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APPROXIMATE IMPEDANCE OPERATOR OF AN INFINITE THIN STRIP AND STABILIZATION IN THE CONTEXT OF LINEAR MICRO-DILATATION ELASTICITY

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Abstract. The aim of this paper is to give asymptotic models for the impedance of an infinite thin strip in the framework of linear elasticity with voids. We start from a two dimensional transmission problem which models the wave propagation between an elastic body with small distributed voids Ω_- and a thin coating strip Ω_+^δ (δ is supposed to be small enough). We show how to model the effect of the thin coating by an impedance boundary condition on the junction of the elastic two bodies. To this end, we use the techniques of abstract differential equations and asymptotic expansion. We prove also an error estimate.

Keywords: linear micro-dilatation elasticity, thin strip, impedance operator, abstract differential equations, asymptotic expansion, stability result.

AMS Subject Classification: Primary 35B40, 35C20; Secondary 74G10, 74K30.

1. Introduction

1.1. **Problem setting and motivations.** In this paper, the physical model under consideration is the linear micro-dilatation elasticity or the so called linear elasticity with voids that was developed by Nunziato and Cowin [14] as a specialization of the non-linear theory [28]. The unknowns of this mathematical model are the displacement vector u_i ($i = 1, 2$) and the scalar micro-dilatation ω, it is adapted to describe the behavior of solids with small distributed voids or pores such as granular and manufactured porous bodies where the theory of classical elasticity is inadequate. Porous materials have voids or pores that increase the surface area, improving heat transfer rates and thermal efficiency [20]. Porosity also affects the mechanical properties of materials, such as compressive, bending,

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and tensile strength [24]. The decrease of porosity generally leads to an improvement in mechanical properties, and the relationship between mechanical properties and porosity can be described by mathematical models.

The theory of elasticity with voids has been extensively studied, and a multitude of contributions addressing both its fundamental principles and practical applications have been published. Basic results can be found in [13, 17, 23] and therein. Theorems concerning the existence and uniqueness of the solution have been established, see for instance [22]. Among the very important recent works, let us mention [10] where the author has obtained explicit solutions of the Dirichlet type boundary value problems of elasticity for porous infinite strip by using the Fourier method, see also [11, 12]. We also mention that in [15], the equilibrium problem of porous elastic spherical bodies under radial surface traction has been solved and its solution has been given in closed form.

The structures studied in engineering are often made of materials covered with thin layers. Their mathematical modeling is a problem of outstanding practical importance. However, from a numerical point of view, the resolution of such problems can not be computed accurately since the small thickness δ of the thin layer creates instabilities related to the parameter δ (see for instance [8]). To avoid these numerical instabilities, we use a well-known approach consists in deriving approximate boundary conditions called approximate impedance conditions that incorporate in an approximate way the effect of the thin layer and doesn't take it into account any more. More precisely, we seek an approximate problem set on the fixed domain (i.e. not including the thin layer) but taking into account its effect via these approximate impedance conditions.

The concept of impedance boundary condition is widely used in numerous studies, mainly in electro-magnetics and mechanics, see for instance [9, 18] for the Helmholtz equation in acoustics, [6, 21] for Maxwell equations, [26, 27, 29, 30] in structure mechanics, and [4, 7, 19, 25] for linear classical elasticity. In physics, the term impedance is used with specific meanings such as mechanical impedance and electrical impedance. In elastostatics, we specifically refer to mechanical impedance, which serves as a metric to measure a material's or structure's resistance to deformation or its reaction to external forces. It provides a quantification of the density of applied forces.

This paper falls within the framework of applications of the techniques of asymptotic expansions with scaling for modeling the impedance of an infinite thin strip in the framework of linear micro-dilatation elasticity. It is a continuation of the researches of the first author in the modeling of the impedance of thin layers of elastic deformable bodies, see for instance [1, 2, 3], where the authors have derived first order approximations of the impedance in asymmetric elasticity. To begin with, we consider a two-dimensional transmission model problem, of linear micro-dilatation elasticity in a domain Ω^δ consisting of two homogeneous and isotropic bonded porous elastic bodies: Ω_- and a thin strip Ω_+^δ. Thus we set

$$\Omega^\delta = \mathbb{R} \times \left]-1, \delta\right[, \ \Omega_- = \mathbb{R} \times \left]-1, 0\right[, \ \Omega_+^\delta = \mathbb{R} \times \left]0, \delta\right[,$$
$$\Gamma_- = \mathbb{R} \times \{-1\} , \ \Sigma = \mathbb{R} \times \{0\} , \ \Gamma_+^\delta = \mathbb{R} \times \{\delta\} .$$

We restrict our consideration to the case of elastostatics, and we denote by the index $+$ (resp. $-$) to the restriction on Ω_+^δ (resp. on Ω_-). The governing equations of the transmission model problem $\left(P^\delta\right)$ are as follows (see [14, 17]):

(1) Equilibrium equations in Ω_-

$$\begin{cases} \sigma_{-ij,j}\left(u_-^\delta, \omega_-^\delta\right) = p_{i-}, \ i,j = 1,2, \\ h_{-i,i}\left(\omega_-^\delta\right) - g_-\left(u_-^\delta, \omega_-^\delta\right) = q_-, \ i = 1,2. \end{cases} \tag{1}$$

(2) Equilibrium equations in Ω_+^δ

$$\left\{\begin{array}{c}\sigma_{+ij,j}\left(u_+^\delta,\omega_+^\delta\right)=0,\ i,j=1,2,\\ h_{+i,i}\left(\omega_+^\delta\right)-g_+\left(u_+^\delta,\omega_+^\delta\right)=0,\ i=1,2.\end{array}\right. \tag{2}$$

(3) Dirichlet boundary conditions on Γ_-

$$\left\{\begin{array}{l}u_-^\delta=0,\\ \omega_-^\delta=0.\end{array}\right. \tag{3}$$

(4) Neumann boundary conditions on Γ_+^δ

$$\left\{\begin{array}{c}\sigma_{+12}\left(u_+^\delta,\omega_+^\delta\right)=0,\\ \sigma_{+22}\left(u_+^\delta,\omega_+^\delta\right)=0,\\ h_{+2}\left(\omega_+^\delta\right)=0.\end{array}\right. \tag{4}$$

(5) Transmission conditions at the interface Σ

$$\left\{\begin{array}{c}u_-^\delta=u_+^\delta,\\ \omega_-^\delta=\omega_+^\delta,\\ \sigma_{-12}\left(u_-^\delta,\omega_-^\delta\right)=\sigma_{+12}\left(u_+^\delta,\omega_+^\delta\right),\\ \sigma_{-22}\left(u_-^\delta,\omega_-^\delta\right)=\sigma_{+22}\left(u_+^\delta,\omega_+^\delta\right),\\ h_{-2}\left(\omega_-^\delta\right)=h_{+2}\left(\omega_+^\delta\right),\end{array}\right. \tag{5}$$

where $\sigma_\pm$ represents the stress tensor, p_- is the body force vector, $g_\pm\left(u_\pm^\delta,\omega_\pm^\delta\right)$ is the intrinsic equilibrated body force, $h_\pm\left(\omega_\pm^\delta\right)$ is the equilibrated stress vector, q_- is the extrinsic equilibrated body force. For the sake of simplicity in the next sections, we adopt the following notations:

$$\left(u_\pm^\delta,\omega_\pm^\delta\right)=\left(u_{\pm1}^\delta,u_{\pm2}^\delta,\omega_\pm^\delta\right),$$
$$\sigma_\pm\left(u_\pm^\delta,\omega_\pm^\delta\right)\nu=\left(\sigma_{\pm12}\left(u_\pm^\delta,\omega_\pm^\delta\right),\sigma_{\pm22}\left(u_\pm^\delta,\omega_\pm^\delta\right)\right)\ \text{where } \nu=(0,1),$$
$$\left(\sigma_\pm\left(u_\pm^\delta,\omega_\pm^\delta\right)\nu,h_{\pm2}\left(\omega_\pm^\delta\right)\right)=\left(\sigma_{\pm12}\left(u_\pm^\delta,\omega_\pm^\delta\right),\sigma_{\pm22}\left(u_\pm^\delta,\omega_\pm^\delta\right),h_{\pm2}\left(\omega_\pm^\delta\right)\right).$$

The constitutive equations for the linear isotropic micro-dilatation elasticity are defined by:

$$\begin{aligned}\sigma_{\pm ij}\left(u_\pm^\delta,\omega_\pm^\delta\right)&=2\mu_\pm e_{\pm ij}+\lambda_\pm e_{\pm kk}\delta_{ij}+\beta_\pm\omega_\pm^\delta\delta_{ij},\ i,j,k=1,2,\\ h_{\pm i}\left(\omega_\pm^\delta\right)&=\alpha_\pm\omega_{\pm,i}^\delta,\ i=1,2,\\ g_\pm\left(u_\pm^\delta,\omega_\pm^\delta\right)&=\beta_\pm e_{\pm kk}+\zeta_\pm\omega_\pm^\delta,\ k=1,2,\end{aligned}$$

where δ_{ij} is the Kronecker delta, $e_{\pm ij}$ is the strain tensor defined by:

$$e_{\pm ij}\left(u_\pm^\delta\right)=\frac{1}{2}\left(u_{\pm j,i}^\delta+u_{\pm i,j}^\delta\right),\ i,j=1,2,$$

and $\mu_\pm$, $\alpha_\pm$, $\zeta_\pm$, $\lambda_\pm$ and $\beta_\pm$ are material constants satisfying the inequalities:

$$\mu_\pm>0,\ \alpha_\pm>0,\ \zeta_\pm>0,\ \beta_\pm>0,$$
$$2\mu_\pm+3\lambda_\pm>0,\ \left(2\mu_\pm+3\lambda_\pm\right)\zeta_\pm>3\beta_\pm^2.$$

It is well known [22] that the transmission problem (P^δ) has a unique solution in the canonical Sobolev space:

$$W=\left\{\begin{array}{c}\left(v_\pm,\varphi_\pm\right)\in\mathbb{H}^1\left(\Omega_\pm\right)=\left[H^1\left(\Omega_\pm\right)\right]^3:\\ v_-=0\text{ on }\Gamma_-,\ \varphi_-=0\text{ on }\Gamma_-,\\ v_-=v_+\text{ on }\Sigma,\ \varphi_-=\varphi_+\text{ on }\Sigma.\end{array}\right\}$$

As was already indicated, our aim in this paper is to derive an approximate impedance boundary condition on the interface Σ that incorporates in an approximate way the effect of the thin strip Ω_+^δ on Ω_- to reduce the transmission problem (P^δ) to a boundary value problem set in the fixed domain Ω_-, i.e. the equilibrium equations in Ω_+^δ, the transmission conditions on Σ and the Neumann boundary conditions on Γ_+^δ are embodied in the form of an impedance boundary conditions on Σ and depending on δ.

1.2. **Impedance of the thin strip Ω_+^δ.** Our goal is to reduce the transmission problem (P^δ) set in $\Omega^\delta = \Omega_- \cup \Omega_+^\delta$ to a boundary value problem set only on the fixed domain Ω_-. The exact effect of the thin strip Ω_+^δ on the domain Ω_- is given by the impedance operator T_δ defined by:

$$T_\delta\left(v^\delta,\phi^\delta\right) := \left(\sigma_+\left(u_+^\delta,\omega_+^\delta\right)\nu_{|\Sigma}, h_{+2}\left(\omega_+^\delta\right)_{|\Sigma}\right),$$

where $\left(u_+^\delta,\omega_+^\delta\right)$ is the solution of the following problem:

$$(P_+^\delta):\begin{cases}\text{Equations (2) in } \ \Omega_+^\delta,\\ \text{Boundary conditions (4) on } \ \Gamma_+^\delta,\\ u_+^\delta = v^\delta \ \text{ on } \Sigma,\ \omega_+^\delta=\phi^\delta \ \text{ on } \Sigma,\end{cases}$$

from the transmission conditions (5), it follows that:

$$\left(\sigma_-\left(u_-^\delta,\omega_-^\delta\right)\nu_{|\Sigma}, h_{-2}\left(\omega_-^\delta\right)_{|\Sigma}\right) = T_\delta\left(\left(u_-^\delta,\omega_-^\delta\right)_{|\Sigma}\right),$$

and the transmission problem (P^δ) is then equivalent to the following impedance problem set in Ω_-:

$$(P_-^\delta):\begin{cases}\text{Equations (1) in } \ \Omega_-,\\ \text{Boundary conditions (3) on } \ \Gamma_-,\\ \left(\sigma_-\left(u_-^\delta,\omega_-^\delta\right)\nu_{|\Sigma}, h_{-2}\left(\omega_-^\delta\right)_{|\Sigma}\right) = T_\delta\left(\left(u_-^\delta,\omega_-^\delta\right)_{|\Sigma}\right) \ \text{ on } \Sigma.\end{cases}$$

Since an explicit expression of the exact impedance operator T_δ is not reachable for the general case, we will just derive an effective approximation $T_{*\delta}$ of T_δ with:

$$T_{*\delta} = \delta T_* \ \text{ and } \ T_*\left(v^\delta,\phi^\delta\right) = \left(C_1\left(v^\delta,\phi^\delta\right), C_2\left(v^\delta,\phi^\delta\right), C_3\left(v^\delta,\phi^\delta\right)\right),$$

where

$$C_1\left(v^\delta,\phi^\delta\right) = \tfrac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}v_{1,11}^\delta + \tfrac{2\mu_+\beta_+}{2\mu_++\lambda_+}\phi_{,1}^\delta,$$
$$C_2\left(v^\delta,\phi^\delta\right) = 0,$$
$$C_3\left(v^\delta,\phi^\delta\right) = \alpha_+\phi_{,11}^\delta - \tfrac{\zeta_+(2\mu_++\lambda_+)-\beta_+^2}{2\mu_++\lambda_+}\phi_{,1}^\delta - \tfrac{2\beta_+\mu_+}{2\mu_++\lambda_+}v_{1,1}^\delta.$$

The solution $(u_-^\delta,\omega_-^\delta)$ of (P^δ) in Ω_- is then approximated by the solution $(u_{-*}^\delta,\omega_{-*}^\delta)$ of the following approximate impedance problem:

$$\left(P_{-*}^\delta\right):\begin{cases}\sigma_{-ij,j}\left(u_{-*}^\delta,\omega_{-*}^\delta\right) = p_{-i},\ i,j=1,2,\\ h_{-i,i}\left(\omega_{-*}^\delta\right) - g_-\left(u_{-*}^\delta,\omega_{-*}^\delta\right) = q_-,\ i=1,2,\\ u_{-*}^\delta = 0 \text{ on } \Gamma_-,\ \omega_{-*}^\delta = 0 \text{ on } \Gamma_-,\\ \left(\sigma_-\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\nu_{|\Sigma}, h_{-2}\left(\omega_{-*}^\delta\right)_{|\Sigma}\right) = T_{*\delta}\left(u_{-*}^\delta,\omega_{-*}^\delta\right) \ \text{ on } \Sigma,\end{cases}$$

and we prove the following main result of the paper:

Theorem 1.1. For given (p_-, q_-) in $\left[L^2(\Omega_-)\right]^3$, the boundary value problem $\left(P^\delta_{-*}\right)$ has a unique solution in the space

$$W_* = \left\{ \begin{array}{c} (v_-, \varphi_-) \in \mathbb{H}^1(\Omega_-) = \left[H^1(\Omega_-)\right]^3 : \\ (v_{-1,1}, v_{-2,1}, \varphi_{-,1}) \in \mathbb{L}^2(\Sigma) = \left[L^2(\Sigma)\right]^3 \\ v_- = \varphi_- = 0 \ \text{ on } \ \Gamma_-, \end{array} \right\}$$

and the following error estimate holds

$$\left\| u^\delta_- - u^\delta_{-*} \right\|_{(H^1(\Omega_-))^2} + \left\| \omega^\delta_- - \omega^\delta_{-*} \right\|_{H^1(\Omega_-)} \leq C\delta^2,$$

where the constant C depends only on p_-, q_- and the elasticity coefficients.

The paper is structured as follows: In the next section, we derive an approximate impedance boundary condition of the thin strip by using the technique of abstract differential equations. In section 3, by using the techniques of asymptotic expansion with scaling, we obtain the same approximate impedance boundary condition which is derived in section 2. In section 4, we establish a stability estimate for the transmission problem. In section 5, we state and prove a stability result for the approximate impedance problem. Finally, in section 6, we estimate the error between $\left(u^\delta_-, \omega^\delta_-\right)$ and $\left(u^\delta_{-*}, \omega^\delta_{-*}\right)$ in an appropriate space.

2. Abstract differential equation and approximate impedance operator

Here we construct a first-order approximation of the impedance by using the techniques of abstract differential equations. The first step for obtaining an approximate impedance condition is to express explicitly the normal derivatives of u^δ_+, ω^δ_+, σ_{+12}, σ_{+22} and h_{+2} on Σ in terms of traces on Σ of σ_{+12}, σ_{+22}, u^δ_+ and ω^δ_+. We look at σ_{+12}, σ_{+22} and h_{+2} as functions of variables x_1 and x_2 with $x = (x_1, x_2)$ in Ω^δ_+. Since we have

$$\begin{aligned} \sigma_{+12}(x) &= \sigma_{+21}(x) = \mu_+ \left(u^\delta_{+1,2}(x) + u^\delta_{+2,1}(x)\right), \\ \sigma_{+22}(x) &= (2\mu_+ + \lambda_+)\, u^\delta_{+2,2}(x) + \lambda_+ u^\delta_{+1,1}(x) + \beta_+ \omega^\delta_+(x), \end{aligned}$$

it follows that

$$\begin{aligned} u^\delta_{+1,2}(x) &= \frac{1}{\mu_+}\left[\sigma_{+12}(x) - \mu_+ u^\delta_{+2,1}(x)\right], \\ u^\delta_{+2,2}(x) &= \frac{1}{2\mu_+ + \lambda_+}\left[\sigma_{+22}(x) - \lambda_+ u^\delta_{+1,1}(x) - \beta_+ \omega^\delta_+(x)\right], \end{aligned}$$

and then

$$\begin{aligned} \sigma_{+11}(x) &= (2\mu_+ + \lambda_+)\, u^\delta_{+1,1}(x) + \lambda_+ u^\delta_{+2,2}(x) + \beta_+ \omega^\delta_+(x) \\ &= (2\mu_+ + \lambda_+)\, u^\delta_{+1,1}(x) + \beta_+ \omega^\delta_+(x) \\ &\qquad + \frac{\lambda_+}{2\mu_+ + \lambda_+}\left[\sigma_{+22}(x) - \lambda_+ u^\delta_{+1,1}(x) - \beta_+ \omega^\delta_+(x)\right] \\ &= \frac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)} u^\delta_{+1,1}(x) + \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+}\omega^\delta_+(x) + \frac{\lambda_+}{2\mu_+ + \lambda_+}\sigma_{+22}(x). \quad (6) \end{aligned}$$

Using equations (2), formula (6), and the fact that $\sigma_{+12} = \sigma_{+21}$, we get

$$\left\{\begin{array}{c} \sigma_{+12,2}(x) = -\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)} u^\delta_{+1,11}(x) - \frac{2\mu_+\beta_+}{2\mu_++\lambda_+}\omega^\delta_{+,1}(x) - \frac{\lambda_+}{2\mu_++\lambda_+}\sigma_{+22,1}(x), \\ \sigma_{+22,2}(x) = -\sigma_{+12,1}(x), \\ h_{+2,2}(x) = -\alpha_+\omega^\delta_{+,11}(x) + \frac{2\beta_+\mu_+}{2\mu_++\lambda_+}u^\delta_{+1,1}(x) + \frac{\zeta_+(2\mu_++\lambda_+)-\beta_+^2}{2\mu_++\lambda_+}\omega^\delta_+(x) \\ +\frac{\beta_+}{2\mu_++\lambda_+}\sigma_{+22}(x). \end{array}\right. \tag{7}$$

The second step for obtaining an approximate impedance condition is to use Taylor expansion of σ_{+12}, σ_{+22} and h_{+2} in Ω^δ_+ with respect to δ, we write then the boundary conditions on Γ^δ_+ as follows:

$$\left\{\begin{array}{c} \sigma_{+12}(x_1,\delta) = \sigma_{+12}(x_1,0) + \delta\sigma_{+12,2}(x_1,0) + ... \\ \sigma_{+22}(x_1,\delta) = \sigma_{+22}(x_1,0) + \delta\sigma_{+22,2}(x_1,0) + ... \\ h_{+2}(x_1,\delta) = h_{+2}(x_1,0) + \delta h_{+2,2}(x_1,\delta) + ... \end{array}\right. \tag{8}$$

Using (8), (7) and the boundary conditions (4) on Γ^δ_+, we obtain at order one the following approximate impedance $T_{*\delta}$ defined by its expression:

$$T_{*\delta}\left(v^\delta,\phi^\delta\right) = \delta\left(C_1\left(v^\delta,\phi^\delta\right), C_2\left(v^\delta,\phi^\delta\right), C_3\left(v^\delta,\phi^\delta\right)\right),$$

and thus we get problem $\left(P^\delta_{-*}\right)$ in Ω_-.

3. Asymptotic expansion and approximate impedance operator

Here, we will built a first approximation of the impedance by the techniques of asymptotic expansion with scaling as follows:

3.1. **Scaling in the thin strip and the scaled transmission problem.** When the parameter δ varies, the domain Ω^δ_+ also varies. The solution $\left(u^\delta_+, \omega^\delta_+\right)$ of the problem $\left(P^\delta\right)$ depends on δ and we cannot compare the solutions corresponding to different values of the parameter δ. We are therefore going to make a change of scale to bring back the transmission problem set in Ω^δ to a transmission problem set on a fixed domain, let us symbolize it with Ω. So we perform a dilatation in the normal direction of Ω^δ_+ of ratio δ^{-1} to get a fixed geometry. Accordingly, we set $\Omega_+ = \mathbb{R}\times\,]0,1[$, $\Gamma_+ = \mathbb{R}\times\{1\}$, $\Omega = \mathbb{R}\times\,]-1,1[$ and for each point $(x_1,x_2)\in\Omega$, we associate the point $\chi^\delta(x_1,x_2)\in\Omega^\delta$ as the following:

$$\begin{array}{llll} \chi^\delta: & \Omega & \to & \Omega^\delta \\ & (x_1,x_2) & \mapsto & \chi^\delta(x_1,x_2) = \left\{\begin{array}{l} (x_1,x_2) \text{ if } x_2 \le 0 \\ (x_1,\delta x_2) \text{ if } x_2 > 0 \end{array}\right., \end{array}$$

and we define the function $\left(\widetilde{u}^\delta_\pm, \widetilde{\omega}^\delta_\pm\right)$ by:

$$\begin{array}{c} \widetilde{u}^\delta_-(x_1,x_2) = u^\delta_-(x_1,x_2), \text{ for all } (x_1,x_2)\in\Omega_-, \\ \widetilde{\omega}^\delta_-(x_1,x_2) = \omega^\delta_-(x_1,x_2), \text{ for all } (x_1,x_2)\in\Omega_-, \end{array}$$

and

$$\begin{array}{llll} \widetilde{u}^\delta_+: & \Omega_+ = \mathbb{R}\times\,]0,1[& \to & \mathbb{R}^2 \\ & (x_1,x_2) & \mapsto & \left(u^\delta_{+1}(x_1,\delta x_2), \delta u^\delta_{+2}(x_1,\delta x_2)\right), \\ \widetilde{\omega}^\delta_+: & \Omega_+ = \mathbb{R}\times\,]0,1[& \to & \mathbb{R} \\ & (x_1,x_2) & \mapsto & \omega^\delta_+(x_1,\delta x_2). \end{array}$$

Then we obtain the following scaled problem:

(1) The equations in Ω_+^δ are rewritten in Ω_+ as:

$$(E_{+1}):\ 0=\left[(2\mu_++\lambda_+)\,\widetilde{u}^\delta_{+1,11}+\frac{\lambda_+}{\delta^2}\widetilde{u}^\delta_{+2,12}+\beta_+\widetilde{\omega}^\delta_{+,1}\right]+\frac{\mu_+}{\delta^2}\left(\widetilde{u}^\delta_{+1,22}+\widetilde{u}^\delta_{+2,12}\right),$$

$$(E_{+2}):\ 0=\frac{\mu_+}{\delta}\left(\widetilde{u}^\delta_{+1,12}+\widetilde{u}^\delta_{+2,11}\right)+\frac{1}{\delta}\left[\frac{(2\mu_++\lambda_+)}{\delta^2}\widetilde{u}^\delta_{+2,22}+\lambda_+\widetilde{u}^\delta_{+1,12}+\beta_+\widetilde{\omega}^\delta_{+,2}\right],$$

$$(E_{+3}):\ 0=\alpha_+\widetilde{\omega}^\delta_{+,11}+\alpha_+\frac{1}{\delta^2}\widetilde{\omega}^\delta_{+,22}-\beta_+\left(\widetilde{u}^\delta_{+1,1}+\frac{1}{\delta^2}\widetilde{u}^\delta_{+2,2}\right)-\zeta_+\widetilde{\omega}^\delta_+.$$

(2) The boundary conditions on Γ_+^δ are rewritten in Γ_+ as:

$$\begin{aligned}
(BC\Gamma_{+1})\quad &:\quad 0=\mu_+\frac{1}{\delta}\widetilde{u}^\delta_{+1,2}+\mu_+\frac{1}{\delta}\widetilde{u}^\delta_{+2,1},\\
(BC\Gamma_{+2})\quad &:\quad 0=\frac{(2\mu_++\lambda_+)}{\delta^2}\widetilde{u}^\delta_{+2,2}+\lambda_+\widetilde{u}^\delta_{+1,1}+\beta_+\widetilde{\omega}^\delta_+,\\
(BC\Gamma_{+3})\quad &:\quad 0=\frac{1}{\delta^2}h_{+2}\left(\widetilde{\omega}^\delta_+\right).
\end{aligned}$$

(3) The transmission conditions on Σ are rewritten as:

$$\begin{aligned}
(CT\Sigma_1)\quad &:\quad \widetilde{u}^\delta_{+1}=\widetilde{u}^\delta_{-1},\\
(CT\Sigma_2)\quad &:\quad \frac{1}{\delta}\widetilde{u}^\delta_{+2}=\widetilde{u}^\delta_{-2},\\
(CT\Sigma_3)\quad &:\quad \widetilde{\omega}^\delta_+=\widetilde{\omega}^\delta_-,
\end{aligned}$$

$$(CT\Sigma_4):\ \sigma_{-12}\left(\widetilde{u}^\delta_-,\widetilde{\omega}^\delta_-\right)=\frac{\mu_+}{\delta}\left(\widetilde{u}^\delta_{+1,2}+\widetilde{u}^\delta_{+2,1}\right),$$

$$(CT\Sigma_5):\ \sigma_{-22}\left(\widetilde{u}^\delta_-,\widetilde{\omega}^\delta_-\right)=\frac{(2\mu_++\lambda_+)}{\delta^2}\widetilde{u}^\delta_{+2,2}+\lambda_+\widetilde{u}^\delta_{+1,1}+\beta_+\widetilde{\omega}^\delta_+,$$

$$(CT\Sigma_6):\ h_{-2}\left(\widetilde{\omega}^\delta_-\right)=\frac{1}{\delta^2}h_{+2}\left(\widetilde{\omega}^\delta_+\right).$$

3.2. **Asymptotic expansion of the scaled transmission problem.** We seek the solution $\left(\widetilde{u}^\delta_\pm,\widetilde{\omega}^\delta_\pm\right)$ of the transmission problem $\left(P^\delta\right)$ after the scaling in the form

$$\begin{aligned}
\widetilde{u}^\delta_-=u^\delta_-=u^0_-+\delta u^1_-+\delta^2u^2_-+...,\ \widetilde{\omega}^\delta_-=\omega^\delta_-=\omega^0_-+\delta\omega^1_-+\delta^2\omega^2_-+...,\\
\widetilde{u}^\delta_+=\widetilde{u}^0_++\delta\widetilde{u}^1_++\delta^2\widetilde{u}^2_++...,\ \widetilde{\omega}^\delta_+=\widetilde{\omega}^0_++\delta\widetilde{\omega}^1_++\delta^2\widetilde{\omega}^2_++...,
\end{aligned}$$

with u^k_-, ω^k_-, $\widetilde{u}^k_+$ and $\widetilde{\omega}^k_+$ for all $k\in\mathbb{N}$ are independent of δ. Inserting these asymptotic expansions in the transmission problem after scaling and identifying the terms with the same power of δ, we obtain

$$\left\{\begin{array}{c}
(EE_{+1})_{k-2}:\ 0=(2\mu_++\lambda_+)\,\widetilde{u}^{k-2}_{+1,11}+\lambda_+\widetilde{u}^k_{+2,12}+\beta_+\widetilde{\omega}^{k-2}_{+,1}\\
+\mu_+\left(\widetilde{u}^k_{+1,22}+\widetilde{u}^k_{+2,12}\right)\ \text{in}\ \ \Omega_+,\\
(EBC\Gamma_{+1})_{k-1}:\ 0=\mu_+\widetilde{u}^k_{+1,2}+\mu_+\widetilde{u}^k_{+2,1}\ \ \text{on}\ \Gamma_+,\\
(ECT\Sigma_4)_{k-1}:\ \sigma_{-12}\left(\widetilde{u}^{k-1}_-,\widetilde{\omega}^{k-1}_-\right)=\mu_+\left(\widetilde{u}^k_{+1,2}+\widetilde{u}^k_{+2,1}\right)\ \ \text{on}\ \Sigma,\\
(ECT\Sigma_1)_k:\ \widetilde{u}^k_{+1}=\widetilde{u}^k_{-1},\ \ \text{on}\ \Sigma,
\end{array}\right. \tag{9}$$

$$\left\{\begin{array}{c}(EE_{+2})_{k-3}:\ 0=\mu_{+}\left(\widetilde{u}_{+1,12}^{k-2}+\widetilde{u}_{+2,11}^{k-2}\right)+(2\mu_{+}+\lambda_{+})\,\widetilde{u}_{+2,22}^{k}\\ +\lambda_{+}\widetilde{u}_{+1,12}^{k-2}+\beta_{+}\widetilde{\omega}_{+,2}^{k-2}\ \text{ in }\Omega_{+},\\ (EBC\Gamma_{+2})_{k-2}:\ 0=(2\mu_{+}+\lambda_{+})\,\widetilde{u}_{+2,2}^{k}+\lambda_{+}\widetilde{u}_{+1,1}^{k-2}+\beta_{+}\widetilde{\omega}_{+}^{k-2}\ \text{ on }\Gamma_{+},\\ (ECT\Sigma_{5})_{k-2}:\ \sigma_{-22}\left(\widetilde{u}_{-}^{k-2},\widetilde{\omega}_{-}^{k-2}\right)=(2\mu_{+}+\lambda_{+})\,\widetilde{u}_{+2,2}^{k}\\ +\lambda_{+}\widetilde{u}_{+1,1}^{k-2}+\beta_{+}\widetilde{\omega}_{+}^{k-2}\ \text{ on }\Sigma,\\ (ECT\Sigma_{2})_{k-1}:\widetilde{u}_{+2}^{k}=\widetilde{u}_{-2}^{k-1}\ \text{ on }\Sigma,\end{array}\right. \tag{10}$$

and

$$\left\{\begin{array}{c}(EE_{+3})_{k-2}:0=\alpha_{+}\widetilde{\omega}_{+,11}^{k-2}+\alpha_{+}\widetilde{\omega}_{+,22}^{k}-\beta_{+}\widetilde{u}_{+1,1}^{k-2}\\ -\beta_{+}\widetilde{u}_{+2,2}^{k}-\zeta_{+}\widetilde{\omega}_{+}^{k-2}\ \text{in }\Omega_{+},\\ (EBC\Gamma_{+3})_{k-1}:\ 0=h_{+2}\left(\widetilde{\omega}_{+}^{k}\right),\ \ \text{on }\Gamma_{+},\\ (ECT\Sigma_{6})_{k-1}:\ h_{-2}\left(\widetilde{\omega}_{-}^{k-1}\right)=h_{+2}\left(\widetilde{\omega}_{+}^{k}\right)\ \text{ on }\Sigma,\\ (ECT\Sigma_{3})_{k}:\ \widetilde{\omega}_{+}^{k}=\widetilde{\omega}_{-}^{k},\ \text{ on }\Sigma,\end{array}\right. \tag{11}$$

where we set

$$\widetilde{u}_{+1}^{-2}=\widetilde{u}_{+1}^{-1}=\widetilde{\omega}_{+}^{-2}=\widetilde{\omega}_{+}^{-1}=0\ \text{ in }\Omega_{+},\ \widetilde{u}_{+}^{-2}=\widetilde{u}_{+}^{-1}=0_{1\times 2}\ \text{ in }\Omega_{+},$$

$$\widetilde{u}_{+1}^{-2}=\widetilde{u}_{+1}^{-1}=\widetilde{\omega}_{+}^{-2}=\widetilde{\omega}_{+}^{-1}=0\ \text{ on }\Gamma_{+},\ \widetilde{u}_{+1}^{-2}=\widetilde{u}_{+1}^{-1}=\widetilde{u}_{-2}^{-1}=\widetilde{\omega}_{+}^{-2}=\widetilde{\omega}_{+}^{-1}=0\ \text{ on }\Sigma,$$

$$\sigma_{-12}\left(\widetilde{u}_{-}^{-1},\widetilde{\omega}_{-}^{-1}\right)=\sigma_{-22}\left(\widetilde{u}_{-}^{-2},\widetilde{\omega}_{-}^{-2}\right)=\sigma_{-22}\left(\widetilde{u}_{-}^{-1},\widetilde{\omega}_{-}^{-1}\right)=D_{2}\widetilde{\omega}_{-}^{-1}=0\ \text{ on }\Sigma.$$

Remark 3.1. Thanks to the technique of scaling in the thin strip, the terms of the asymptotic expansion of $\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)$ can be calculated explicitly by recurrence in function of terms of the asymptotic expansion of $\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)$. Equations in problems (9)−(11) are second order linear differential equations with respect to the variable x_2. For $k=0$, an integration by part in x_2 in problems (9)−(11), gives the following results:

$$\begin{aligned}\widetilde{u}_{2+}^{0}&=0\text{ in }\Omega_{+},\\ \widetilde{u}_{+1}^{0}&=u_{-1|\Sigma}^{0}\ \text{ in }\ \Omega_{+},\\ \widetilde{\omega}_{+}^{0}&=\omega_{-|\Sigma}^{0}\text{ in }\Omega_{+}.\end{aligned}$$

For $k=1$, we get

$$\begin{aligned}\widetilde{u}_{+2}^{1}&=u_{-2|\Sigma}^{0}\ \text{ in }\Omega_{+},\\ \widetilde{u}_{+1}^{1}&=u_{-1|\Sigma}^{1}-x_{2}u_{-2,1|\Sigma}^{0}\ \text{ in }\Omega_{+},\\ \widetilde{\omega}_{+}^{1}&=\omega_{-|\Sigma}^{1}\ \text{ in }\Omega_{+}.\end{aligned}$$

For $k=2$, we get

$$\widetilde{u}_{+2}^{2}=u_{-2|\Sigma}^{1}-\frac{x_{2}}{(2\mu_{+}+\lambda_{+})}\left(\lambda_{+}u_{-1,1|\Sigma}^{0}+\beta_{+}\omega_{-|\Sigma}^{0}\right)\ \text{ in }\Omega_{+},$$

$$\begin{aligned}\widetilde{u}_{+1}^{2}&=u_{-1|\Sigma}^{2}+x_{2}\left[\frac{4\left(\lambda_{+}+\mu_{+}\right)}{(2\mu_{+}+\lambda_{+})}u_{-1,11|\Sigma}^{0}+\frac{2\beta_{+}}{(2\mu_{+}+\lambda_{+})}\omega_{-,1|\Sigma}^{0}-u_{-2,1|\Sigma}^{1}\right]\\ &-\frac{x_{2}^{2}}{2}\left[\frac{(3\lambda_{+}+4\mu_{+})}{(2\mu_{+}+\lambda_{+})}u_{-1,11|\Sigma}^{0}+\frac{\beta_{+}}{(2\mu_{+}+\lambda_{+})}\omega_{-,1|\Sigma}^{0}\right]\ \text{ in }\ \Omega_{+},\end{aligned}$$

$$\widetilde{\omega}_{+}^{2}=\omega_{-|\Sigma}^{2}+\left(\frac{x_{2}^{2}}{2\alpha_{+}}-\frac{x_{2}}{\alpha_{+}}\right)\left[\begin{array}{c}-\alpha_{+}\omega_{-,11|\Sigma}^{0}+\frac{2\mu_{+}\beta_{+}}{(2\mu_{+}+\lambda_{+})}u_{-1,1|\Sigma}^{0}\\ +\frac{\zeta_{+}(2\mu_{+}+\lambda_{+})-\beta_{+}^{2}}{(2\mu_{+}+\lambda_{+})}\omega_{-|\Sigma}^{0}\end{array}\right]\ \text{in }\Omega_{+}.$$

For $k=3$, we get

$$\sigma_{-22}\left(u_{-}^{1},\omega_{-}^{1}\right)_{|\Sigma}=\left[(2\mu_{+}+\lambda_{+})\,\widetilde{u}_{+2,2}^{3}+\lambda_{+}\widetilde{u}_{+1,1}^{1}+\beta_{+}\widetilde{\omega}_{+}^{1}\right]_{|\Sigma}=0.$$

Then, the asymptotic expansion of the equations in Ω_-, the boundary conditions on Γ_- and the transmission conditions $(TC\Sigma_4)-(TC\Sigma_6)$ on Σ, allow us to obtain the following results:

Terms of order 0: At order 0, the terms $\left(u_-^0,\omega_-^0\right)$ satisfy the following boundary value problem in Ω_-**:**

$$(P_-)_0:\begin{cases}\sigma_{-ij,j}\left(u_-^0,\omega_-^0\right)=p_{-i},\ i,j=1,2,\\ h_{-i,i}\left(\omega_-^0\right)-g_-\left(u_-^0,\omega_-^0\right)=q_-,\ i=1,2,\\ u_-^0=0 \text{ on } \Gamma_-,\ \omega_-^0=0 \text{ on } \Gamma_-,\\ \left(\sigma_-\left(u_-^0,\omega_-^0\right)\nu_{|\Sigma},h_{-2}\left(\omega_-^0\right)_{|\Sigma}\right)=0 \text{ on } \Sigma,\end{cases}$$

which means that at order 0, the thin strip Ω_+^δ has no effect on Ω_-.

Terms of order 1: At order 1, the terms $\left(u_-^1,\omega_-^1\right)$ satisfy the following boundary value problem in Ω_-**:**

$$(P_-)_1:\begin{cases}\sigma_{-ij,j}\left(u_-^1,\omega_-^1\right)=0,\ i,j=1,2 \text{ in } \Omega_-,\\ h_{-i,i}\left(\omega_-^1\right)-g_-\left(u_-^1,\omega_-^1\right)=0,\ i=1,2 \text{ in } \Omega_-,\\ u_-^1=0 \text{ on } \Gamma_-,\ \omega_-^1=0 \text{ on } \Gamma_-,\\ \left(\sigma_-\left(u_-^1,\omega_-^1\right)\nu_{|\Sigma},h_{-2}\left(\omega_-^1\right)_{|\Sigma}\right)=T_*\left(u_-^0,\omega_-^0\right) \text{ on } \Sigma.\end{cases}$$

which implies that at order 1, the effect of the thin strip Ω_+^δ on Ω_- is represented by forces and stresses exerted on Σ.

First order approximation of the impedance. The function $\left(u_{-*}^\delta,\omega_{-*}^\delta\right)$ defined by: $\left(u_{-*}^\delta,\omega_{-*}^\delta\right)=\left(u_-^0+\delta u_-^1,\omega_-^0+\delta\omega_-^1\right)$ satisfies

$$\left(\sigma_-\left(u_{-*}^\delta,\omega_{-*}^\delta\right)\nu_{|\Sigma},h_{-2}\left(\omega_{-*}^\delta\right)_{|\Sigma}\right)=\delta T_*\left(\left(u_{-*}^\delta,\omega_{-*}^\delta\right)_{|\Sigma}\right)-\delta^2T_*\left(\left(u_-^1,\omega_-^1\right)_{|\Sigma}\right).$$

Thus at order 1, with respect to δ, we recover the same approximate impedance which is obtained by Taylor expansion in section 2.

4. Stability result for the transmission problem

4.1. **Weak formulation of the scaled transmission problem.** The canonical space for the study of the transmission problem in the fixed domain $\Omega_-\cup\Sigma\cup\Omega_+$ is:

$$W_\delta=\left\{\begin{array}{c}(v_\pm,\varphi_\pm)\in\mathbb{H}^1(\Omega_\pm)=\left[H^1(\Omega_\pm)\right]^3:\\ v_-=0 \text{ on } \Gamma_-,\ \varphi_-=0 \text{ on } \Gamma_-,\\ v_{-1}=v_{+1} \text{ on } \Sigma,\ \delta v_{-2}=v_{+2} \text{ on } \Sigma,\ \varphi_-=\varphi_+ \text{ on } \Sigma,\end{array}\right\}$$

and its weak formulation is written:

$$\begin{aligned}\text{Find }\left(\widetilde{u}_\pm^\delta,\widetilde{\omega}_\pm^\delta\right)&\in W_\delta,\text{ such that }\forall(v_\pm,\varphi_\pm)\in W_\delta:\\ L_\delta(v_\pm,\varphi_\pm)&=a^-\left[\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right),(v_-,\varphi_-)\right]+a^+\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),(v_+,\varphi_+)\right],\end{aligned}\qquad(12)$$

with

$$a^-\left[\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right),(v_-,\varphi_-)\right]=\int_{\Omega_-}\left[\begin{array}{c}\operatorname{tr}_2\left(\sigma_-\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right)e_-(v_-,\varphi_-)\right)\\ +\alpha_-\nabla\widetilde{\omega}_-^\delta\nabla\varphi_-+\zeta_-\widetilde{\omega}_-^\delta\varphi_-\\ +\beta_-\left(\widetilde{u}_{-1,1}^\delta+\widetilde{u}_{-2,2}^\delta\right)\varphi_-\end{array}\right]d\Omega_-,$$

and

$$\begin{aligned}a^+\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),(v_+,\varphi_+)\right]=\delta a_{+1}^+\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),(v_+,\varphi_+)\right]+\tfrac{1}{\delta}a_{-1}^+\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),(v_+,\varphi_+)\right]\\ +\tfrac{1}{\delta^3}a_{-3}^+\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),(v_+,\varphi_+)\right]\ ,\end{aligned}$$

where

$$a_{+1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right]=\int_{\Omega_{+}}\left[\begin{array}{c}(2\mu_{+}+\lambda_{+})\,\widetilde{u}_{+1,1}^{\delta}v_{+1,1}+\alpha_{+}\widetilde{\omega}_{+,1}^{\delta}\varphi_{+,1}\\ +\zeta_{+}\widetilde{\omega}_{+}^{\delta}\varphi_{+}+\beta_{+}\left(\widetilde{u}_{+1,1}^{\delta}\varphi_{+}+\widetilde{\omega}_{+}^{\delta}v_{+1,1}\right)\end{array}\right]d\Omega,$$

$$a_{-1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right]=\int_{\Omega_{+}}\left[\begin{array}{c}\lambda_{+}\left(\widetilde{u}_{+2,2}^{\delta}v_{+1,1}+\widetilde{u}_{+1,1}^{\delta}v_{+2,2}\right)\\ +\beta_{+}\left(\widetilde{u}_{+2,2}^{\delta}\varphi_{+}+\widetilde{\omega}_{+}^{\delta}v_{+2,2}\right)\\ +\mu_{+}\left(\widetilde{u}_{+2,1}^{\delta}+\widetilde{u}_{+1,2}^{\delta}\right)(v_{+2,1}+v_{+1,2})\\ +\alpha_{+}\widetilde{\omega}_{+,2}^{\delta}\varphi_{+,2}\end{array}\right]d\Omega_{+},$$

$$a_{-3}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right]=\int_{\Omega_{+}}(2\mu_{+}+\lambda_{+})\,\widetilde{u}_{+2,2}^{\delta}v_{+2,2}d\Omega_{+},$$

and

$$L_{\delta}(v_{\pm},\varphi_{\pm})=\int_{\Omega_{-}}(p_{-}.v_{-}+q_{-}\varphi_{-})\,d\Omega_{-}.$$

4.2. **Stability result for the transmission problem.** Here, we state and prove the following stability result:

Theorem 4.1. Let L_δ be a continuous linear form on W_δ such that

$$|L_{\delta}(v_{\pm},\varphi_{\pm})|\leq l_{\delta}\left[\mathbb{A}(v_{-},\varphi_{-})+\sqrt{\delta}\mathbb{B}(v_{+},\varphi_{+})+\frac{1}{\sqrt{\delta}}\mathbb{C}(v_{+},\varphi_{+})+\frac{1}{\delta\sqrt{\delta}}\mathbb{D}(v_{+},\varphi_{+})\right],$$

where l_δ is any function of $\delta>0$ being possibly not bounded as δ goes to 0, and

$$\begin{array}{rcl}\mathbb{A}(v_{-},\varphi_{-}) & = & \|v_{-}\|_{(H^{1}(\Omega_{-}))^{2}}+\|\varphi_{-}\|_{H^{1}(\Omega_{-})},\\ \mathbb{B}(v_{+},\varphi_{+}) & = & \|v_{+1,1}\|_{L^{2}(\Omega_{+})}+\|\varphi_{+,1}\|_{L^{2}(\Omega_{+})}+\|\varphi_{+}\|_{L^{2}(\Omega_{+})},\\ \mathbb{C}(v_{+},\varphi_{+}) & = & \|v_{+2,1}+v_{+1,2}\|_{L^{2}(\Omega_{+})}+\|\varphi_{+,2}\|_{L^{2}(\Omega_{+})},\\ \mathbb{D}(v_{+},\varphi_{+}) & = & \|v_{+2,2}\|_{L^{2}(\Omega_{+})}.\end{array}$$

Then there exists a constant $C>0$ (not depending on δ) such that the solution $\left(\widetilde{u}_{\pm}^{\delta},\widetilde{\omega}_{\pm}^{\delta}\right)$ of the problem

$$\begin{array}{rcl}\text{Find }\left(\widetilde{u}_{\pm}^{\delta},\widetilde{\omega}_{\pm}^{\delta}\right) & \in & W_{\delta},\text{ such that }\forall(v_{\pm},\varphi_{\pm})\in W_{\delta}:\\ L_{\delta}(v_{\pm},\varphi_{\pm}) & = & a^{-}\left[\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right),(v_{-},\varphi_{-})\right]+\ a^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right],\end{array}\quad(13)$$

satisfies the estimates

$$\mathbb{A}\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)\leq Cl_{\delta},\quad(14)$$

$$\mathbb{B}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\leq C\delta^{-\frac{1}{2}}l_{\delta},\quad(15)$$

$$\mathbb{C}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\leq C\delta^{\frac{1}{2}}l_{\delta},\quad(16)$$

$$\mathbb{D}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\leq C\delta^{\frac{3}{2}}l_{\delta}.\quad(17)$$

Proof. The expression $\mathbb{A}^2\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)$ is equivalent to $a^{-}\left[\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right),\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)\right]$, and we have

$$a_{-1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\right]=\int_{\Omega_{+}}\left[\begin{array}{c}2\lambda_{+}\widetilde{u}_{+2,2}^{\delta}\widetilde{u}_{+1,1}^{\delta}+2\beta_{+}\widetilde{u}_{+2,2}^{\delta}\widetilde{\omega}_{+}^{\delta}\\ +\mu_{+}\left(\widetilde{u}_{+2,1}^{\delta}+\widetilde{u}_{+1,2}^{\delta}\right)^{2}+\alpha_{+}\left(\widetilde{\omega}_{+,2}^{\delta}\right)^{2}\end{array}\right]d\Omega_{+},$$

from the expressions of $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$, we get

$$a_{-1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\right]\leq(2\lambda_{+}+2\beta_{+})\,\mathbb{B}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\mathbb{D}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)+(\mu_{+}+\alpha_{+})\,\mathbb{C}^{2}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right).$$

As we have

$$(2\lambda_+ + 2\beta_+)\frac{1}{\delta}(\mathbb{B}.\mathbb{D})\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right) \leq (\lambda_+ + \beta_+)\left[\delta\mathbb{B}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right) + \frac{1}{\delta^3}\mathbb{D}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\right],$$

then by taking $(v_\pm,\varphi_\pm) = \left(\widetilde{u}_\pm^\delta,\widetilde{\omega}_\pm^\delta\right)$ in the weak formulation (13), we get the following estimate:

$$\left[\begin{array}{c}\mathbb{A}^2\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right) + \delta\mathbb{B}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\\ +\frac{1}{\delta}\mathbb{C}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right) + \frac{1}{\delta^3}\mathbb{D}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\end{array}\right] \leq Cl_\delta\left[\begin{array}{c}\mathbb{A}\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right) + \sqrt{\delta}\mathbb{B}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\\ +\frac{1}{\sqrt{\delta}}\mathbb{C}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right) + \frac{1}{\delta\sqrt{\delta}}\mathbb{D}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\end{array}\right],$$

where $C > 0$ is a positive constant independent of δ. This leads to the estimates (14)-(17).

4.3. **Application of theorem 4.1.** Setting

$$u_-^{(\delta,1)} = u_-^0 + \delta u_-^1,\quad u_+^{(\delta,1)} = \widetilde{u}_+^0 + \delta\widetilde{u}_+^1 + \delta^2\left(0,\widetilde{u}_{+2}^2\right),$$
$$\omega_-^{(\delta,1)} = \omega_-^0 + \delta\omega_-^1,\quad \omega_+^{(\delta,1)} = \widetilde{\omega}_+^0 + \delta\widetilde{\omega}_+^1,$$

and making use of roblems at zeroth order $(P_-)_0$ and at first order $(P_-)_1$ satisfied by $\left(u_-^0,\omega_-^0\right)$ and $\left(u_-^1,\omega_-^1\right)$ respectively, and also the problems $(9)-(11)$ for $k \in \{0,1,2,3\}$, we get

$$\begin{aligned} L_\delta(v_\pm,\varphi_\pm) \;=\;& a_{-3}^+\left[\left(\widetilde{u}_+^3,\widetilde{\omega}_+^3\right),(v_+,\varphi_+)\right] + \delta\int_{\Omega_+}\mu_+\left[D_2\widetilde{u}_{+1}^2 + D_1\widetilde{u}_{+2}^2\right]D_2v_{+1}d\Omega_+\\ &+\delta\int_{\Omega_+}\alpha_+D_2\widetilde{\omega}_+^2D_2\varphi_+d\Omega_+ - \delta\int_{\Omega_+}\mu_+D_1\widetilde{u}_{+2}^2\left(D_1v_{+2} + D_2v_{+1}\right)d\Omega_+\\ &-\delta^2a_{+1}^+\left[\left(\widetilde{u}_+^1,\widetilde{\omega}_+^1\right),(v_+,\varphi_+)\right],\end{aligned}$$

where

$$\begin{aligned} L_\delta(v_\pm,\varphi_\pm) \;=\;& a^-\left[\left(\widetilde{u}_-^\delta - u_-^{(\delta,1)},\widetilde{\omega}_-^\delta - \omega_-^{(\delta,1)}\right),(v_-,\varphi_-)\right]\\ &+ a^+\left[\left(\widetilde{u}_+^\delta - u_+^{(\delta,1)},\widetilde{\omega}_+^\delta - \omega_+^{(\delta,1)}\right),(v_+,\varphi_+)\right],\end{aligned}$$

and thus

$$|L_\delta(v_\pm,\varphi_\pm)| \leq C\delta^{3/2}\left[\mathbb{A}(v_-,\varphi_-) + \sqrt{\delta}\mathbb{B}(v_+,\varphi_+) + \frac{1}{\sqrt{\delta}}\mathbb{C}(v_+,\varphi_+) + \frac{1}{\delta\sqrt{\delta}}\mathbb{D}(v_+,\varphi_+)\right],$$

which implies, by the stability result (see Theorem 4.1), the following error estimate:

$$\left\|u_-^\delta - u_-^{(\delta,1)}\right\|_{(H^1(\Omega_-))^2} + \left\|\omega_-^\delta - \omega_-^{(\delta,1)}\right\|_{H^1(\Omega_-)} \leq C\delta^{3/2}, \tag{18}$$

where C is a positive constant independent of δ.

Remark 4.1. We took

$$u_{+2}^{(\delta,1)} = \widetilde{u}_{+2}^0 + \delta\widetilde{u}_{+2}^1 + \delta^2\widetilde{u}_{+2}^2,$$

in order to satisfy the transmission condition

$$u_{+2}^{(\delta,1)} = \delta u_{-2}^{(\delta,1)} \text{ on } \Sigma,$$

and thus we get $\left(u_\pm^{(\delta,1)},\omega_\pm^{(\delta,1)}\right)$ in the space W_δ.

Remark 4.2. If p_- and q_- are smooth enough, we can obtain all the terms of the asymptotic expansion $\left(u_-^k,\omega_-^k\right)$ for $k \geq 2$, and then we can prove an analogous error estimate. For instance we can prove

$$\left\|u_-^\delta - u_-^{(\delta,2)}\right\|_{(H^1(\Omega_-))^2} + \left\|\omega_-^\delta - \omega_-^{(\delta,2)}\right\|_{H^1(\Omega_-)} \leq C\delta^{5/2}, \tag{19}$$

where

$$u_{-}^{(\delta,2)}=u_{-}^{0}+\delta u_{-}^{1}+\delta^{2}u_{-}^{2}\quad\text{and}\quad\omega_{-}^{(\delta,2)}=\omega_{-}^{0}+\delta\omega_{-}^{1}+\delta^{2}\omega_{-}^{2}.$$

5. Stability result for the approximate impedance problem

5.1. **Weak formulation of the impedance problem.** The canonical space for the study of the problem $\left(P_{-*}^{\delta}\right)$ is:

$$W_{*}=\left\{\begin{array}{c}(v_{-},\varphi_{-})\in\mathbb{H}^{1}(\Omega_{-})=\left[H^{1}(\Omega_{-})\right]^{3}:\\(v_{-1,1},v_{-2,1},\varphi_{-,1})\in\mathbb{L}^{2}(\Sigma)=\left[L^{2}(\Sigma)\right]^{3}\\v_{-}=\varphi_{-}=0\ \text{ on }\ \Gamma_{-}.\end{array}\right\}$$

Let

$$\begin{aligned}&a^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),(v_{-},\varphi_{-})\right]\\=&\int_{\Omega_{-}}\left[\begin{array}{c}\operatorname{tr}_{2}\left(\sigma_{-}\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)e_{-}(v_{-},\varphi_{-})\right)+\alpha_{-}\nabla\omega_{-*}^{\delta}\nabla\varphi_{-}\\+\zeta_{-}\omega_{-*}^{\delta}\varphi_{-}+\beta_{-}\left(u_{-*1,1}^{\delta}+u_{-*2,2}^{\delta}\right)\varphi_{-}\end{array}\right]d\Omega_{-},\end{aligned}$$

$$\begin{aligned}&a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),(v_{-},\varphi_{-})\right]\\=&\int_{\Sigma}\left[\begin{array}{c}\frac{4\mu_{+}(\mu_{+}+\lambda_{+})}{(\lambda_{+}+2\mu_{+})}u_{-*1,1}^{\delta}v_{-1,1}+\alpha_{+}\omega_{-*,1}^{\delta}\varphi_{-,1}\\+\frac{\zeta_{+}(2\mu_{+}+\lambda_{+})-\beta_{+}^{2}}{2\mu_{+}+\lambda_{+}}\omega_{-*}^{\delta}\varphi_{-}+\frac{2\mu_{+}\beta_{+}}{2\mu_{+}+\lambda_{+}}\left(\omega_{-*}^{\delta}v_{-1,1}+\varphi_{-}u_{-*1,1}^{\delta}\right)\end{array}\right]d\Sigma.\end{aligned}$$

The approximate impedance problem admits the following weak formulation

$$\left\{\begin{array}{c}\text{Find }\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\in W_{*}\text{ such that }\ \forall(v_{-},\varphi_{-})\in W_{*}:\\a^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),(v_{-},\varphi_{-})\right]+\ \delta a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),(v_{-},\varphi_{-})\right]\\=\int_{\Omega_{-}}(p_{-}.v_{-}+q_{-}\varphi_{-})\,d\Omega_{-}.\end{array}\right.\tag{20}$$

5.2. **Stability result for the approximate impedance problem.** The existence and uniqueness of the solution of $\left(P_{-*}^{\delta}\right)$ in the space W_{*} is easily obtained and we have the following stability result.

Theorem 5.1. Let L_{δ} be a continuous linear form on W_{*} satisfying the following bound in δ:

$$|L_{\delta}(v_{-},\varphi_{-})|\leq l_{\delta}\left[\mathbb{A}(v_{-},\varphi_{-})+\sqrt{\delta}\,\mathbb{B}(v_{-},\varphi_{-})\right],$$

where l_{δ} is any function of $\delta>0$ possibly not bounded as δ goes to 0, and

$$\begin{aligned}\mathbb{A}(v_{-},\varphi_{-})&=\|v_{-}\|_{(H^{1}(\Omega_{-}))^{2}}+\|\varphi_{-}\|_{H^{1}(\Omega_{-})},\\\mathbb{B}(v_{-},\varphi_{-})&=\|v_{-1,1}\|_{L^{2}(\Sigma)}+\|\varphi_{-,1}\|_{L^{2}(\Sigma)}+\|\varphi_{-}\|_{L^{2}(\Sigma)}.\end{aligned}$$

Then there exists $C>0$ (not depending on δ) such that the solution $\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ of the problem

$$\left\{\begin{array}{l}\text{Find }\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\in W_{*},\text{ such that for all }\ (v_{-},\varphi_{-})\in W_{*}:\\a^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),(v_{-},\varphi_{-})\right]+\ \delta a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),(v_{-},\varphi_{-})\right]=L_{\delta}(v_{-},\varphi_{-}),\end{array}\right.\tag{21}$$

satisfies the following estimates

$$\mathbb{A}\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\ \leq\ Cl_{\delta},\tag{22}$$

$$\mathbb{B}\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\ \leq\ C\,\delta^{-1/2}l_{\delta}.\tag{23}$$

Proof. Using Gauss decomposition, we can show that

$$\forall\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\in W_{*}:\ a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]\geq0,$$

and by Korn inequality (see [5], [31]), the expression $A^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ is equivalent to $a^-\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]$, and we have

$$a_\Sigma\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]\geq C\ \mathbb{B}^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),,$$

where $C>0$ is a constant depending only on the elasticity coefficients. Then by taking $(v_-,\varphi_-)=\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ in the weak formulation (21), we get with some positive constant $K>0$ not depending on δ, the estimate

$$\mathbb{A}^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)+\delta\ \mathbb{B}^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\leq Kl_\delta\left[\mathbb{A}\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)+\sqrt{\delta}\mathbb{B}\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right],$$

which in turn gives directly the estimates (22) and (23).

5.3. **Asymptotic expansion for the approximate impedance problem.** Setting

$$\begin{aligned} u_{-*}^{\delta} &= u_{-*}^{0}+\delta u_{-*}^{1}+\delta^2 u_{-*}^{2}+...,\\ \omega_{-*}^{\delta} &= \omega_{-*}^{0}+\delta \omega_{-*}^{1}+\delta^2 \omega_{-*}^{2}+..., \end{aligned}$$

and plugging the expansion in the approximate impedance problem $\left(P_{-*}^{\delta}\right)$, we get a hierarchy of equations and boundary conditions. At order 0, we have

$$(P_{-*})_0:\left\{\begin{array}{l}\sigma_{-ij,j}\left(u_{-*}^{0},\omega_{-*}^{0}\right)=-p_{-i},\ i,j=1,2\text{ in }\Omega_-,\\ h_{-i,i}\left(\omega_{-*}^{0}\right)-g_-\left(u_{-*}^{0},\omega_{-*}^{0}\right)=-q_-,\ i=1,2\text{ in }\Omega_-,\\ u_{-*}^{0}=0\text{ on }\Gamma_-,\ \omega_{-*}^{0}=0\text{ on }\Gamma_-,\\ \left(\sigma_-\left(u_{-*}^{0},\omega_{-*}^{0}\right)\nu_{|\Sigma},h_{-2}\left(\omega_{-*}^{0}\right)_{|\Sigma}\right)=0\text{ on }\Sigma.\end{array}\right.$$

At order 1, we have

$$(P_{-*})_1:\left\{\begin{array}{l}\sigma_{-ij,j}\left(u_{-*}^{1},\omega_{-*}^{1}\right)=0,\ i,j=1,2\text{ in }\Omega_-,\\ h_{-i,i}\left(\omega_{-*}^{1}\right)-g_-\left(u_{-*}^{1},\omega_{-*}^{1}\right)=0,\ i=1,2\text{ in }\Omega_-,\\ u_{-*}^{1}=0\text{ on }\Gamma_-,\ \omega_{-*}^{1}=0\text{ on }\Gamma_-,\\ \left(\sigma_-\left(u_{-*}^{1},\omega_{-*}^{1}\right)\nu_{|\Sigma},h_{-2}\left(\omega_{-*}^{1}\right)_{|\Sigma}\right)=T_*\left(u_{-*}^{0},\omega_{-*}^{0}\right)\ \text{ on }\Sigma.\end{array}\right.$$

5.4. **Application of theorem 5.1.** Setting

$$\begin{aligned} u_{-*}^{(\delta,1)} &= u_{-*}^{0}+\delta u_{-*}^{1},\\ \omega_{-*}^{(\delta,1)} &= \omega_{-*}^{0}+\delta \omega_{-*}^{1}, \end{aligned}$$

$$\begin{aligned} L_\delta\left(v_-,\varphi_-\right) &= a^-\left[\left(u_{-*}^{\delta}-u_{-*}^{(\delta,1)},\omega_{-*}-\omega_{-*}^{(\delta,1)}\right),\left(v_-,\varphi_-\right)\right]\\ &+\ \delta a_\Sigma\left[\left(u_{-*}^{\delta}-u_{-*}^{(\delta,1)},\omega_{-*}-\omega_{-*}^{(\delta,1)}\right),\left(v_-,\varphi_-\right)\right], \end{aligned}$$

and making use of the problems at zeroth order $(P_{-*})_0$ and at first order $(P_{-*})_1$ satisfied by $(u_{-*}^{0},\omega_{-*}^{0})$ and $(u_{-*}^{1},\omega_{-*}^{1})$, respectively, we get

$$L_\delta\left(v_-,\varphi_-\right)=-\delta^2 a_\Sigma\left[\left(u_{-*}^{1},\omega_{-*}^{1}\right),\left(v_-,\varphi_-\right)\right],$$

and thus

$$\left|L_\delta\left(v_-,\varphi_-\right)\right|\leq C\delta^{3/2}\left[\mathbb{A}\left(v_-,\varphi_-\right)+\sqrt{\delta}\ \mathbb{B}\left(v_-,\varphi_-\right)\right],$$

which implies, by the stability result (see Theorem 5.1), the following error estimate:

$$\left\|u_{-*}^{\delta}-u_{-*}^{(\delta,1)}\right\|_{[H^1(\Omega_-)]^2}+\left\|\omega_{-*}^{\delta}-\omega_{-*}^{(\delta,1)}\right\|_{H^1(\Omega_-)}\leq C\delta^{3/2}, \tag{24}$$

where C is a positive constant independent of δ.

Remark 5.1. If p_- and q_- are smooth enough, we can obtain all the terms of the asymptotic expansion $\left(u^k_{-*}, \omega^k_{-*}\right)$ for $k \geq 2$, and prove an analogous error estimate. For instance we can prove

$$\left\| u^\delta_{-*} - u^{(\delta,2)}_{-*} \right\|_{(H^1(\Omega_-))^3} + \left\| \omega^\delta_{-*} - \omega^{(\delta,2)}_{-*} \right\|_{H^1(\Omega_-)} \leq C\delta^{5/2}, \tag{25}$$

where

$$u^{(\delta,2)}_{-*} = u^0_{-*} + \delta u^1_{-*} + \delta^2 u^2_{-*} \text{ and } \omega^{(\delta,2)}_{-*} = \omega^0_{-*} + \delta\omega^1_{-*} + \delta^2\omega^2_{-*}.$$

Remark 5.2. The terms $\left(u^0_{-*}, \omega^0_{-*}\right)$ and $\left(u^0_-, \omega^0_-\right)$ (respectively $\left(u^1_{-*}, \omega^1_{-*}\right)$ and $\left(u^1_-, \omega^1_-\right)$) of the expansion of $\left(u^\delta_{-*}, \omega^\delta_{-*}\right)$ and $\left(u^\delta_-, \omega^\delta_-\right)$ are solutions of the same boundary value problem at zeroth order (respectively at first order). Then by uniqueness we have

$$\left(u^0_{-*}, \omega^0_{-*}\right) = \left(u^0_-, \omega^0_-\right) \text{ and } \left(u^1_{-*}, \omega^1_{-*}\right) = \left(u^1_-, \omega^1_-\right).$$

6. Final error estimate between the solution of $\left(P^\delta\right)$ and the solution of $\left(P^\delta_{-*}\right)$

As we have $(\left(u^0_{-*}, \omega^0_{-*}\right) = \left(u^0_-, \omega^0_-\right)$ and $\left(u^1_{-*}, \omega^1_{-*}\right) = \left(u^1_-, \omega^1_-\right)$ (see Remark 5.2) then by triangular inequality, we can write

$$\begin{aligned} & \left\| \widetilde{u}^\delta_- - u^\delta_{-*} \right\|_{(H^1(\Omega_-))^2} + \left\| \widetilde{\omega}^\delta_- - \omega^\delta_{-*} \right\|_{H^1(\Omega_-)} \\ \leq & \left\| \widetilde{u}^\delta_- - u^0_- - \delta u^1_- \right\|_{(H^1(\Omega_-))^2} + \left\| u^\delta_{-*} - u^0_{-*} - \delta u^1_{-*} \right\|_{(H^1(\Omega_-))^2} \\ & + \left\| \widetilde{\omega}^\delta_- - \omega^0_- - \delta\omega^1_- \right\|_{H^1(\Omega_-)} + \left\| \omega^\delta_{-*} - \omega^0_{-*} - \delta\omega^1_{-*} \right\|_{H^1(\Omega_-)}, \end{aligned}$$

and in virtue of (18) and (24), we find

$$\left\| \widetilde{u}^\delta_- - u^\delta_{-*} \right\|_{(H^1(\Omega_-))^2} + \left\| \widetilde{\omega}^\delta_- - \omega^\delta_{-*} \right\|_{H^1(\Omega_-)} \leq C\delta^{3/2},$$

where the constant C depends only on p_-, q_- and the elasticity coefficients. Actually, if the data p_- and q_- are smooth enough to allow the determination of $\left(u^2_-, \omega^2_-\right)$ and $\left(u^2_{-*}, \omega^2_{-*}\right)$, then in virtue of (19), the last error estimate (which is not optimal) may be improved as follows:

$$\begin{aligned} \left\| \widetilde{u}^\delta_- - u^0_- - \delta u^1_- \right\|_{(H^1(\Omega_-))^2} & \leq \left\| \widetilde{u}^\delta_- - u^0_- - \delta u^1_- - \delta^2 u^2_- \right\|_{(H^1(\Omega_-))^2} + \delta^2 \left\| u^2_- \right\|_{(H^1(\Omega_-))^2} \\ & \leq C_1\delta^{5/2} + C_2\delta^2 \leq C\delta^2, \end{aligned}$$

and

$$\begin{aligned} \left\| \widetilde{\omega}^\delta_- - \omega^0_- - \delta\omega^1_- \right\|_{H^1(\Omega_-)} & \leq \left\| \widetilde{\omega}^\delta_- - \omega^0_- - \delta\omega^1_- - \delta^2\omega^2_- \right\|_{H^1(\Omega_-)} + \delta^2 \left\| \omega^2_- \right\|_{H^1(\Omega_-)} \\ & \leq C_1\delta^{5/2} + C_2\delta^2 \leq C\delta^2. \end{aligned}$$

Using (25), similar estimates for $\left(u^\delta_{-*} - u^0_{-*} - \delta u^1_{-*}, \omega^\delta_{-*} - \omega^0_{-*} - \delta\omega^1_{-*}\right)$ can be proved in the same manner.

7. Conclusion

In this paper, we have derived an approximate impedance for a planar porous thin layer by means of two different approaches: The first approach consists in using a Taylor expansion, while the second approach based on the asymptotic expansion with scaling. The resulting approximate impedance allowed us to replace the transmission model problem which set in $\Omega_- \cup \Omega_+^\delta$ by an approximate impedance boundary problem set only in the fixed domain Ω_-, and then, we have proved an error estimate between the solution of the transmission problem in $\Omega_-$and the solution of the approximate impedance problem in the norm of Sobolev space.

Declarations

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