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EDGE IRREGULAR REFLEXIVE LABELING ON DOUBLE BROOM GRAPH AND COMB OF CYCLE AND STAR GRAPH

A. R. S. VINATIH[1*], B. D. INDRIATI[1], §

Abstract. Assume that G is a connected, undirected, simple graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. A labeling technique known as edge irregular reflexive labeling allows each vertex to have a label that is a non-negative even number from 0 to $2k_v$, and each edge to have a label that is a positive integer from 1 to k_e, with distinct weights for each edge. The smallest k of the largest label in graph G, represented by $res(G)$, is the reflexive edge strength. The paper's contents determine the reflexive edge strength of double broom graph $B(r,s,s)$ with $r,s \geq 2$, and comb of cycle and star graph $C_r \rhd S_s$ with $r \geq 3$, $s \equiv 2, 5 \pmod 6$.

Keywords: Graph labeling, double broom, comb operation, reflexive edge strength.

AMS Subject Classification: 05C78

1. Introduction

Graph theory is a branch of mathematics with a wide application in everyday life and various other sciences. Graph G is an alternating sequence between the non-empty finite vertex set $V(G)$ and the edge set $E(G)$. Graph labeling is one of the frequently discussed subjects in graph theory. Wallis [13] defines labeling as a mapping from a graph's components to a positive or non-negative integer value. According to a survey by Galian [8], there are various ways to label graphs, including irregular total k-labeling. The two types of irregular total k-labeling that Bača et al. [2] defined are vertex and edge irregular total k-labeling.

Vertex and edge irregular reflexive k-labeling was first used in 2017 by Ryan et al. [3] to describe a new concept about irregular total k-labeling. The function $f_v : V(G) \to \{0, 2, 4, \ldots, 2k_v\}$ and $f_e : E(G) \to \{1, 2, 3, \ldots, k_e\}$, where $k = max\{k_e, 2k_v\}$ such that the weights for all edges are different, are considered to be edge irregular reflexive k-labeling. The sum of the edge label with all the vertex label that is incident to the edge is known as the edge weight. The weight of the edge uv to the labeling ψ on graph G is denoted by

[1] Sebelas Maret University - Faculty of Mathematics and Natural Sciences - Department of Mathematics - Surakarta 57126 - Indonesia.
e-mail: vinatih17@gmail.com; ORCID no. 0009-0004-3189-6820,
e-mail: diari_indri@yahoo.co.id; ORCID no. 0000-0002-2889-0557.
∗ Corresponding author.
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$wt(uv)$ where $wt(uv) = \psi(u) + \psi(uv) + \psi(v)$. Reflexive edge strength, given by $res(G)$, is the lowest value of k in the graph G that can be labeled with irregular reflexive labeling. This lemma was proven by Ryan et al. [3].

Lemma 1.1. For all graph G,

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{when } |E(G)| \not\equiv 2,3 \pmod 6, \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{when } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

A lot of research has been conducted to find $res(G)$ in various classes of graphs, some of which are prism graph D_n and wheel graph W_n [12], corona of path and other graphs [9], corona of cycle and null graph with two vertices $C_n \odot N_2$ and sun graph sun_n [11], double star graph $DS_{n,m}$ and broom graph $B(m,n)$ [1], palm tree graph $C_3B_{q,r}$, swing graph S_n^3 and comb product vertex of $S_n \rhd C_3$ graph [7]. In contrast to Agustin's research et al. [1], namely the broom graph, this paper's contents determine reflexive edge strength on double broom graph $B(r,s,s)$ with $r,s \geq 2$, and comb of cycle and star graph $C_r \rhd S_s$ with $r \geq 3$, $s \equiv 2,5 \pmod 6$.

2. Reflexive Edge Strength on Double Broom Graph $B(r,s,s)$

By considering the definition of broom graph from Brualdi and Goldwasser [5], Purwanto and Lestari [10] use a definition of double broom graph as follows. The path P_r is transformed into the double broom graph $B(r,s,s)$ by s pendant edges being added to each end vertex of P_r. The path's vertices are denoted as u_j, the left pendant's vertices as v_i, and the right pendant's vertices as w_i. So, set of vertices $V(B(r,s,s)) = \{v_i : 1 \leq i \leq s\} \cup \{\mathrm{u}_j : 1 \leq j \leq r\} \cup \{\mathrm{w}_i : 1 \leq i \leq s\}$, consequently set of edges $E(B(r,s,s)) = \{u_1v_i : 1 \leq i \leq s\} \cup \{u_ju_{j+1} : 1 \leq j \leq r-1\} \cup \{u_rw_i : 1 \leq i \leq s\}$. As a result, $B(r,s,s)$ have $r+2s$ vertices and $r+2s-1$ edges. Theorem 2.1 can be used to calculate the $res(G)$ of $B(r,s,s)$.

Theorem 2.1. *For $B(r,s,s)$ with $r,s \geq 2$,*

$$res(B(r,s,s)) = \begin{cases} \lceil \frac{r+2s-1}{3} \rceil, & r+2s-1 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{r+2s-1}{3} \rceil + 1, & r+2s-1 \equiv 2,3 \pmod 6. \end{cases} \quad (1)$$

Proof. Firstly, verify a lower bound for $res(B(r,s,s))$. By reason of total edges of $B(r,s,s)$ are $r+2s-1$, subsequently by Lemma 1.1 was gained:

$$res(B(r,s,s)) \geq \begin{cases} \lceil \frac{r+2s-1}{3} \rceil, & r+2s-1 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{r+2s-1}{3} \rceil + 1, & r+2s-1 \equiv 2,3 \pmod 6. \end{cases} \quad (2)$$

This is equivalent to the lower bound in (1).

Furthermore, to obtain the upper bound of $res(B(r,s,s))$, ψ-labeling is constructed on double broom graph as follows, For $1 \leq j \leq r$

$$\psi(u_j) = \begin{cases} 0, & j = 1; r, s \geq 2, \\ & \text{and } j = 1; r \geq 4; s = 2, \\ 2, & j = r = 3; s = 2, \\ \lfloor \frac{s}{3} \rfloor, & j = 2; r = 3; s \equiv 2 \pmod 6 \\ & \text{and } j = 3; r = 4; s \equiv 1 \pmod 6 \\ & \text{with } s \geq 8, \\ 2\lceil \frac{r+s+4}{6} \rceil, & j = r \equiv 4 \pmod 6; s = 4, \\ & j = r \equiv 2, 3 \pmod 6; s = 5, \\ & j = r \equiv 0, 1, 2 \pmod 6; s = 6, \\ & j = r \not\equiv 2, 3 \pmod 6; s = 7, \\ & \text{and } j = r \not\equiv 1 \pmod 6; s = 8, \\ 2\lceil \frac{r+2s-5}{6} \rceil, & j = r; s \geq 9, \\ 2\lceil \frac{j+s-2}{6} \rceil, & \text{for others.} \end{cases}$$

$$\psi(u_j u_{j+1}) = \begin{cases} 1, & j = 2; r = 3; s = 2, \\ & \text{and } j = 2; s = 3, \\ 2, & j = r-1 = 2; s \equiv 2 \pmod 6; s > 8, \\ & \text{and } j = r-1 = 3; s \equiv 1 \pmod 6; s > 8, \\ 3, & j = 3; s = 2, \\ s - 2\lfloor \frac{s-2}{5} \rfloor, & j = 1; r \geq 2; s = 2, 3, 4, \\ & \text{and } j = 1; r \neq 2; s > 4, \\ \lceil \frac{s}{3} \rceil, & j = 2; r = 4; s \equiv 1 \pmod 6, \\ s - \lfloor \frac{s}{3} \rfloor, & j = 1; r = 3; s \equiv 2 \pmod 6, \\ s + 1 - 2\lceil \frac{s-1}{3} \rceil, & j = 1; r = 2; s > 4, \\ s + j - 2 - 4\lceil \frac{s+j-2}{6} \rceil, & j = r-1 \equiv 3 \pmod 6; s = 4, \\ & j = r-1 \equiv 1, 2 \pmod 6; s = 5, \\ & j = r-1 \equiv 0, 1, 5 \pmod 6; s = 6; j \neq 1, \\ & j = r-1 \not\equiv 2, 3 \pmod 6; s = 7; j \neq 1, 2, \\ & \text{and } j = r-1 \not\equiv 0 \pmod 6; s = 8; j \neq 3, \\ s + j - 2 - 4\lceil \frac{s+j-3}{6} \rceil, & s + j \equiv 2 \pmod 6, \\ j - 1 - 2\lfloor \frac{j}{3} \rfloor, & j = r-1 \equiv 2, 3 \pmod 6; s > 8; \\ & s \equiv 1 \pmod 6, \\ r - 4\lceil \frac{j-3}{6} \rceil, & j = r-1 \not\equiv 2, 3 \pmod 6; s > 8; \\ & s \equiv 1 \pmod 6, \\ j - 2 - 4\lfloor \frac{j-1}{6} \rfloor, & j = r-1; j \neq 2; s > 8; s \equiv 2 \pmod 6, \\ r - 4\lceil \frac{j}{6} \rceil, & j = r-1 \equiv 5 \pmod 6; s > 8; \\ & s \equiv 3 \pmod 6, \\ r - 2 - 4\lfloor \frac{j}{6} \rfloor, & j = r-1 \not\equiv 5 \pmod 6; s > 8; \\ & s \equiv 3 \pmod 6, \\ 2\lceil \frac{j}{6} \rceil - 1, & j \text{ is odd}; j = r-1; s > 8; s \equiv 0, 4 \pmod 6, \\ 2\lceil \frac{j}{6} \rceil, & j \text{ is even}; j = r-1; s > 8; s \equiv 0, 4 \pmod 6, \\ r - 2\lceil \frac{j}{3} \rceil, & j = r-1; s > 8; s \equiv 5 \pmod 6, \\ s + j - 4\lceil \frac{s+j-2}{6} \rceil, & \text{for others.} \end{cases}$$

For $1 \leq i \leq s$

$$\psi(v_i) = 2\lfloor\frac{i-1}{3}\rfloor, i = 1, 2, \dots s.$$

$$\psi(u_1v_i) = \begin{cases} s - \lceil\frac{s}{3}\rceil, & i = 8; r = 3, s \geq 8; s \equiv 2 \pmod 6, \\ i - 2\lceil\frac{i-1}{3}\rceil, & \text{for others.} \end{cases}$$

$$\psi(w_i) = \begin{cases} 0, & i = 1; r = 2, 3; s = 2, \\ 2\lfloor\frac{s}{6}\rfloor, & i = 1; r = 2; s \equiv 2 \pmod 6; s \neq 2, \\ \lfloor\frac{r+s+i-2}{3}\rfloor, & \text{applies to the following 9 boundary cases:} \\ & i, s \equiv 0 \pmod 3; r \equiv 0 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 3 \pmod 6 \text{ with } s \text{ is odd,} \\ & i, s \equiv 1 \pmod 3; r \equiv 4 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 1 \pmod 6 \text{ with } s \text{ is odd,} \\ & i, s \equiv 2 \pmod 3; r \equiv 5 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 2 \pmod 6 \text{ with } s \text{ is odd,} \\ & i \equiv 0 \pmod 3; r \equiv 4 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 1 \pmod 6 \text{ with } s \text{ is odd; } s \equiv 2 \pmod 6, \\ & i \equiv 1 \pmod 3; r \equiv 2 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 5 \pmod 6 \text{ with } s \text{ is odd; } s \equiv 0 \pmod 6, \\ & i \equiv 2 \pmod 3; r \equiv 3 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 0 \pmod 6 \text{ with } s \text{ is odd; } s \equiv 1 \pmod 6, \\ & i \equiv 0 \pmod 3; r \equiv 2 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 5 \pmod 6 \text{ with } s \text{ is odd; } s \equiv 1 \pmod 6, \\ & i \equiv 1 \pmod 3; r \equiv 0 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 3 \pmod 6 \text{ with } s \text{ is odd; } s \equiv 2 \pmod 6, \\ & \text{and } i \equiv 2 \pmod 3; r \equiv 1 \pmod 6 \text{ with } s \text{ is even;} \\ & r \equiv 4 \pmod 6 \text{ with } s \text{ is odd; } s \equiv 0 \pmod 6, \\ 2\lceil\frac{r+s+i+3}{6}\rceil, & \text{for others.} \end{cases}$$

$$\psi(u_rw_i) = \begin{cases} 2, & i = 1; r = 2; s \equiv 2 \pmod 6, \\ & \text{and } i = 1; r = 3; s = 2, \\ i + \lfloor\frac{r-2}{6}\rfloor, & i = 1, 2; r \equiv 1 \pmod 6; s = 2, \\ 2\lceil\frac{r+i-1}{6}\rceil + 2\lfloor\frac{s-1}{6}\rfloor, & s \equiv 0, 3 \pmod 6 \text{ with} \\ & i = s; r \equiv 3 \pmod 6; \\ & i = s-1; r \equiv 4 \pmod 6; \\ & i = s-2; r \equiv 5 \pmod 6, \\ & s \equiv 1, 4 \pmod 6 \text{ with} \\ & i = s; r \equiv 1 \pmod 6; \\ & i = s-1; r \equiv 2 \pmod 6; \\ & i = s-2; r \equiv 3 \pmod 6, \\ & s \equiv 2, 5 \pmod 6 \text{ with} \\ & i = s; r \equiv 5 \pmod 6; \\ & i = s-1; r \equiv 0 \pmod 6; \\ & i = s-2; r \equiv 1 \pmod 6, \\ r + i + 1 - 2\lceil\frac{r+i+3}{6}\rceil - 2\lfloor\frac{r}{6}\rfloor, & s \equiv 0 \pmod 6, \end{cases}$$

continued $\psi(u_r w_i)$

$$\psi(u_r w_i) = \begin{cases} r+i-2\lceil\frac{r+i-2}{6}\rceil - 2\lfloor\frac{r+2}{6}\rfloor, & s \equiv 1 \pmod 6, \\ r+i-1-2\lceil\frac{r+i-1}{6}\rceil - 2\lfloor\frac{r-2}{6}\rfloor, & s \equiv 2 \pmod 6, \\ r+i-2\lceil\frac{r+l}{6}\rceil - 2\lfloor\frac{r}{6}\rfloor, & s \equiv 3 \pmod 6, \\ r+i+1-2\lceil\frac{r+i+1}{6}\rceil - 2\lfloor\frac{r+2}{6}\rfloor, & s \equiv 4 \pmod 6, \\ r+i-2\lceil\frac{r+i+2}{6}\rceil - 2\lfloor\frac{r-2}{6}\rfloor, & s \equiv 5 \pmod 6. \end{cases}$$

The upper bound is the maximum vertex and edge labels constructed with ψ-labeling on a double broom graph $B(r,s,s)$ for $r,s \geq 2$. Then take the largest label, that is at vertex w_i with $i = s$.

For $r+2s-1 \equiv 0 \pmod 6$,

$$2\left\lfloor\frac{r+s+i+3}{6}\right\rfloor = \frac{r+2s+5}{3} = \left\lceil\frac{r+2s-1}{3}\right\rceil.$$

For $r+2s-1 \equiv 2 \pmod 6$,

$$2\left\lfloor\frac{r+s+i+3}{6}\right\rfloor = \frac{r+2s}{3} + \frac{3}{3} = \left\lceil\frac{r+2s-1}{3}\right\rceil + 1.$$

For $r+2s-1 \equiv 3 \pmod 6$,

$$2\left\lfloor\frac{r+s+i+3}{6}\right\rfloor = \frac{r+2s-1}{3} + \frac{3}{3} = \left\lceil\frac{r+2s-1}{3}\right\rceil + 1.$$

For $r+2s-1 \equiv 4 \pmod 6$,

$$2\left\lfloor\frac{r+s+i+3}{6}\right\rfloor = \frac{r+2s+1}{3} = \left\lceil\frac{r+2s-1}{3}\right\rceil.$$

For $r+2s-1 \equiv 5 \pmod 6$,

$$2\left\lfloor\frac{r+s+i+3}{6}\right\rfloor = \frac{r+2s}{3} = \left\lceil\frac{r+2s-1}{3}\right\rceil.$$

For $r+2s-1 \equiv 1 \pmod 6$ there are six cases as follows,
$s \equiv 0 \pmod 6$ and $r \equiv 2 \pmod 3$,

$$r+i+1-2\left\lceil\frac{r+i+3}{6}\right\rceil - 2\left\lfloor\frac{r}{6}\right\rfloor = \frac{r+2s+1}{3} = \left\lceil\frac{r+2s-1}{3}\right\rceil.$$

$s \equiv 1 \pmod 6$ and $r \equiv 0 \pmod 3$,

$$\begin{aligned} r+i-2\left\lceil\frac{r+i-2}{6}\right\rceil-2\left\lfloor\frac{r+2}{6}\right\rfloor &= r+s-\frac{2r+s-1}{3} \\ &= \frac{r+2s-1}{3} \\ &= \left\lceil\frac{r+2s-1}{3}\right\rceil. \end{aligned}$$

$s \equiv 2 \pmod 6$ and $r \equiv 1 \pmod 3$,

$$\begin{aligned} r+i-1-2\left\lceil\frac{r+i-1}{6}\right\rceil-2\left\lfloor\frac{r-2}{6}\right\rfloor &= r+s-1-\frac{2r+s-4}{3} \\ &= \frac{r+2s+1}{3} \\ &= \left\lceil\frac{r+2s-1}{3}\right\rceil. \end{aligned}$$

$s \equiv 3 \pmod 6$ and $r \equiv 2 \pmod 3$,

$$\begin{aligned} r+i-2\left\lceil\frac{r+l}{6}\right\rceil-2\left\lfloor\frac{r}{6}\right\rfloor &= r+s-\frac{2r+s-1}{3} \\ &= \frac{r+2s-1}{3} \\ &= \left\lceil\frac{r+2s-1}{3}\right\rceil. \end{aligned}$$

$s \equiv 4 \pmod 6$ and $r \equiv 0 \pmod 3$,

$$\begin{aligned} r+i+1-2\left\lceil\frac{r+i+1}{6}\right\rceil-2\left\lfloor\frac{r+2}{6}\right\rfloor &= r+s+1-\frac{2r+s+2}{3} \\ &= \frac{r+2s+1}{3} \\ &= \left\lceil\frac{r+2s-1}{3}\right\rceil. \end{aligned}$$

$s \equiv 5 \pmod 6$ and $r \equiv 1 \pmod 3$,

$$\begin{aligned} r+i-2\left\lceil\frac{r+i+2}{6}\right\rceil-2\left\lfloor\frac{r-2}{6}\right\rfloor &= r+s-\frac{2r+s-1}{3} \\ &= \frac{r+2s-1}{3} \\ &= \left\lceil\frac{r+2s-1}{3}\right\rceil. \end{aligned}$$

Since the largest label equals (1), we get

$$res(B(r,s,s)) \le \begin{cases} \lceil\frac{r+2s-1}{3}\rceil, & r+2s-1 \not\equiv 2,3 \pmod 6, \\ \lceil\frac{r+2s-1}{3}\rceil+1, & r+2s-1 \equiv 2,3 \pmod 6. \end{cases} \tag{3}$$

From (2) and (3) it is evident that

$$res(B(r,s,s)) = \begin{cases} \lceil\frac{r+2s-1}{3}\rceil, & r+2s-1 \not\equiv 2,3 \pmod 6, \\ \lceil\frac{r+2s-1}{3}\rceil+1, & r+2s-1 \equiv 2,3 \pmod 6. \end{cases}$$

Then the weight of edges are,

$$wt(u_1v_i) = \begin{cases} i+1, & i = s \equiv 2 \pmod 6; s \neq 2; r = 3, \\ i, & \text{for others.} \end{cases}$$

$$wt(u_ju_{j+1}) = \begin{cases} 5, & j = 2; r = 3; s = 2, \\ s, & j = 1; r = 3; s \equiv 2 \pmod 6; s > 7, \\ j+s, & \text{for others.} \end{cases}$$

$$wt(u_rw_i) = \begin{cases} 4, & i = 1; r = 3; s = 2, \\ r+s+i-1, & \text{for others.} \end{cases}$$

We know that all edges in the double broom graph $B(r, s, s)$ are distinct based on the weights of the edges that have been shown. As a result, ψ possesses the necessary component on an edge irregular reflexive k-labeling and follows Theorem 2.1 by $res(B(r, s, s))$. So, the $res(B(r, s, s))$ proof is finished.

□

Figure 1 evidences the illustration of edge irregular reflexive 8-labeling on double broom graph $B(8, 7, 7)$. Black color represents the names and labels of the vertices, red color represents the edge weight, and blue color represents the edges.

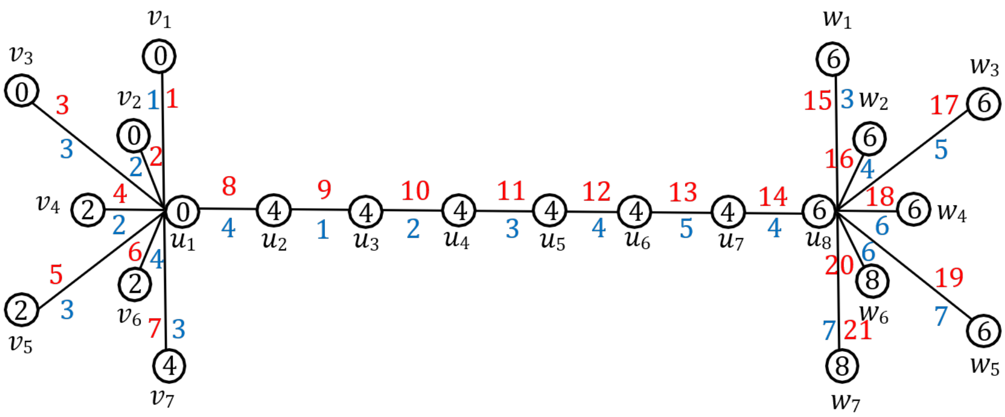

Figure 1. Edge irregular reflexive 8-labeling on double broom graph $B(8, 7, 7)$

3. Reflexive Edge Strength on Comb of Cycle and Star Graph $C_r \rhd S_s$

Acording to Darmaji and Alfarisi [6], comb of cycle and star graph denoted by $C_r \rhd S_s$ is a graph obtained by taking one copy of C_r and $|V(C_r)|$ copies of S_s graph and identify the i-th copy of S_s with the i-th vertex of C_r. The cycle's vertices are denoted as u_t, the apex of the star's vertices as v_t, and the star's vertices as $v_{t,l}$. So, set of vertices $V(C_r \rhd S_s) = \{u_t : 1 \leq t \leq r\} \cup \{v_t : 1 \leq t \leq r\} \cup \{v_{t,l} : 1 \leq t \leq r, 1 \leq l \leq s-1\}$, consequently set of edges $E(C_r \rhd S_s) = \{u_tu_{t+1} : 1 \leq t \leq r, t \equiv \pmod r\} \cup \{u_tv_t : 1 \leq t \leq r\} \cup \{v_tv_{t,l} : 1 \leq t \leq r, 1 \leq l \leq s-1\}$. As a result, $C_r \rhd S_s$ has $rs + r$ vertices and edges.

For comb of cycle and star graph $C_r \rhd S_s$ with $s \equiv 2, 5 \pmod 6$ are presented in the following subsections.

3.1. **For $C_r \rhd S_s$ with $s \equiv 2 \pmod 6$.** Theorem 3.1 can be used to calculate the $res(G)$ of $C_r \rhd S_s$ with $s \equiv 2 \pmod 6$.

Theorem 3.1. *For $C_r \rhd S_s$ with $r \geq 3, s \equiv 2 \pmod 6$,*

$$res(C_r \rhd S_s) = \begin{cases} \lceil \frac{rs+r}{3} \rceil, & rs+r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{rs+r}{3} \rceil + 1, & rs+r \equiv 2,3 \pmod 6. \end{cases} \tag{4}$$

Proof. Firstly, verify a lower bound for $res(C_r \rhd S_s)$. By reason of total edges of $C_r \rhd S_s$ are $rs+r$, subsequently by Lemma 1.1 was gained:

$$res(C_r \rhd S_s) \geq \begin{cases} \lceil \frac{rs+r}{3} \rceil, & rs+r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{rs+r}{3} \rceil + 1, & rs+r \equiv 2,3 \pmod 6. \end{cases} \tag{5}$$

This is equivalent to the lower bound in (4).

Next, to obtain the upper bound of $res(C_r \rhd S_s)$ with $s \equiv 2 \pmod 6$, ψ-labeling is constructed on comb of cycle and star graph as follows,

For $1 \leq t \leq r$, $1 \leq l \leq s-1$

$$\psi(u_t) = \begin{cases} 3\lceil \frac{s}{6} \rceil - 1, & \frac{s}{2} \text{ is odd; } t = 2; \\ & t < \lceil \frac{r+1}{2} \rceil + 1, \\ 3\lceil \frac{s}{6} \rceil, & \frac{s}{2} \text{ is even; } t = 2; \\ & t < \lceil \frac{r+1}{2} \rceil + 1, \\ 6(t-1) + 4(t-1)\lfloor \frac{s-3}{6} \rfloor, & t \neq 2; t < \lceil \frac{r+1}{2} \rceil + 1, \\ 6\lfloor \frac{r-1}{2} \rfloor + 4\lceil \frac{r-2}{2} \rceil \lfloor \frac{s-3}{6} \rfloor + 2\lfloor \frac{s-3}{6} \rfloor + 2, & r \geq 3; \\ & t = \lceil \frac{r+1}{2} \rceil + 1, \\ 6(r-t+1) + 4(r-t+1)\lfloor \frac{s-3}{6} \rfloor + 2\lfloor \frac{s-3}{6} \rfloor + 2, & \text{for others.} \end{cases}$$

$$\psi(u_t u_{t+1}) = \begin{cases} \frac{s}{2} + 4(t-1)\lfloor \frac{s}{6} \rfloor + 2(t-1), & \frac{s}{2} \text{ is odd; } t = 1,2; \\ & t < \lfloor \frac{r}{2} \rfloor + 1, \\ \frac{s}{2} + 4(t-1)\lfloor \frac{s}{6} \rfloor + 2(t-1) - 1, & \frac{s}{2} \text{ is even; } t = 1,2; \\ & t < \lfloor \frac{r}{2} \rfloor + 1, \\ 6t + 4(t-1)\lfloor \frac{s-3}{6} \rfloor + 2\lfloor \frac{s-3}{6} \rfloor - 3, & t \neq 1,2; t < \lfloor \frac{r}{2} \rfloor + 1, \\ \frac{s}{2} + 2, & \frac{s}{2} \text{ is odd; } r = 3; \\ & t = \lfloor \frac{r}{2} \rfloor + 1, \\ \frac{s}{2} + 1, & \frac{s}{2} \text{ is even; } r = 3; \\ & t = \lfloor \frac{r}{2} \rfloor + 1, \\ 3(r-1) + 2(r-2)\lfloor \frac{s-3}{6} \rfloor - 1, & r \neq 3; t = \lfloor \frac{r}{2} \rfloor + 1, \\ s + 2, & r - t = 0; t > \lfloor \frac{r}{2} \rfloor + 1, \\ 6(r-t+1) + 4(r-t+1)\lfloor \frac{s-3}{6} \rfloor + 2, & \text{for others.} \end{cases}$$

$$\psi(v_t) = \begin{cases} 3(\frac{s-2}{6}), & \frac{s}{2} \text{ is odd; } t = 2; \\ & t < \lceil \frac{r+1}{2} \rceil + 1, \\ 3(\frac{s-2}{6}) - 1, & \frac{s}{2} \text{ is even; } t = 2; \\ & t < \lceil \frac{r+1}{2} \rceil + 1, \end{cases}$$

continued $\psi(v_t)$

$$\psi(v_t)=\begin{cases}6(t-1)+4(t-1)\lfloor\frac{s-3}{6}\rfloor, & t\neq 2; t<\lceil\frac{r+1}{2}\rceil+1,\\ 6\lfloor\frac{r-1}{2}\rfloor+4\lceil\frac{r-2}{2}\rceil\lfloor\frac{s-3}{6}\rfloor+2\lfloor\frac{s-3}{6}\rfloor+2, & r\geq 3; t=\lceil\frac{r+1}{2}\rceil+1,\\ 6(r-t+1)+4(r-t+1)\lfloor\frac{s-3}{6}\rfloor+2\lfloor\frac{s-3}{6}\rfloor+2, & \text{for others.}\end{cases}$$

$$\psi(u_tv_t)=\begin{cases}t, & t=1,2; t<\lceil\frac{r}{2}\rceil+1,\\ 6(t-2)+4(t-3)\lfloor\frac{s-3}{6}\rfloor+2\lfloor\frac{s-3}{6}\rfloor-2, & t\neq 1,2; t<\lceil\frac{r}{2}\rceil+1,\\ 6\lceil\frac{r-2}{2}\rceil+2(r-3)\lfloor\frac{s-3}{6}\rfloor-2, & r \text{ is odd; } t=\lceil\frac{r}{2}\rceil+1,\\ 6\lceil\frac{r-2}{2}\rceil+2(r-3)\lfloor\frac{s-3}{6}\rfloor-1, & r \text{ is even; } t=\lceil\frac{r}{2}\rceil+1,\\ 6(r-t)+4(r-t)\lfloor\frac{s-3}{6}\rfloor+3, & \text{for others.}\end{cases}$$

$$\psi(v_{t,l})=\begin{cases}3\lceil\frac{s}{6}\rceil-1, & \frac{s}{2} \text{ is odd; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ 3\lceil\frac{s}{6}\rceil, & \frac{s}{2} \text{ is even; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ 6(t-1)+4(t-1)\lfloor\frac{s-3}{6}\rfloor, & t\neq 2; t<\lceil\frac{r+1}{2}\rceil+1,\\ 6\lceil\frac{r}{2}\rceil+4\lceil\frac{r-2}{2}\rceil\lfloor\frac{s-3}{6}\rfloor+2\lfloor\frac{s-3}{6}\rfloor-2, & r\geq 3; t=\lceil\frac{r+1}{2}\rceil+1,\\ 6(r-t+2)+4(r-t+1)\lfloor\frac{s-3}{6}\rfloor+2\lfloor\frac{s-3}{6}\rfloor-2, & \text{for others.}\end{cases}$$

$$\psi(v_tv_{t,l})=\begin{cases}t+l, & t=1,2; 1\leq l\leq s-1;\\ & t<\lceil\frac{r}{2}\rceil+1,\\ 6(t-2)+4(t-3)\lfloor\frac{s-3}{6}\rfloor+2\lfloor\frac{s-3}{6}\rfloor+l-2, & t\neq 1,2; 1\leq l\leq s-1;\\ & t<\lceil\frac{r}{2}\rceil+1,\\ 3(r-2)+2(r-3)\lfloor\frac{s-3}{6}\rfloor+l-1, & r\neq 3; 1\leq l\leq s-1;\\ & t=\lceil\frac{r}{2}\rceil+1,\\ 6(r-t)+4(r-t)\lfloor\frac{s-3}{6}\rfloor+l+1, & \text{for others.}\end{cases}$$

The upper bound is the maximum vertex and edge labels constructed with ψ-labeling on the comb of cycle and star graph $C_r \rhd S_s$ for $r\geq 3, s\equiv 2 \pmod 6$. Then take the largest label, that is at vertex $v_{t,l}$.

For r is even with $rs+r\equiv 0 \pmod 6$ and $t=\frac{r}{2}+1$,

$$\begin{aligned}6(t-1)+4(t-1)\left\lfloor\frac{s-3}{6}\right\rfloor &= \left(\frac{r}{2}\right)+4\left(\frac{r}{2}\right)\left(\frac{s-8}{6}\right)\\ &= \frac{rs+r}{3}\\ &= \left\lceil\frac{rs+r}{3}\right\rceil.\end{aligned}$$

For r is odd with $rs+r\equiv 3 \pmod 6$,

$$\begin{aligned}6\left\lceil\frac{r}{2}\right\rceil+4\left\lceil\frac{r-2}{2}\right\rceil\left\lfloor\frac{s-3}{6}\right\rfloor+2\left\lfloor\frac{s-3}{6}\right\rfloor-2 &= 3(r+1)+(r-1)\left(\frac{s-8}{3}\right)+\frac{s-8}{3}-2\\ &= \frac{rs+r}{3}+1\\ &= \left\lceil\frac{rs+r}{3}\right\rceil+1.\end{aligned}$$

Because of the largest label equals (4), we get

$$res(C_r \rhd S_s) \leq \begin{cases} \lceil \frac{rs+r}{3} \rceil, & \text{For } rs+r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{rs+r}{3} \rceil + 1, & \text{For } rs+r \equiv 2,3 \pmod 6. \end{cases} \tag{6}$$

From (5) and (6) it can be concluded that

$$res(C_r \rhd S_s) = \begin{cases} \lceil \frac{rs+r}{3} \rceil, & \text{For } rs+r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{rs+r}{3} \rceil + 1, & \text{For } rs+r \equiv 2,3 \pmod 6. \end{cases}$$

Then the weight of edges are,

$$wt(u_t u_{t+1}) = \begin{cases} 18t + 12(t-1)\lfloor \frac{s-3}{6} \rfloor + 6\lfloor \frac{s-3}{6} \rfloor - 9, & t < \lfloor \frac{r}{2} \rfloor + 1, \\ 9(r-1) + 6(r-1)\lfloor \frac{s-3}{6} \rfloor + 1, & t = \lfloor \frac{r}{2} \rfloor + 1, \\ 18(r-t+1) + 12(r-t+1)\lfloor \frac{s-3}{6} \rfloor, & t > \lfloor \frac{r}{2} \rfloor + 1. \end{cases}$$

$$wt(u_t v_t) = \begin{cases} 9(t-1) + 6(t-1)\lfloor \frac{s-3}{6} \rfloor + 1, & t = 1,2; t < \lceil \frac{r}{2} \rceil + 1, \\ 18(t-1) + 12(t-2)\lfloor \frac{s-3}{6} \rfloor + 6\lfloor \frac{s-3}{6} \rfloor - 8, & t \neq 1,2; t < \lceil \frac{r}{2} \rceil + 1, \\ 9(r-1) + 6(r-1)\lfloor \frac{s-3}{6} \rfloor + 2, & t = \lceil \frac{r}{2} \rceil + 1, \\ 18(r-t+1) + 12(r-t+1)\lfloor \frac{s-3}{6} \rfloor + 1, & t > \lceil \frac{r}{2} \rceil + 1. \end{cases}$$

$$wt(v_t v_{t,l}) = \begin{cases} 9(t-1) + 6(t-1)\lfloor \frac{s-3}{6} \rfloor + l + 1, & t = 1,2; 1 \leq l \leq s-1; \\ & t < \lceil \frac{r}{2} \rceil + 1, \\ 18(t-1) + 12(t-2)\lfloor \frac{s-3}{6} \rfloor + 6\lfloor \frac{s-3}{6} \rfloor + l - 8, & t \neq 1,2; 1 \leq l \leq s-1; \\ & t < \lceil \frac{r}{2} \rceil + 1, \\ 9(r-1) + 6(r-1)\lfloor \frac{s-3}{6} \rfloor + l + 2, & 1 \leq l \leq s-1; t = \lceil \frac{r}{2} \rceil + 1, \\ 18(r-t+1) + 12(r-t+1)\lfloor \frac{s-3}{6} \rfloor + l + 1, & 1 \leq l \leq s-1; t > \lceil \frac{r}{2} \rceil + 1. \end{cases}$$

From the weight of edges that have been presented, all edges in the comb of cycle and star graph $C_r \rhd S_s$ with $s \equiv 2 \pmod 6$ are different. Accordingly, ψ has the required component on an edge irregular reflexive k-labeling, then $res(C_r \rhd S_s)$ as in Theorem 3.1. So, the proof of $res(C_r \rhd S_s)$ with $s \equiv 2 \pmod 6$ is completed.

□

3.2. **For $C_r \rhd S_s$ with $s \equiv 5 \pmod 6$.** Theorem 3.2 can be used to calculate the $res(G)$ of $C_r \rhd S_s$ with $s \equiv 5 \pmod 6$.

Theorem 3.2. *For $C_r \rhd S_s$ with $r \geq 3, s \equiv 5 \pmod 6$,*

$$res(C_r \rhd S_s) = \lceil \frac{rs+r}{3} \rceil, rs+r \not\equiv 2,3 \pmod 6. \tag{7}$$

Proof. Firstly, verify a lower bound for $res(C_r \rhd S_s)$. By reason of total edges of $C_r \rhd S_s$ are $rs+r$, subsequently by Lemma 1.1 was gained:

$$res(C_r \rhd S_s) \geq \lceil \frac{rs+r}{3} \rceil, rs+r \not\equiv 2,3 \pmod 6. \tag{8}$$

This is equivalent to the lower bound in (7).

Furthermore, to obtain the upper bound of $res(C_r \rhd S_s)$ with $s \equiv 5 \pmod 6$, ψ-labeling is constructed on comb of cycle and star graph as follows, For $1 \le t \le r$, $1 \le l \le s-1$

$$\psi(u_t)=\begin{cases}\lceil\frac{s}{2}\rceil, & \lfloor\frac{s}{2}\rfloor \text{ is odd; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ \lfloor\frac{s}{2}\rfloor+2, & \lfloor\frac{s}{2}\rfloor \text{ is even; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ 4(t-1)+4(t-1)\lfloor\frac{s}{6}\rfloor, & t\neq 2; t<\lceil\frac{r+1}{2}\rceil+1,\\ 4\lceil\frac{r}{2}\rceil+4\lceil\frac{r-2}{2}\rceil\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor-2, & r\geq 3; t=\lceil\frac{r+1}{2}\rceil+1,\\ 4(r-t)+4(r-t+1)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor+6, & \text{for others.}\end{cases}$$

$$\psi(u_tu_{t+1})=\begin{cases}\lceil\frac{s}{2}\rceil+4(t-1)\lceil\frac{s}{6}\rceil, & \lfloor\frac{s}{2}\rfloor \text{ is odd; } t=1,2;\\ & t<\lfloor\frac{r}{2}\rfloor+1,\\ \lfloor\frac{s}{2}\rfloor+4(t-1)\lceil\frac{s}{6}\rceil, & \lfloor\frac{s}{2}\rfloor \text{ is even; } t=1,2;\\ & t<\lfloor\frac{r}{2}\rfloor+1,\\ 4t+4(t-1)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor-2, & t\neq 1,2; t<\lfloor\frac{r}{2}\rfloor+1,\\ \lceil\frac{s}{2}\rceil+1, & \lfloor\frac{s}{2}\rfloor \text{ is odd; } r=3;\\ & t=\lfloor\frac{r}{2}\rfloor+1,\\ \lceil\frac{s}{2}\rceil, & \lfloor\frac{s}{2}\rfloor \text{ is even; } r=3;\\ & t=\lfloor\frac{r}{2}\rfloor+1,\\ 2(r-2)+2(r-2)\lfloor\frac{s}{6}\rfloor+1, & r\neq 3; t=\lfloor\frac{r}{2}\rfloor+1,\\ s+1, & r-t=0; t>\lfloor\frac{r}{2}\rfloor+1,\\ 4(r-t+1)+4(r-t+1)\lfloor\frac{s}{6}\rfloor, & \text{for others.}\end{cases}$$

$$\psi(v_t)=\begin{cases}\lceil\frac{s}{2}\rceil, & \lfloor\frac{s}{2}\rfloor \text{ is odd; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ \lfloor\frac{s}{2}\rfloor, & \lfloor\frac{s}{2}\rfloor \text{ is even; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ 4(t-1)+4(t-1)\lfloor\frac{s}{6}\rfloor, & t\neq 2; t<\lceil\frac{r+1}{2}\rceil+1,\\ 4\lceil\frac{r}{2}\rceil+4(r-2)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor-2, & r\geq 3; t=\lceil\frac{r+1}{2}\rceil+1,\\ 4(r-t)+4(r-t+1)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor+6, & \text{for others.}\end{cases}$$

$$\psi(u_tv_t)=\begin{cases}1, & t=1,2; t<\lceil\frac{r}{2}\rceil+1,\\ 4(t-2)+4(t-3)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor-1, & t\neq 1,2; t<\lceil\frac{r}{2}\rceil+1,\\ 2(r-2)+2(r-3)\lfloor\frac{s}{6}\rfloor, & r\neq 3; t=\lceil\frac{r}{2}\rceil+1,\\ 4(r-t)+4(r-t)\lfloor\frac{s}{6}\rfloor+1, & \text{for others.}\end{cases}$$

$$\psi(v_{t,l})=\begin{cases}\lceil\frac{s}{2}\rceil, & \lfloor\frac{s}{2}\rfloor \text{ is odd; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ \lfloor\frac{s}{2}\rfloor+2, & \lfloor\frac{s}{2}\rfloor \text{ is even; } t=2;\\ & t<\lceil\frac{r+1}{2}\rceil+1,\\ 4(t-1)+4(t-1)\lfloor\frac{s}{6}\rfloor, & t\neq 2; t<\lceil\frac{r+1}{2}\rceil+1,\\ 4\lceil\frac{r}{2}\rceil+4\lceil\frac{r-2}{2}\rceil\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor-2, & r\geq 3; t=\lceil\frac{r+1}{2}\rceil+1,\\ 4(r-t)+4(r-t+1)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor+6, & \text{for others.}\end{cases}$$

$$\psi(v_tv_{t,l})=\begin{cases}l+1, & t=1,2; 1\leq l\leq s-1; t<\lceil\frac{r}{2}\rceil+1,\\ 4(t-2)+4(t-3)\lfloor\frac{s}{6}\rfloor+2\lfloor\frac{s}{6}\rfloor+l-1, & t\neq 1,2; 1\leq l\leq s-1; t<\lceil\frac{r}{2}\rceil+1,\\ 2(r-2)+2(r-3)\lfloor\frac{s}{6}\rfloor+l, & r\neq 3; 1\leq l\leq s-1; t=\lceil\frac{r}{2}\rceil+1,\\ 4(r-t)+4(r-t)\lfloor\frac{s}{6}\rfloor+l+1, & \text{for others.}\end{cases}$$

The upper bound is the maximum vertex and edge labels constructed with ψ-labeling on the comb of cycle and star graph $C_r \rhd S_s$ for $r \geq 3, s \equiv 5 \pmod 6$. en take the largest label, that is at edge $v_t v_{t,l}$,

For $rs + r \equiv 0 \pmod 6$ with $l = s - 1$,

$$\begin{aligned} 2(r-2) + 2(r-3)\left\lfloor \frac{s}{6} \right\rfloor + l &= \frac{6r-12}{3} + \frac{rs - 5r - 3s + 15}{6} + s - 1 \\ &= \frac{rs+r}{3} \\ &= \left\lceil \frac{rs+r}{3} \right\rceil . \end{aligned}$$

Because of the largest label equals (7), we get

$$res(C_r \rhd S_s) \leq \left\lceil \frac{rs+r}{3} \right\rceil, rs + r \not\equiv 2, 3 \pmod 6. \tag{9}$$

From (8) and (9) it can be concluded that

$$res(C_r \rhd S_s) = \left\lceil \frac{rs+r}{3} \right\rceil, rs + r \not\equiv 2, 3 \pmod 6. \tag{10}$$

Then the edge weights are,

$$wt(u_t u_{t+1}) = \begin{cases} 12t + 12(t-1)\lfloor \frac{s}{6} \rfloor + 6\lfloor \frac{s}{6} \rfloor - 6, & t < \lfloor \frac{r}{2} \rfloor + 1, \\ 6(r-1) + 6(r-1)\lfloor \frac{s}{6} \rfloor + 1, & t = \lfloor \frac{r}{2} \rfloor + 1, \\ 12(r-t+1) + 12(r-t+1)\lfloor \frac{s}{6} \rfloor, & t > \lfloor \frac{r}{2} \rfloor + 1. \end{cases}$$

$$wt(u_t v_t) = \begin{cases} 6(t-1) + 6(t-1)\lfloor \frac{s}{6} \rfloor + 1, & t = 1, 2; t < \lceil \frac{r}{2} \rceil + 1, \\ 12(t-1) + 12(t-2)\lfloor \frac{s}{6} \rfloor + 6\lfloor \frac{s}{6} \rfloor - 5, & t \neq 1, 2; t < \lceil \frac{r}{2} \rceil + 1, \\ 6(r-1) + 6(r-1)\lfloor \frac{s}{6} \rfloor + 2, & t = \lceil \frac{r}{2} \rceil + 1, \\ 12(r-t+1) + 12(r-t+1)\lfloor \frac{s}{6} \rfloor + 1, & t > \lceil \frac{r}{2} \rceil + 1. \end{cases}$$

$$wt(v_t v_{t,l}) = \begin{cases} 6(t-1) + 6(t-1)\lfloor \frac{s}{6} \rfloor + l + 1, & t = 1, 2; 1 \leq l \leq s-1; \\ & t < \lceil \frac{r}{2} \rceil + 1, \\ 12(t-1) + 12(t-2)\lfloor \frac{s}{6} \rfloor + 6\lfloor \frac{s}{6} \rfloor + l - 5, & t \neq 1, 2; 1 \leq l \leq s-1; \\ & t < \lceil \frac{r}{2} \rceil + 1, \\ 6(r-1) + 6(r-1)\lfloor \frac{s}{6} \rfloor + l + 2, & 1 \leq l \leq s-1; t = \lceil \frac{r}{2} \rceil + 1, \\ 12(r-t+1) + 12(r-t+1)\lfloor \frac{s}{6} \rfloor + l + 1, & 1 \leq l \leq s-1; t > \lceil \frac{r}{2} \rceil + 1. \end{cases}$$

We can infer that all edges in the comb of cycle and star graph $C_r \rhd S_s$ with $s \equiv 5 \pmod 6$ are different based on the weights of the edges that have been provided. As a result, $res(C_r \rhd S_s)$ as in Theorem 3.2, and ψ possesses the necessary component on an edge irregular reflexive k-labeling. So, the proof of $res(C_r \rhd S_s)$ with $s \equiv 5 \pmod 6$ is finished.

□

Figure 2 illustrates a comb of cycle and star graph $C_4 \rhd S_5$ graph with an edge irregular reflexive 8-labeling. The names and labels of the vertices are indicated by the black color, the weight of the edge by the red color, and the edges by the blue color.

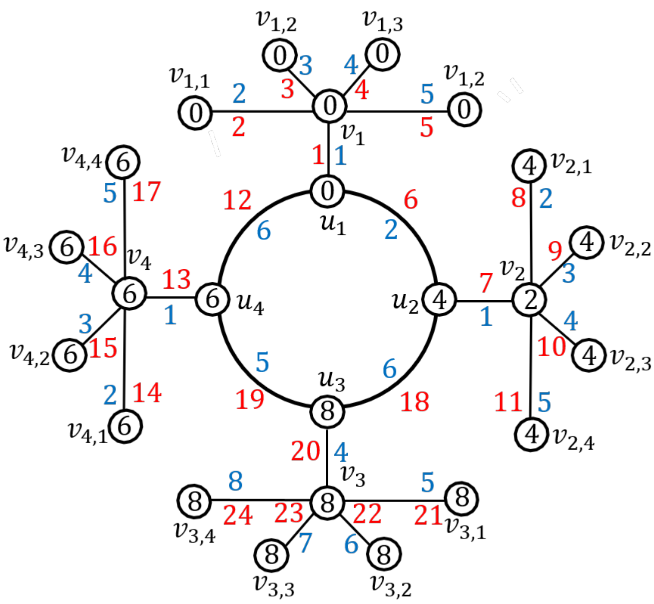

FIGURE 2. Edge irregular reflexive 8-labeling on comb of cycle and star graph $C_4 \triangleright S_5$

4. Conclusions

After more research, we determine that

(1) Reflexive edge strength on double broom graph $B(r, s, s)$ with $r, s \geq 2$ are

$$res(B(r,s,s)) = \begin{cases} \lceil \frac{r+2s-1}{3} \rceil, & r+2s-1 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{r+2s-1}{3} \rceil + 1, & r+2s-1 \equiv 2,3 \pmod 6. \end{cases}$$

(2) Reflexive edge strength on comb of cycle and star graph $C_r \triangleright S_s$ with $r \geq 3$, $s \equiv 2,5 \pmod 6$ are

$$res(C_r \triangleright S_s) = \begin{cases} \lceil \frac{rs+r}{3} \rceil, & rs+r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{rs+r}{3} \rceil + 1, & rs+r \equiv 2,3 \pmod 6. \end{cases}$$

Open Problem. The readers can continue this research for double broom graph $B(r, s, t)$ with $r, s, t \geq 2, s \neq t$ and comb of cycle and star graph $C_r \triangleright S_s$ with $r \geq 3$, $s \not\equiv 2,5 \pmod 6$.

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Rahma Shinta Vinatih is graduated from Department of Mathematics, Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Indonesia as bachelor in 2023. Her research group is combinatorial mathematics, especially graph labeling. In 2022, she became a paper presenter in the international conference that is the 6th International Conference on Combinatorics, Graph Theory, and Network Topology (ICCGANT) at Jember University, Indonesia. She participated actively in both national and international conferences.

Diari Indriati is a mathematics lecturer and researcher in the field of graph theory at Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Indonesia. Bachelor, master, and doctoral courses are taken at the Department of Mathematics, Faculty of Mathematics and Natural Sciences, Gadjah Mada University, Yogyakarta, Indonesia. She is also a reviewer for several international journals in the field of graph theory.