_i, v_j) \leq min\,\{\alpha_V(v_i), \alpha_V(v_j)\},$$
$$\beta_E(v_i, v_j) \leq min\{\beta_V(v_i), \beta_V(v_j)\},$$
$$\gamma_E(v_i, v_j) \leq max\{\gamma_V(v_i), \gamma_V(v_j)\},\ \forall v_i, v_J \in V$$

and $0 \leq \alpha_E^2(v_i, v_J) + \beta_E^2(v_i, v_J) + \gamma_E^2(v_i, v_J) \leq 1,\ \forall (v_i, v_J) \in V \times V$.

Where $\alpha_E : V \times V \to [0,1]$, $\beta_E : V \times V \to [0,1]$, *and* $\gamma_E : V \times V \to [0,1]$ *represent the membership, neutral and non-membership grades of E respectively.*

Definition 2.2. [5] *A spherical fuzzy graph (SFG) $G = (V, E)$ defined as a complete Spherical fuzzy graph if*

$$\alpha_E(v_i, v_j) = min\,\{\alpha_V(v_i), \alpha_V(v_j)\},$$
$$\beta_E(v_i, v_j) = min\{\beta_V(v_i), \beta_V(v_j)\},$$
$$\gamma_E(v_i, v_j) = max\{\gamma_V(v_i), \gamma_V(v_j)\}\ \forall v_i, v_j \in V$$

and $0 \leq \alpha_E^2(v_iv_j) + \beta_E^2(v_iv_j) + \gamma_E^2(v_iv_j) \leq 1, \forall (v_i, v_j) \in V \times V.$

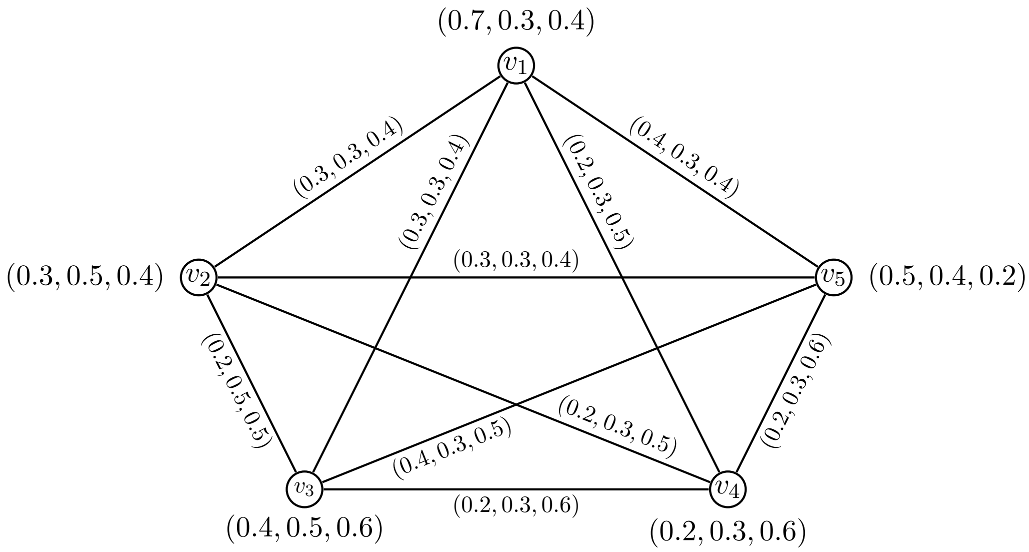


FIGURE 1. Spherical fuzzy graph

Definition 2.3. [5] *A spherical fuzzy graph (SFG) on a non empty set X is a pairs $G = (V, E)$ defined as a strong spherical fuzzy graph if*

$$\alpha_E(v_i, v_J) = min\,\{\alpha_V(v_i), \alpha_V(v_J)\},$$
$$\beta_E(v_i, v_J) = min\{\beta_V(v_i), \beta_V(v_J)\},$$
$$\gamma_E(v_i, v_J) = max\{\gamma_V(v_i), \gamma_V(v_J)\},\ \forall v_i, v_J \in V$$

and $0 \leq \alpha_E^2(v_i v_J) + \beta_E^2(v_i v_J) + \gamma_E^2(v_i v_J) \leq 1,\ \forall (v_i, v_J) \in V \times V.$

Definition 2.4. *A spherical fuzzy graph (SFG) $G = (V, E)$ is called single valued average spherical fuzzy graph if*

$$\alpha_E(v_i, v_j) = \frac{1}{2} min\,\{\alpha_V(v_i), \alpha_V(v_j)\},$$
$$\beta_E(v_i, v_j) = \frac{1}{2} min\{\beta_V(v_i), \beta_V(v_j)\}$$
$$\gamma_E(v_i, v_j) = \frac{1}{2} max\{\gamma_V(v_i), \gamma_V(v_j)\},\ \forall v_i, v_j \in V$$

and $0 \leq \alpha_E^2(v_i, v_J) + \beta_E^2(v_i, v_j) + \gamma_E^2(v_i, v_j) \leq 1,\ \forall (v_i, v_j) \in V \times V.$

Definition 2.5. *The complement of a spherical fuzzy graph $G = (V, E)$ is a spherical fuzzy graph $G^C = (V^C, E^C)$ if and only if*

(1) $V = V^C$

(2) $\alpha_V^C(v_i) = \alpha_V(v_i), \beta_V^C(v_i) = \beta_V(v_i), \gamma_V^C(v_i) = \gamma_V(v_i).\ \forall v_i \in V.$

(3) $\alpha_E^C(v_i, v_J) = \alpha_V(v_i) \wedge \alpha_V(v_j) - \alpha_E(v_i, v_J)$ $\beta_E^C(v_i, v_J) = \beta_V(v_i) \wedge \beta_V(v_j) - \beta_V(v_i, v_J)$ $\gamma_E^C(v_i, v_J) = \gamma_V(v_i) \vee \gamma_V(v_j) - \gamma_V(v_i, v_J),\ \forall (v_i, v_J) \in V \times V.$

Theorem 2.1. *Let $G = (V, E)$ be a spherical fuzzy graph. G is called self-complementary if and only if G is an average spherical fuzzy graph.*

Proof. Since $G = (V, E)$ be a spherical fuzzy graph and $G^C = (V^C, E^C)$ be its complement of G. Then $\alpha_V^C = \alpha_V$,$\beta_V^C = \beta_V$, $\gamma_V^C = \gamma_V$ and

$$\alpha_E^C(u,v) = \alpha_V(u) \wedge \alpha_V(v) - \alpha_E(u,v)$$
$$\beta_E^C(u,v) = \beta_V(u) \wedge \beta_V(v) - \beta_E(u,v)$$
$$\gamma_E^C(u,v) = \gamma_V(u) \vee \gamma_V(v) - \gamma_E(u,v),\ \forall u,v \in V.$$

Let G be a single valued average spherical fuzzy graph, then

$$\alpha_E(u,v) = \frac{1}{2} min\{\alpha_V(u), \alpha_V(v)\},$$
$$\beta_E(u,v) = \frac{1}{2} min\{\beta_V(u), \beta_V(v)\},$$
$$\gamma_E(u,v) = \frac{1}{2} max\{\gamma_V(u), \gamma_V(v)\},\ \forall u,v \in V.$$

$$\begin{aligned}\alpha_E^c(u,v) &= \alpha_V(u) \wedge \alpha_V(v) - \alpha_V(u,v)\\ &= min\{\alpha_V(u), \alpha_V(v)\} - \frac{1}{2} min\{\alpha_V(u), \alpha_V(v)\}\\ &= \frac{1}{2} min\{\alpha_V(u), \alpha_V(v)\},\ \forall u,v \in V.\end{aligned}$$

Similarly, $\beta_E^C(u,v) = \beta_E(u,v)$, $\gamma_E^C(u,v) = \gamma_E(u,v)$. i.e G^C is similar to G. Hence G is self complementary spherical fuzzy graph.

Conversely, let G is self complementary spherical fuzzy graph, then $\alpha_E^C(u,v) = \alpha_E(u,v)$, $\beta_E^C(u,v) = \beta_E(u,v)$, $\gamma_E^C(u,v) = \gamma_E(u,v)$. $\forall u,v \in V$.

Now,

$$\begin{aligned}\alpha_E(u,v) = \alpha_E^c(u,v) &= min\{\alpha_V(u), \alpha_V(v)\} - \alpha_V(u,v)\\ \Rightarrow 2\alpha_E(u,v) &= min\{\alpha_V(u), \alpha_V(v)\}\\ \Rightarrow \alpha_E(u,v) &= \frac{1}{2} min\{\alpha_V(u), \alpha_V(v)\}.\end{aligned}$$

Similarly, it can be shown that

$$\beta_E(u,v) = \frac{1}{2}\ min\{\beta_V(u), \beta_V(v)\}$$
$$and\ \gamma_(u,v) = \frac{1}{2}\ max\{\gamma_V(u), \gamma_V(v)\}, \forall u,v \in V.$$

Hence, G is a single valued average spherical fuzzy graph. □

3. Balance Spherical Fuzzy Graph

In this section, the primary emphasis has been on elucidating the concepts and attributes associated with balanced spherical fuzzy graphs and strictly balanced fuzzy graphs. Furthermore, an investigation into the prerequisites necessary for a graph to be classified as balanced spherical fuzzy has been conducted.

Definition 3.1. *Let $G = (V, E)$ be a spherical fuzzy graph, then the size of G is denoted by Λ_G and is defined by $\Lambda_G = (\Lambda_\alpha(G), \Lambda_\beta(G), \Lambda_\gamma(G))$, where*

$$\Lambda_\alpha(u) = \sum_{v_i \neq v_j} \alpha_E(v_i, v_j),$$
$$\Lambda_\beta(u) = \sum_{v_i \neq v_j} \beta_E(v_i, v_j),$$
$$\Lambda_\gamma(u) = \sum_{v_i \neq v_j} \gamma_E(v_i, v_j) \ \forall v_i, v_j \in V.$$

Definition 3.2. *Let $G = (V, E)$ be a spherical fuzzy graph. Then the weight of G is denoted by $M_G = (M_\alpha(G), M_\beta(G), M_\gamma(G))$, where*

$$M_\alpha(G) = \sum_{(v_i, v_j) \in E} min\{\alpha_V(v_i), \alpha_V(v_j)\},$$
$$M_\beta(G) = \sum_{(v_i, v_j) \in E} min\{\beta_V(v_i), \beta_V(v_j)\},$$
$$M_\gamma(G) = \sum_{(v_i, v_j) \in E} max\{\gamma_V(v_i), \gamma_V(v_j)\} \ \forall v_i, v_j \in V.$$

Definition 3.3. *Let $G = (V, E)$ be a spherical fuzzy graph. Then the density of G is denoted by $\kappa(G) = (\kappa_\alpha(G), \kappa_\beta(G), \kappa_\gamma(G))$, where $\kappa_\alpha(G) = \dfrac{\Lambda_\alpha(G)}{M_\alpha(G)}$, $\kappa_\beta(G) = \dfrac{\Lambda_\beta(G)}{M_\beta(G)}$; $\kappa_\gamma(G) = \dfrac{\Lambda_\gamma(G)}{M_\gamma(G)}$ for all $v_i, v_j \in V$.*

Definition 3.4. *A spherical fuzzy graph $G = (V, E)$ is said to be balanced if all its subgraphs are in G i.e $\kappa(S) \leq \kappa(G)$ for any subgraph S of G. Now $\kappa(S) \leq \kappa(G)$ holds if $\kappa_\alpha(S) \leq \kappa_\alpha(G)$, $\kappa_\beta(S) \leq \kappa_\beta(G)$, $\kappa_\gamma(S) \leq \kappa_\gamma(G)$.*

Example 1. *Consider a SFG, $G = (V, E)$, such that $V = \{s_1, s_2, s_3, s_4, s_5\}$, $E = \{(s_1, s_2), (s_2, s_3), (s_3, s_4), (s_4, s_5), (s_5, s_1), (s_2, s_5), (s_1, s_4), (s_2, s_4)\}$.*
Then

$$\kappa_\alpha(G) = \frac{0.272 + 0.272 + 0.136 + 0.136 + 0.408 + 0.272 + 0.136 + 0.272}{0.4 + 0.4 + 0.2 + 0.2 + 0.6 + 0.4 + 0.2 + 0.2} = 0.68.$$
$$\kappa_\beta(G) = \frac{0.064 + 0.096 + 0.096 + 0.064 + 0.096 + 0.064 + 0.096 + 0.128}{0.2 + 0.3 + 0.4 + 0.3 + 0.2 + 0.2 + 0.3 + 0.3} = 0.32.$$
$$\kappa_\gamma(G) = \frac{0.425 + 0.425 + 0.34 + 0.34 + 0.255 + 0.34 + 0.425 + 0.136}{0.5 + 0.5 + 0.4 + 0.4 + 0.3 + 0.5 + 0.4 + 0.5} = 0.85.$$

Then the density of G is denoted by $\kappa(G) = (0.68, 0.32, 0.85)$. Let $\wp_i, (i = 1, 2, 3, \cdots, 26)$ be a non empty subgraphs of G, and density for all subgraphs of G are either $(0, 0, 0)$ or $(0.68, 0.32, 0.85)$ shown in the Table 1. So $\kappa(\wp(G)) \leq \kappa(G)$ for all subgraphs $\wp_i$ of G. Hence G is balanced SFG.

Definition 3.5. *A spherical fuzzy graph $G = (V, E)$ is said to be strictly balanced if $\kappa(S) = \kappa(G)$ for all subgraph S of G holds if $\kappa_\alpha(S) = \kappa_\alpha(G)$, $\kappa_\beta(S) = \kappa_\beta(G)$, $\kappa_\gamma(S) = \kappa_\gamma(G)$.*

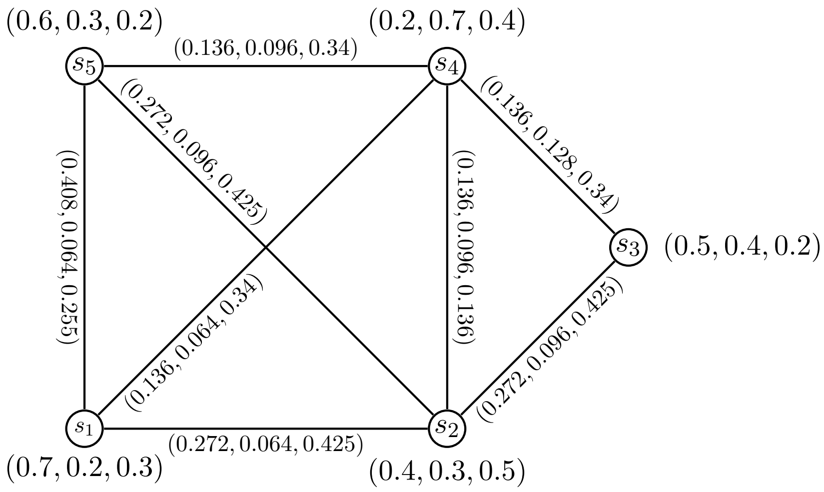


FIGURE 2. Balance Spherical fuzzy graph with density $(0.68, 0.32, 0.85)$

Example 2. *Consider a SFG $G = (V, E)$, such that $V = \{n_1, n_2, n_3, n_4\}$, $E = \{(n_1, n_2), (n_2, n_3), (n_3, n_4), (n_4, n_1), (n_1, n_3), (n_2, n_4)\}$. Then*

$$\kappa_\alpha(G) = \frac{0.16 + 0.128 + 0.128 + 0.192 + 0.16 + 0.128}{0.4 + 0.5 + 0.5 + 0.6 + 0.5 + 0.4} = 0.32, \text{ Similarly } \kappa_\beta(G) = 0.25$$

and $\kappa_\gamma(G) = 0.45$.

Then the density of G is denoted by $\kappa(G) = (0.32, 0.25, 0.45)$. Let $\Re_i, (i = 1, 2, 3, \cdots, 11)$ be a non empty subgraphs of G, and density for all subgraphs of G is $(0.68, 0.32, 0.85)$ shown in the Table 2 . So $\kappa(\Re_i(G)) = \kappa(G)$ for all subgraphs $\Re_i$ of G. Hence G is strictly balanced SFG.

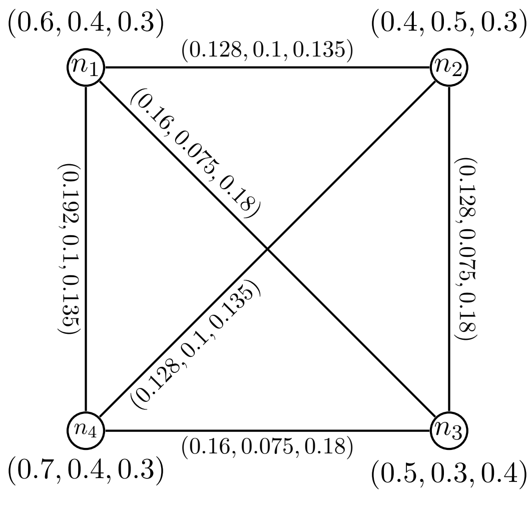


FIGURE 3. Strictly Balance Spherical Fuzzy Graph with density $(0.32, 0.25, 0.45)$

Theorem 3.1. *Let $G = (V, E)$ be an average spherical fuzzy graph and $\kappa(G)$ is the density of G, then $\kappa(G) = (1, 1, 1)$.*

Table 1. Density of all subgraph of the SFG in Fig-2

Example of Balance Spherical Fuzzy Graph		
Subgraph	Vertex Set	Density
$\wp_1$	$\{s_1, s_2, s_3, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_2$	$\{s_1, s_2, s_3, s_4\}$	$(0.68, 0.32, 0.85)$
$\wp_3$	$\{s_1, s_2, s_3, s_5\}$	$(0, 0, 0)$
$\wp_4$	$\{s_1, s_2, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_5$	$\{s_1, s_3, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_6$	$\{s_2, s_3, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_7$	$\{s_1, s_2, s_3\}$	$(0.68, 0.32, 0.85)$
$\wp_8$	$\{s_1, s_2, s_4\}$	$(0.68, 0.32, 0.85)$
$\wp_9$	$\{s_1, s_2, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{10}$	$\{s_1, s_3, s_4\}$	$(0, 0, 0)$
$\wp_{11}$	$\{s_1, s_3, s_5\}$	$(0, 0, 0)$
$\wp_{12}$	$\{s_1, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{13}$	$\{s_2, s_3, s_4\}$	$(0.68, 0.32, 0.85)$
$\wp_{14}$	$\{s_2, s_3, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{15}$	$\{s_2, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{16}$	$\{s_3, s_4, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{17}$	$\{s_1, s_2\}$	$(0.68, 0.32, 0.85)$
$\wp_{18}$	$\{s_1, s_3\}$	$(0, 0, 0)$
$\wp_{19}$	$\{s_1, s_4\}$	$(0.68, 0.32, 0.85)$
$\wp_{20}$	$\{s_1, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{21}$	$\{s_2, s_3\}$	$(0.68, 0.32, 0.85)$
$\wp_{22}$	$\{s_2, s_4\}$	$(0.68, 0.32, 0.85)$
$\wp_{23}$	$\{s_2, s_5\}$	$(0.68, 0.32, 0.85)$
$\wp_{24}$	$\{s_3, s_4\}$	$(0.68, 0.32, 0.85)$
$\wp_{25}$	$\{s_3, s_5\}$	$(0, 0, 0)$
$\wp_{26}$	$\{s_4, s_5\}$	$(0.68, 0.32, 0.85)$

Table 2. Density of all subgraph of the SFG in Fig-3

Example of Strictly Balance Spherical Fuzzy Graph		
Subgraph	Vertex Set	Density
$\Re_1$	$\{n_1, n_2\}$	$(0.32, 0.25, 0.45)$
$\Re_2$	$\{n_1, n_3\}$	$(0.32, 0.25, 0.45)$
$\Re_3$	$\{n_1, n_4\}$	$(0.32, 0.25, 0.45)$
$\Re_4$	$\{n_2, n_3\}$	$(0.32, 0.25, 0.45)$
$\Re_5$	$\{n_2, n_4\}$	$(0.32, 0.25, 0.45)$
$\Re_6$	$\{n_3, n_4\}$	$(0.32, 0.25, 0.45)$
$\Re_7$	$\{n_1, n_2, n_3\}$	$(0.32, 0.25, 0.45)$
$\Re_8$	$\{n_1, n_3, n_4\}$	$(0.32, 0.25, 0.45)$
$\Re_9$	$\{n_1, n_2, n_4\}$	$(0.32, 0.25, 0.45)$
$\Re_{10}$	$\{n_2, n_3, n_4\}$	$(0.32, 0.25, 0.45)$
$\Re_{11}$	$\{n_1, n_2, n_3, n_4\}$	$(0.32, 0.25, 0.45)$

Proof. Since $\kappa(G)$ is the density of spherical fuzzy graph G, then the density of G is given by $\kappa(G) = (\kappa_\alpha(G), \kappa_\beta(G), \kappa_\gamma(G))$. We know that $\kappa_\alpha(G) = \dfrac{B_\alpha(G)}{W_\alpha(G)} = 1$. Similarly $\kappa_\beta(G) = 1$, $\kappa_\gamma(G) = 1$. Therefore $\kappa(G) = (1, 1, 1)$. □

Theorem 3.2. *A spherical fuzzy graph $G = (V, E)$ is said to be strictly balanced iff*

$$\alpha_E(v_i, v_j) = \xi_1 \ min\{\alpha_V(v_i), \alpha_V(v_j)\}, \ \beta_E(v_i, v_j) = \xi_2 \ min\{\beta_V(v_i), \beta_V(v_j)\},$$
$$\gamma_E(v_i, v_j) = \xi_3 \ max\{\gamma_V(v_i), \gamma_V(v_j)\}, \ \ \forall v_i, v_j \in V \ \ where \ \kappa(G) = (\xi_1, \xi_2, \xi_3).$$

Proof. Assume that the SFG $G = (V, E)$ has n nodes and is strictly balanced, and that $V = \{v_1, v_2, ..., v_n\}$. Then $\kappa(S) = \kappa(G)$ for all subgraph S of G. Also given that $\kappa(G) = (\xi_1, \xi_2, \xi_3)$. Since, G contains n nodes and $V = \{v_1, v_2, v_3, \cdots, v_n\}$ then G has $2^n - (n+1) = \aleph$ subgraph. Among the $\aleph$ sub-graphs of G, ${}^nC_2 = \aleph_1$ subgraps are the subgraphs, each containing 2 nodes. In $\aleph_1$ subgraph, let us consider any arbitrary subgraph, say S_λ of G. Let $S_\lambda = (V_\lambda, E_\lambda)$ where $V_\lambda = \left\{v_{\lambda_i}, v_{\lambda_j}\right\}$. Now, the density of S_λ is given by $\kappa(S_\lambda) = (\kappa_\alpha(S_\lambda), \ \kappa_\beta(S_\lambda), \ \kappa_\gamma(S_\lambda))$.
Therefore, we have

$$\kappa_\alpha(S_\lambda) = \frac{\alpha_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j})}{min\{\alpha_V(v_{\lambda_i}), \alpha_V(v_{\lambda_j})\}},$$
$$\kappa_\beta(S_\lambda) = \frac{\beta_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j})}{min\{\beta_V(v_{\lambda_i}), \beta_V(v_{\lambda_j})\}},$$
$$\kappa_\gamma(S_\lambda) = \frac{\gamma_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j})}{max\{\gamma_V(v_{\lambda_i}), \gamma_V(v_{\lambda_j})\}} \ \forall v_{\lambda_i}, v_{\lambda_j} \in V_\lambda.$$

Since, S_λ is an arbitrary subgraph G with two nodes and G is strictly balanced, so $\kappa(S_\lambda) = \kappa(G)$. This implies that $\kappa_\alpha(S_\lambda) = \xi_1$, $\kappa_\beta(S_\lambda) = \xi_2$, $\kappa_\gamma(S_\lambda) = \xi_3$. That is $\alpha_E(v_{\lambda_i}, v_{\lambda_j}) = \xi_1 \times min\{\alpha_V(v_{\lambda_i}), \alpha_V(v_{\lambda_j})\}$, $\beta_E(v_{\lambda_i}, v_{\lambda_j}) = \xi_2 \times min\{\beta_V(v_{\lambda_i}), \beta_V(v_{\lambda_j})\}$, $\gamma_E(v_{\lambda_i}, v_{\lambda_j}) = \xi_3 \times max\{\gamma_V(v_{\lambda_i}), \gamma_V(v_{\lambda_j})\}$. As S_λ represents any arbitrary subgraph consisting of two nodes from the set of nodes in graph G, the relation mentioned above holds true for all pairs of nodes v_{λ_i} and v_{λ_j} belonging to the vertex set V.

Conversely, let $G = (V, E)$ be a SFG where $V = \{v_1, v_2, v_3, \cdots, v_n\}$. $\kappa(G)$ be the density of G where $\kappa(G) = (\xi_1, \xi_2, \xi_3)$. The membership function of all edges are satisfied the relation $\alpha_E(v_i, v_j) = \xi_1 \times min\{\alpha_V(v_i), \alpha_V(v_j)\}$, $\beta_E(v_i, v_j) = \xi_2 \times min\{\beta_V(v_i), \beta_V(v_j)\}$, $\gamma_E(v_i, v_j) = \xi_3 \times max\{\gamma_V(v_i), \gamma_V(v_j)\}$, $\forall v_i, v_j \in V$. Our objective now is to establish that graph G strictly balanced. Let $S_m = (V_m, E_m)$ be any subgraph of G, where $V_m = \{v_{m_1}, v_{m_2}, v_{m_3}, \cdots, v_{m_r}\}$, $m_1, m_2, m_3, \cdots, m_r \in \{1, 2, 3, \cdots, n\}$ and $m_i \neq m_j$, $\forall i, j$. Let $\kappa(S_m) = (\kappa_\alpha(S_m), \kappa_\beta(S_m), \kappa_\gamma(S_m))$ be the density of the subgraph S_m.

Then

$$\begin{aligned}
\kappa_\alpha(S_m) &= \frac{\sum\limits_{v_{m_i}\neq v_{m_j}} \alpha_{E_m}(v_{m_i}, v_{m_j})}{\sum\limits_{(v_{m_i},v_{m_j})\in E} \min\{\alpha_{V_m}(v_{m_i}), \alpha_{V_m}(v_{m_j})\}} \\
&= \xi_1 \times \frac{\sum\limits_{(v_{m_i},v_{m_j})\in E} \min\{\alpha_{V_m}(v_{m_i}), \alpha_{V_m}(v_{m_j})\}}{\sum\limits_{(v_{m_i},v_{m_j})\in E} \min\{\alpha_{V_m}(v_{m_i}), \alpha_{V_m}(v_{m_j})\}} \\
&= \xi_1.
\end{aligned}$$

Similarly, one can easily obtained that $\kappa_\beta(S_\xi) = \xi_2$ and $\kappa_\gamma(S_\xi) = \xi_3$.

Therefore, $\kappa(S_m) = (\kappa_\alpha(S_m), \kappa_\beta(S_m), \kappa_\gamma(S_m)) = (\xi_1, \xi_2, \xi_3)$. Since S_m is an arbitrary subgraph of SFG of G, So $\kappa(S) = \kappa(G)$ for all subgraph S of G. Hence G is strictly balanced. □

Corollary 1. *A SFGs $G = (V, E)$ is balanced iff*

$$\begin{aligned}
&\alpha_E(v_i, v_j) = \xi_1 \times min\,\{\alpha_V(v_i), \alpha_V(v_j)\}, \\
&\beta_E(v_i, v_j) = \xi_2 \times min\,\{\beta_V(v_i), \beta_V(v_j)\}, \\
&\gamma_E(v_i, v_j) = \xi_3 \times min\,\{\gamma_V(v_i), \gamma_V(v_j)\},\ \forall (v_i, v_j) \in E,
\end{aligned}$$

where $\kappa(G) = (\xi_1, \xi_2, \xi_3)$.

Corollary 2. *Let $G = (V, E)$ be a balanced SFGs and S be any subgraph of G then $\kappa(S) = \kappa(G)$ or $\kappa(S) = (0, 0, 0)$.*

4. To determine whether a spherical fuzzy graph is balanced or not

Step-1: Consider a spherical fuzzy graph $G = (V, E)$

Step-2: Compute

$$\begin{aligned}
\xi_1 &= \frac{\sum\limits_{v_i\neq v_j} \alpha_E(v_i, v_j)}{\sum\limits_{(v_i,v_j)\in E} \min\{\alpha_V(v_i), \alpha_V(v_j)\}}, \\
\xi_2 &= \frac{\sum\limits_{v_i\neq v_j} \beta_E(v_i, v_j)}{\sum\limits_{(v_i,v_j)\in E} \min\{\beta_V(v_i), \beta_V(v_j)\}}, \\
\xi_3 &= \frac{\sum\limits_{v_i\neq v_j} \gamma_E(v_i, v_j)}{\sum\limits_{(v_i,v_j)\in E} \min\{\gamma_V(v_i), \gamma_V(v_j)\}}.
\end{aligned}$$

where, ξ_1, ξ_2, ξ_3 are respectively α-density, β-density, γ-density of G, i.e $\kappa(G) = (\xi_1, \xi_2, \xi_3)$.

Step-3: for $i = 1$ to n for $j = 1$ to n $(i \neq j)$ if $(\alpha_E(v_i, v_j), \beta_E(v_i, v_j), \gamma_E(v_i, v_j)) = (0, 0, 0)$ or $(\xi_1\ min\{\alpha_V(v_i), \alpha_V(v_j)\}, \xi_2\ min\{\beta_V(v_i), \beta_V(v_j)\}, \xi_3\ max\{\gamma_V(v_i), \gamma_V(v_j)\})$. Then G is balanced.

Step-4: If not satisfied the condition then G is not balanced.

Step-5: Conclusion G is balanced or not balanced.

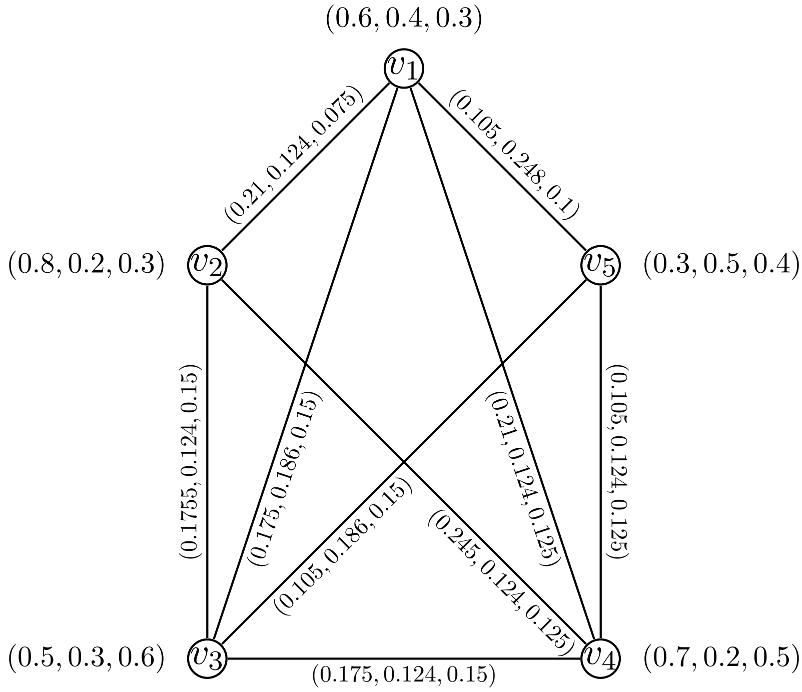


FIGURE 4. A BSFG with density $\kappa(G) = (0.35, 0.62, 0.25)$.

Example of Balance Spherical Fuzzy Graph. For each of two alternative SFGs, the BSFG is calculated while taking into account every scenario that might arise.

Let us consider a SFG $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ are shown in Fig-4, for this graph, $\kappa(G) = (0.35, 0.62, 0.25) = (\xi_1, \xi_2, \xi_3)$. Therefore, $\xi_1 = 0.35$, $\xi_2 = 0.62$, $\xi_3 = 0.25$.

Step-1

For $i = 1$

Let us consider a node v_1, we compute $(\alpha_E(v_1, v_j), \beta_E(v_1, v_j), \gamma_E(v_1, v_j))$ for $j = 2, 3, 4, 5$.

when j=2

$$\alpha_E(v_1, v_2) = 0.21 = 0.35\ min\{0.6, 0.8\} = 0.35\ min\{\alpha_V(v_1), \alpha_V(v_2)\}$$
$$\beta_E(v_1, v_2) = 0.124 = 0.62\ min\{0.4, 0.2\} = 0.62\ min\{\beta_V(v_1), \beta_V(v_2)\}$$
$$\gamma_E(v_1, v_2) = 0.075 = 0.25\ max\{0.3, 0.3\} = 0.25\ max\{\gamma_V(v_1), \gamma_V(v_2)\}.$$

Therefore,
$(\alpha_E(v_1,v_2),\beta_E(v_1,v_2),\gamma_E(v_1,v_2)) = (\xi_1\{\alpha_V(v_1),\alpha_V(v_2)\},\xi_2\{\beta_V(v_1),\beta_V(v_2)\},\xi_3\{\gamma_V(v_1),\gamma_V(v_2)\})$.

when j=3

$$\alpha_E(v_1,v_3) = 0.175 = 0.35\ min\{0.6,0.5\} = 0.35\ min\{\alpha_V(v_1),\alpha_V(v_3)\}$$
$$\beta_E(v_1,v_3) = 0.186 = 0.62\ min\{0.4,0.3\} = 0.62\ min\{\beta_V(v_1),\beta_V(v_3)\}$$
$$\gamma_E(v_1,v_3) = 0.15 = 0.25\ max\{0.3,0.6\} = 0.25\ max\{\gamma_V(v_1),\gamma_V(v_3)\}$$

Therefore,
$(\alpha_E(v_1,v_3),\beta_E(v_1,v_3),\gamma_E(v_1,v_3)) = (\xi_1\ min\{\alpha_V(v_1),\alpha_V(v_3)\},\xi_2\ min\{\beta_V(v_1),\beta_V(v_3)\},\xi_3\ min\{\gamma_V(v_1),\gamma_V(v_3)\})$.

Similarly, we can check for $j = 4, 5$.

$(\alpha_E(v_1,v_5),\beta_E(v_1,v_5),\gamma_E(v_1,v_5)) = (\xi_1\ min\{\alpha_V(v_1),\alpha_V(v_5)\},\xi_2\ min\{\beta_V(v_1),\beta_V(v_5)\},\xi_3\ min\{\gamma_V(v_1),\gamma_V(v_5)\})$.
$(\alpha_E(v_1,v_j),\beta_E(v_1,v_j),\gamma_E(v_1,v_j)) = (\xi_1\ min\{\alpha_V(v_1),\alpha_V(v_j)\},\xi_2\ min\{\beta_V(v_1),\beta_V(v_j)\},\xi_3\ min\{\gamma_V(v_1),\gamma_V(v_j)\})$, for $j = 2,3,4,5$.

Step-2

For $i = 1$
Let us consider a node v_2. Now compute $(\alpha_E(v_2,v_j),\beta_E(v_2,v_j),\gamma_E(v_2,v_j))$ for $j = 3,4,5$.
when j=3

$$\alpha_E(v_2,v_3) = 0.175 = 0.35\ min\{0.8,0.5\} = 0.35\ min\{\alpha_V(v_2),\alpha_V(v_3)\}$$
$$\beta_E(v_2,v_3) = 0.186 = 0.62\ min\{0.2,0.3\} = 0.62\ min\{\beta_V(v_2),\beta_V(v_3)\}$$
$$\gamma_E(v_2,v_3) = 0.15 = 0.25\ max\{0.3,0.6\} = 0.25\ max\{\gamma_V(v_2),\gamma_V(v_3)\}$$

Therefore,

$(\alpha_E(v_2,v_3),\beta_E(v_2,v_3),\gamma_E(v_2,v_3)) = (\xi_1 min\ \{\alpha_V(v_2),\alpha_V(v_3)\},\xi_2\ min\{\beta_V(v_2),\beta_V(v_3)\},\xi_3\ min\{\gamma_V(v_2),\gamma_V(v_3)\})$.

Similarly we can check for $j = 4, 5$.

$(\alpha_E(v_2,v_j),\beta_E(v_2,v_j),\gamma_E(v_2,v_j)) = (\xi_1\ min\{\alpha_V(v_2),\alpha_V(v_j)\},\xi_2 min\ \{\beta_V(v_2),\beta_V(v_j)\},\xi_3\ min\{\gamma_V(v_2),\gamma_V(v_j)\})$, for $j = 2,3,4$;=$(0,0,0)$ for, j=5.

Similarly, we can check the following steps.

Step-3

$(\alpha_E(v_3,v_j),\beta_E(v_3,v_j),\gamma_E(v_3,v_j)) = (\xi_1 min\ \{\alpha_V(v_3),\alpha_V(v_j)\},\xi_2 min\ \{\beta_V(v_3),\beta_V(v_j)\},\xi_3\ min\{\gamma_V(v_3),\gamma_V(v_j)\})$. for, $j = 1,2,4,5$.

Step-4

$(\alpha_E(v_4,v_j),\beta_E(v_4,v_j),\gamma_E(v_4,v_j)) = (\xi_1 min\{\alpha_V(v_4),\alpha_V(v_j)\},\xi_2 min\{\beta_V(v_4),\beta_V(v_j)\},\xi_3 min\{\gamma_V(v_4),\gamma_V(v_j)\})$. for, $j = 1,2,3,5$.

Step-5

$(\alpha_E(v_5,v_j),\beta_E(v_5,v_j),\gamma_E(v_5,v_j)) = (\xi_1 min\ \{\alpha_V(v_5),\alpha_V(v_j)\},\xi_2 min\ \{\beta_V(v_5),\beta_V(v_j)\},\xi_3\ min\{\gamma_V(v_5),\gamma_V(v_j)\})$. for,

$j = 1, 2, 4$=$(0, 0, 0)$ for, j=2.

Therefore, we conclude that
$(\alpha_E(v_i, v_j), \beta_E(v_i, v_j), \gamma_E(v_i, v_j)) = (\xi_1 min\ \{\alpha_V(v_i), \alpha_V(v_j)\}, \xi_2 min\ \{\beta_V(v_i), \beta_V(v_j)\}, \xi_3\ min\{\gamma_V(v_i), \gamma_V(v_j)\})$, or $= (0, 0, 0)$ for, $i, j = 1, 2, 3, 4, 5$, and $i \neq j$. Hence, G is a balanced spherical fuzzy graph.

Example of Spherical Fuzzy Graph is not Balanced. Let us consider a SFGs $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ are shown in Figure 5, for this graph, $\kappa(G) = (0.4, 0.5, 0.6) = (\xi_1, \xi_2, \xi_3)$. Therefore, $\xi_1 = 0.4$, $\xi_2 = 0.5$, $\xi_3 = 0.6$. The next few steps are used to determine whether or not G is balanced.

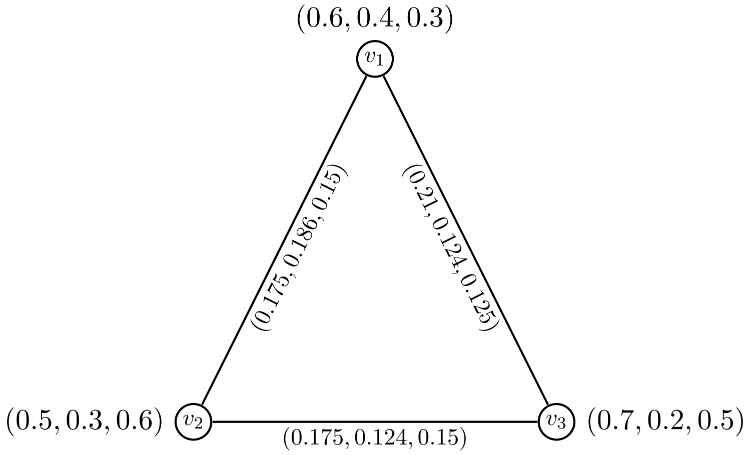


FIGURE 5. An Unbalanced SFG with density $\kappa(G) = (0.4, 0.5, 0.6)$.

Step-1
For $i = 1$
Let us consider a node v_1. Now compute $(\alpha_E(v_1, v_j), \beta_E(v_1, v_j), \gamma_E(v_1, v_j))$ for $j = 2, 3$. clearly for $j = 2, 3$ we can easily check that
$(\alpha_E(v_1, v_3), \beta_E(v_1, v_3), \gamma_E(v_1, v_3)) = (\xi_1\ min\{\alpha_V(v_1), \alpha_V(v_3)\}, \xi_2\ min\{\beta_V(v_1), \beta_V(v_3)\}, \xi_3\ min\{\gamma_V(v_1), \gamma_V(v_3)\})$.

Step-2 For $i = 1$,
Let us consider a node v_2. Now $(\alpha_E(v_2, v_j), \beta_E(v_2, v_j), \gamma_E(v_2, v_j))$ for $j = 1, 3$. when $j = 2\alpha_E(v_2, v_3) = 0.12 = 0.4\ min\{0.5, 0.3\} = 0.4\ min\{\alpha_V(v_1), \alpha_V(v_3)\}$ $\beta_E(v_2, v_3) = 0.10 \neq 0.5\ min\{0.3, 0.5\} \neq 0.5\ min\{\beta_V(v_1), \beta_V(v_3)\}$ $\gamma_E(v_2, v_3) = 0.18 \neq 0.6\ max\{0.4, 0.4\} \neq 0.6\ max\{\gamma_V(v_1), \gamma_V(v_3)\}$. Therefore,we conclude that
$(\alpha_E(v_2, v_3), \beta_E(v_2, v_3), \gamma_E(v_2, v_3)) \neq (\xi_1 min\{\alpha_V(v_2), \alpha_V(v_3)\}, \xi_2 min\{\beta_V(v_2), \beta_V(v_3)\}, \xi_3 min\{\gamma_V(v_2), \gamma_V(v_3)\})$.
Hence, G is not balanced spherical fuzzy graph.

Theorem 4.1. *Let $G = (V, E)$ be a complete spherical fuzzy graph then $\kappa(G) = (1, 1, 1)$.*

Proof. Let $G = (V, E)$ be a complete spherical fuzzy graph, where

$$\alpha_E(v_i, v_j) = min\ \{\alpha_V(v_i), \alpha_V(v_j)\},$$
$$\beta_E(v_i, v_j) = min\ \{\beta_V(v_i), \beta_V(v_j)\},$$
$$\gamma_E(v_i, v_j) = max\ \{\gamma_V(v_i), \gamma_V(v_j)\}, \forall v_i, v_j \in E.$$

Since, $\kappa(G)$ is the density of G then $\kappa(G) = (\kappa_\alpha(G), \kappa_\beta(G), \kappa_\gamma(G))$. Now, $\kappa_\alpha(G) = \dfrac{S_\alpha(G)}{W_\alpha(G)} = 1$. Similarly, $\kappa_\beta(G) = 1$, $\kappa_\gamma(G) = 1$. □

Theorem 4.2. *Any single valued complete spherical fuzzy graph $G = (V, E)$ is strictly balanced.*

Proof. Let $G = (V, E)$ be a complete spherical fuzzy graph. Let $\kappa(G)$ is the density of G. Since G is complete spherical fuzzy graph then

$$\alpha_E(v_i, v_j) = min\{\alpha_V(v_i), \alpha_V(v_j)\}$$
$$\beta_E(v_i, v_j) = min\{\beta_V(v_i), \beta_V(v_j)\}$$
$$\gamma_E(v_i, v_j) = max\{\gamma_V(v_i), \gamma_V(v_j)\},\ \forall v_i, v_j \in V.$$

Now, the above relations can be written as

$$\alpha_E(v_i, v_j) = \xi_1\ min\{\alpha_V(v_i), \alpha_V(v_j)\},$$
$$\beta_E(v_i, v_j) = \xi_2\ min\{\beta_V(v_i), \beta_V(v_j)\},$$
$$\gamma_E(v_i, v_j) = \xi_3\ max\{\gamma_V(v_i), \gamma_V(v_j)\},\ \forall v_i, v_j \in V$$

Where $\xi_1 = \xi_2 = \xi_3 = 1$.

□

In the next section, we will explore a novel application of balanced spherical fuzzy graphs in the context of understanding the inter-relationships between India and its neighboring countries. The primary aim of this study is to identify neighboring nations that could potentially engage in collaborative efforts under specific predefined conditions, which will be elaborated upon below.

5. Application of Balanced Spherical Fuzzy Graph

India is the largest and most significant country in the South Asia region. Its land borders include Nepal, China, Bhutan, Bangladesh, Pakistan, and Myanmar as well as a sea border with Sri Lanka and the Maldives. India's Neighborhood First Policy is a foreign policy initiative that was launched by the Indian government in 2014. India's foreign policy, known as the "Neighborhood First" Policy, proactively focuses on establishing links with India's relations with its neighbors. The Neighborhood First Policy is based on mutual benefit and aims to build stronger economic, political, and cultural ties between India and its neighboring countries. It focuses on improving connectivity, enhancing trade and investment, promoting cultural exchanges, and addressing shared challenges such as terrorism, climate change, and poverty. Also the policy is designed to improve connections with neighbors while promoting global cooperation and peace.

In today's globalized world, where geographical boundaries are increasingly becoming blurred and irrelevant, it is important for neighboring countries in the developing world to understand the development strategies being pursued by their neighbors. This is more so because they share the relatively limited economic space in world markets. The goal of the article is to pinpoint Indian neighbours who will boost India's influence. India's approach is to put its residents' needs first, construct regional frameworks for peace and prosperity that are advantageous to everyone, and to use its neighbours as leverage in regional and international issues. Additionally, it wants to advance global cooperation and peace while strengthening relationships with neighbouring nations.

Here, we consider the neighboring countries of India(I) are Nepal (N), China (C), Bhutan (B), Bangladesh (BA), Pakistan (P), Mayanmar (MY), Sri Lanka (SL) and Maldives (MA). India intends to form an alliance with her neighboring countries with an eye on mutual cooperation in different fields. As a result, we create an SFG among the nine countries, where each nation is represented by a node, and any alliance between two nations is connected by edges. For instance, Bangladesh (BA) and Bhutan (B) have an alliance, then there is an edge between Bhutan (B) and Bangladesh (BA). If there is no alliance between Bhutan (B) and Bangladesh (BA) then there is no edge between Bhutan (B) and Bangladesh (BA). Now, let's examine the membership functions assigned to both nodes and edges, outlined as follows :

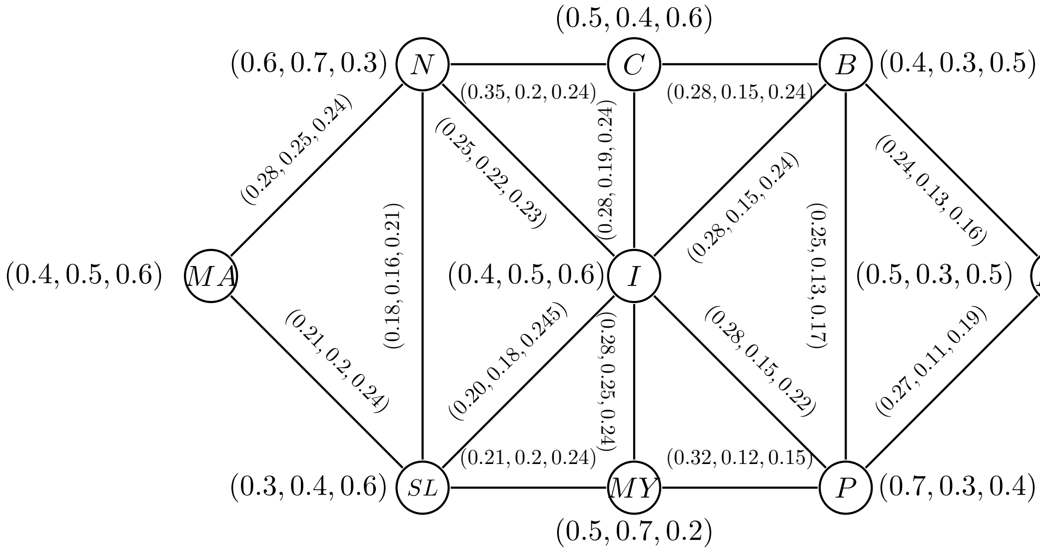


FIGURE 6. A SFG for nine different countries

For vertices/nodes

(1) The GDP (Gross Domestic Product) and Per Capita Income of each nation are used to define positive membership.
(2) Each nation's agriculture, carrying capacity, and inflation are referred to as the neutral membership degree of the node.
(3) The degree of negative-membership of a node is measured by the disinvestment, fertility rate, and sex ratio of each nation.

For edges

(1) A positive membership degree of each edge is referred to as the relationship between the Goods and Services Tax, the commercialization of agriculture, and the Cash Reserve Ratio (CRR).
(2) In each edge country, urbanization and unemployment are referred to as neutral membership degrees.
(3) The negative membership degree of edges of a nation refers to the ratio of infant mortality, poverty, and cascade effect taken together.

The membership values for individual nodes and edges are visually represented in Figure - 6, and their specific values are tabulated in Table - 3. Notably, it can be observed that the nodes Nepal (N) exhibit membership values of $(0.6, 0.7, 0.3)$, while the nodes China (C) demonstrate membership values of $(0.5, 0.4, 0.6)$. Furthermore, by referring to Table - 3,

Table 3. Membership values of edges.

	N	C	B	BA	P
N	$(0,0,0)$	$(0.35,0.2,0.24)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
C	$(0.35,0.2,0.24)$	$(0,0,0)$	$(0.28,0.15,0.24)$	$(0,0,0)$	$(0,0,0)$
B	$(0,0,0)$	$(0.28,0.15,0.24)$	$(0,0,0)$	$(0.24,0.13,0.16)$	$(0.25,0.13,0.17)$
BA	$(0,0,0)$	$(0,0,0)$	$(0.24,0.13,0.16)$	$(0,0,0)$	$(0.27,0.11,0.19)$
P	$(0,0,0)$	$(0,0,0)$	$(0.25,0.13,0.17)$	$(0.27,0.11,0.19)$	$(0,0,0)$
MY	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0.32,0.12,0.15)$
I	$(0.25,0.22,0.23)$	$(0.28,0.19,0.24)$	$(0.28,0.15,0.24)$	$(0,0,0)$	$(0.28,0.15,0.22)$
SL	$(0.18,0.16,0.21)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
MA	$(0.28,0.25,0.24)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

	MY	I	SL	MA
N	$(0,0,0)$	$(0.25,0.22,0.23)$	$(0.18,0.16,0.21)$	$(0.28,0.25,0.24)$
C	$(0,0,0)$	$(0.28,0.19,0.24)$	$(0,0,0)$	$(0,0,0)$
B	$(0,0,0)$	$(0.28,0.15,0.24)$	$(0,0,0)$	$(0,0,0)$
BA	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
P	$(0.32,0.12,0.15)$	$(0.28,0.15,0.22)$	$(0,0,0)$	$(0,0,0)$
MY	$(0,0,0)$	$(0.28,0.25,0.24)$	$(0.21,0.2,0.24)$	$(0,0,0)$
I	$(0.28,0.25,0.24)$	$(0,0,0)$	$(0.2,0.18,0.25)$	$(0,0,0)$
SL	$(0.21,0.2,0.24)$	$(0.2,0.18,0.25)$	$(0,0,0)$	$(0.21,0.2,0.24)$
MA	$(0,0,0)$	$(0,0,0)$	$(0.21,0.2,0.24)$	$(0,0,0)$

we can determine that the membership value associated with the edges connecting Nepal (N) and China (C) is $(0.35, 0.2, 0.24)$. Analogously, we can derive the membership values for the remaining nodes and edges from Figure - 6 and Table - 3. In the context of this graph, the degree of alliance interaction (density) among the countries is represented as $(0.7, 0.5, 0.4)$. It is worth noting that within this graph, there exists a subgraph denoted as $S = \{N, C, B, I, MY, SL, MA\}$, which is characterized by equal relationship rates between every pair of nodes. Hence, the subgraph $S = \{N, C, B, I, MY, SL, MA\}$ is balanced with density $(0.35, 0.2, 0.24)$. Hence these seven countries, Nepal (N), China (C), Bhutan (B), India (I), Myanmar (M), Sri Lanka (SL), and the Maldives (M) should therefore form a legitimate alliance. Our example is helpful for numerous businesses alliances to carry out their individual strategies by taking the aforementioned into account.

6. Conclusion

Fuzzy graph theory has gained significant traction in the modern scientific and technological sphere, with the spherical fuzzy model standing out for its enhanced adaptability. This is attributed to its larger membership degree space compared to that of the PFG, enabling it to effectively address ambiguities inherent in real-world scenarios across various domains. This article introduces novel terminology and delves into the exploration of various properties pertaining to spherical fuzzy graphs (SFGs). Specifically, it defines and expounds upon key concepts like average SFG, balanced SFG, size, order, and density of an SFG. Moreover, the article outlines methods for ascertaining the balance or imbalance of SFGs, accompanied by illustrative examples, and presents notable findings and observations derived from these investigations. Additionally, the article offers an application of balanced SFGs, briefly discussing its practical relevance within a specific context.

This process has certain limitations. For example, verifying the balanced SFG approach requires extensive calculation. We will develop a programming or algorithm in the future to solve this limitation. Also, in future, we will study spherical fuzzy threshold graph, spherical fuzzy k-competition graph, spherical fuzzy planer graph, etc.

Declaration of Competing Interest. The authors affirm that to the best of their knowledge, there are no financial conflicts of interest or close personal relationships that could be perceived as potentially influencing the research conducted in this study.

Ethical approval. The authors affirm that this article does not entail any research or experiments that involve human participants or animals.

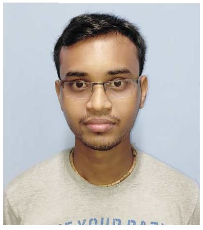
Acknowledgement. The authors wish to express their sincere gratitude to the anonymous reviewers for their invaluable and constructive feedback, which significantly enhanced the quality of the initial manuscript. It is noteworthy that no financial assistance, grants, or any other types of support were received during the preparation of this article.

References

[1] Atanassov, K. T., (1986), Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1), pp. 87-96.
[2] Akram, M., Davvaz, B., (2017), Strong intuitionistic fuzzy graphs, Filomat, 26 (1), pp. 177-196.
[3] Akram, M., Habib, A., Ilyas, F., Mohsan Dar, J.,(2018), Specific types of Pythagorean fyzzy graphs and application to decision-making, Mathematical and Computational Applications, 23(3), pp. 1-28.
[4] Akram, M., Habib, A., (2019), q-rung picture fuzzy graphs : a creative view on regularity with applications, Journal of Applied Mathematics and Computing, 61(1), pp. 235-280.
[5] Akram, M., Saleem, D., T. Al-Hawary., (2020), Spherical fuzzy graphs with application to decision-making, Mathematical and Computational Applications, 25(1), pp. 1-32.
[6] Akram, M., Karunambigai, M. G., Palanivel, K., Sivasankar, S., (2014), Balanced bipolar fuzzy graphs, Journal of advanced research in pure mathematics, 6(4), pp. 58-71.
[7] Amanathulla, S., Bera, B., Pal, M., (2021), Balanced picture fuzzy graph with application, Artificial Intelligence Review, 54(7) pp. 5255-5281 .
[8] Cuong, B. C., Kreinovich, V., (2013), Picture fuzzy sets, Part 1. In Seminar Neuro-Fuzzy Systems with Applications ; Institute of Mathematics, Vietnam Academy of Science and Technology: Hanoi, Vietnam.
[9] Cuong, B. C., (2013), Picture fuzzy sets, Part 2. In Seminar Neuro-Fuzzy Systems with Applications; Institute of Mathematics, Vietnam Academy of Science and Technology: Hanoi, Vietnam .
[10] Cuong, B. C., Kreinovich, V., (2013), Picture Fuzzy Sets - a new concept for computational intelligence problems,Third world congress on information and communication technologies, 1(3), pp. 1-6.
[11] Das, S., Ghorai, G., (2020), Analysis of road map design based on multigraph with picture fuzzy information, International Journal of Applied and Computational Mathematics1, 6(1), pp. 1-17.
[12] Gündogdu, F. K., Kahraman C., (2018), Spherical fuzzy sets and spherical fuzzy TOPSIS method, Journal of intelligent and fuzzy systems, 36(1), pp. 337-352.
[13] Ghorai, G., Pal, M., (2015), On some operations and density of m-polar fuzzy graphs, Pacific Science Review A : Natural Science and Engineering, 17(1), pp. 14-22.
[14] Guleria, A., Bajaj, R. K., (2020), T-spherical fuzzy graphs: operations and applications in various selection processes. Arabian Journal for Science and Engineering, 45(3) pp. 2177–2193.
[15] Al-Hawary, T., (2017), Certain classes of fuzzy graphs., European Journal of Pure and Applied Mathematics, 10(3), pp. 552-560.
[16] Al-Hawary, T., (2011), Complete fuzzy graphs., International Journal of Mathematical Combinatorics, 4(2), pp. 26-34.
[17] Al-Hawary, T., (2000), On balanced graphs and balanced matroids., Math. Sci. Res. Hot-Line, 4(7), pp. 35-45.
[18] Al-Hawary, T., Mahmood, T., Jan, N., Ullah, K., Hussain, A., (2018), On intuitionistic fuzzy graphs and some operations on picture fuzzy graphs, Italian journal of pure and applied mathematics, 32(2018), pp. 1-15 .

[19] Ismayil, M., Bosley, A., (2019), Domination in picture fuzzy graphs, American International Journal of Research in Science, Technology, Engineering and Mathematics, 5th International Conference on Mathematical Methods and Computation, February, pp. 20–21.
[20] Ismayil, M., Rehman, R. U., Tejaskumar, R., (2019), Edge Domination in picture fuzzy graphs, International Journal of Computational Engineering Research, 9(8), pp. 39-45.
[21] Jan, N., Mahmood, T., Zedam, L., Abdullah, L., Ullah, K., (2021), Analysis of double domination by using the concept of spherical fuzzy information with application, Journal of Ambient Intelligence and Humanized Computing, 3(2), pp. 1-16.
[22] Parvathi, R., Karunambigai, M.G., (2006), Intuitionistic fuzzy graphs. In Computational Intelligence, Theory and Applications: International Conference 9th Fuzzy Days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings, pp. 139–150.
[23] Karunambigai, M. G., Akram, M., Sivasankar, S., Palanivel, K., (2013), Balanced intuitionistic fuzzy graphs, Applied Mathematical Sciences, 7(51), pp. 2501-2514.
[24] Liu, C., Nie, F., Wang, R., Li, X., (2022), Graph based soft-balanced fuzzy clustering, IEEE Transactions on Fuzzy Systems, 31(3), pp. 2044-2055.
[25] Mohanta, K., Dey, A., Pal, A., (2020), A study on picture Dombi fuzzy graph, Decision Making : Applications in Management and Engineering, 3(2), pp. 119-130.
[26] Mohamed, S. Y., Ali, A. M., (2020), Energy of spherical fuzzy graphs, Advances in Mathematics: Scientific Journal, 9(1), pp. 321-332.
[27] Naz, S., Ashraf, S., Akram, M., (2018), A novel approach to decision-making with Pythagorean fuzzy information, Mathematics, 6(6), pp. 95-112 .
[28] Samanta, S., Pal, M., Pal, A., (2014), Some more results on fuzzy k-competition graphs, International Journal of Advanced Research in Artificial Intelligence, 3(1), pp. 60-67.
[29] Rosenfeld, A., (1975),"Fuzzy graphs" in Fuzzy Sets and Their Applications, pp. 77-95.
[30] Rashmanlou, H., Pal, M., (2013), Balanced interval-valued fuzzy graphs, Journal of Physical Sciences, 17(1), pp. 43-57.
[31] Santhimaheswari, N. R., Sekar, C., (2016), On strongly edge irregular fuzzy graphs, Kragujevac Journal of Mathematics, 40(1), pp. 125-135.
[32] Samanta, S., Pal, M., Pal, A., (2014), New concepts of fuzzy planar graphs, International Journal of Advanced Research in Artificial Intelligence, 3(1), pp. 52-59.
[33] Samanta, S., Pal, M., (2015), Fuzzy planar graph., IEEE Trans. Fuzzy Syst, 23(6), pp. 1936-1942 .
[34] Sahoo, S., Pal, M., (2015), Intuitionistic fuzzy competition graph, Journal of Applied Mathematics and Computing, 52(1), pp. 37-57.
[35] Sivasankar, S., Broumi, S., (2022), Balanced Neutrosophic Graphs, Neutrosophic Sets and Systems, 50(1), pp. 309-319.
[36] Shoaib, M., Mahmood, W., Albalawi, W., Shami, F.A., (2022), Notion of complex spherical fuzzy graph with application, Journal of Function Spaces, 2022(1) pp. 1-27.
[37] Xiao, W., Dey, A., Son L. H., (2020), A Study on Regular Picture Fuzzy Graph with Applications in Communication Networks, Journal of Intelligent and Fuzzy Systems, 39(3), pp. 3633-3645.
[38] Zuo, C., Dey, A., (2019), New concepts of picture fuzzy graphs with application, Mathematics, 7(5), pp. 470-488.
[39] Yager, R. R., (2014), Pythagorean membership grades in multi-criteria decision making, IEEE Transactions on fuzzy systems, 22(4), pp. 958-965.
[40] Zadeh, L. A., (1965), Fuzzy Sets, Information and Control, 8(3), pp. 338-353.
[41] Zadeh, L. A., (1971), Similarity relations and fuzzy orderings, Information sciences, 3(2), pp. 177-200.
[42] Zedam, L., Jan, N., Rak, E., Mahmood, T., Ullah, K., (2020), An approach towards decision-making and shortest path problems based on T-spherical fuzzy information, International Journal of Fuzzy Systems, 22(1), pp. 1521-1534.
[43] Koczy, L. T., Jan, N., Mahmood, T., Ullah, K., (2020), Analysis of social networks and Wi-Fi networks by using the concept of picture fuzzy graphs, Soft Computing, 24(1), pp. 16551-16563.
[44] Jan, N., Mahmood, T., Zedam, L., Ullah, K., Alcantud, J. C. R., Davvaz, B., (2019), Analysis of social networks, communication networks, and shortest path problems in the environment of interval-valued q-rung ortho pair fuzzy graphs, International Journal of Fuzzy Systems, 21(1), pp. 1687-1708.
[45] Jan, N., Gwak, J., Pamucar, D., Martínez, L., (2023), Hybrid integrated decision-making model for an operating system based on complex intuitionistic fuzzy and soft information, Information Sciences, 651(1), pp. 119592-119604.
[46] Al-Hawary, T.L., Al-Shaladen, S.H., Akram M., (2023), Certain matrices and energies of fuzzy graphs. TWMS JPAM V.14, N.1, pp.50-68

Biswajit Some is a senior research scholar in the department of mathematics, National Institute of Technology Durgapur, West Bengal, India. He is graduated from Syamsundar college under Burdwan University West Bengal, India. Post graduated from IIEST Shibpur, West Bengal, India. His research interests include fuzzy sets, fuzzy graphs, intuitionistic fuzzy sets, intuitionistic fuzzy graph, picture fuzzy graph, spherical fuzzy graph, chemical graph theory, spectral graph theory.



Parikshit Das is graduated from Dinabandhu Mahavidyalaya, Bongaon under West Bengal State University (WBSU), Dist. North 24 Parganas, West Bengal, India. Post graduated from WBSU, India. He is a senior research scholar in the Department of Mathematics, National Institute of Technology Durgapur, India. His research interests include fuzzy sets, intuitionistic fuzzy sets, intuitionistic fuzzy graph, picture fuzzy graph, spherical fuzzy graph, chemical graph theory, spectral graph theory.

Dr. Anita Pal is an Associte Professor of in the department of mathematics, National Institute of Technology Durgapur, West Bengal, India. She has published more than 100 research articles in different international and national journals. She has successfully guided 16 Ph.D scholars. Her specializations is include computational and fuzzy graph theory, chemical graph theory. Her works, which are intended for professionals and undergraduate and graduate students alike, have been published in India.