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EXTENDED RESULTS OF COMMON FIXED POINTS BY USING $\alpha-\psi$-GERAGHTY CONTRACTION AND APPLICATIONS

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Abstract. By introducing an $\alpha-\psi$-Geraghty contraction for α_C-admissible pair of multi-valued operators, we get some new results of common fixed points in complex-valued double-controlled metric spaces. These new results improve and generalize some results mentioned in the literature.

Keywords: Common fixed point, multivalued mapping, double controlled metric space.

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1. Introduction

Azam et al. in [5], presented the notion of complex-valued metric spaces (CMS), and fixed point (FP) results for functions with some rational inequalities were established. Also, an essential class of FP theorems is $\alpha-\psi$–contraction mappings, introduced by Samet and Vetro, were extended to studies on multi-valued mappings by some authors, see for instance [2, 16].
CMS found exciting applications in many branches of mathematics such as algebraic geometry and number theory as well as in fields of study such as physics, thermodynamics, and electrical engineering. Also, the concept of b-metric was presented in 1989 by Bakhtin [8]. Based on this results, Rao et al. [9] introduced the idea of FP theorems on complex-valued b-metric spaces which is a natural extension of the CMS. In 2016, Singh et al. [17] presented a contractive type mapping satisfying some rational inequalities. They obtained the existence of a common fixed point (CFP) for a pair of single-valued mappings satisfying more general contraction conditions in the framework of CMS. By combining $\alpha-\psi$-Geraghty contraction with α_C-admissible pair of multi-valued operators, we get some new results of CFPs in complex-valued double-controlled metric spaces. We expand the results of [17, 20] from CMS to complex-valued b-metric spaces and also generalize the contraction mappings used therein by introducing a multivalued contraction mapping satisfying a general condition in complex-valued b-metric spaces. Our result really improves

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and strengthens the results of [1,3,4,7,10–13,15,18–20] and many other related results in the literature.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbf{C}$ be the set of real and complex numbers, respectively and $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathbf{C}$. A partial order $\preceq$ on $\mathbf{C}$ is defined as below:

$$\vartheta_1 \preceq \vartheta_2 \Longleftrightarrow \mathrm{Re}(\vartheta_1) \leq \mathrm{Re}(\vartheta_2),\ \mathrm{Im}(\vartheta_1) \leq \mathrm{Im}(\vartheta_2).$$

So $\vartheta_1 \preceq \vartheta_2$ if only one of the below four conditions holds:
(**a**) $\mathrm{Re}(\vartheta_1) = \mathrm{Re}(\vartheta_2)$, and $\mathrm{Im}(\vartheta_1) = \mathrm{Im}(\vartheta_2)$,
(**b**) $\mathrm{Re}(\vartheta_1) = \mathrm{Re}(\vartheta_2)$, and $\mathrm{Im}(\vartheta_1) < \mathrm{Im}(\vartheta_2)$,
(**c**) $\mathrm{Re}(\vartheta_1) < \mathrm{Re}(\vartheta_2)$, and $\mathrm{Im}(\vartheta_1) = \mathrm{Im}(\vartheta_2)$,
(**d**) $\mathrm{Re}(\vartheta_1) < \mathrm{Re}(\vartheta_2)$, and $\mathrm{Im}(\vartheta_1) < \mathrm{Im}(\vartheta_2)$.
We use the notation $\vartheta_1 \precnsim \vartheta_2$, if $\vartheta_1 \neq \vartheta_2$ and if one of the properties (**b**), (**c**) or (**d**) hold, then we have $\vartheta_1 \prec \vartheta_2$ if only (**a**) satisfied. If $\mu_1, \mu_2 \in \mathbb{R}$, $0 \leq \mu_1 \leq \mu_2$ and $\vartheta_1 \preceq \vartheta_2$, then $\mu_1\vartheta_1 \preceq \mu_2\vartheta_2$ for all $\vartheta_1, \vartheta_2 \in \mathbf{C}$. Also, $\vartheta_1 \preceq \vartheta_2$, and $\vartheta_2 \prec \vartheta_3$ implies $\vartheta_1 \prec \vartheta_3$.
If $\mathfrak{S}(\vartheta) = \{\mathbf{w} \in \mathbf{C} : \vartheta \preceq \mathbf{w}\}$ for $\vartheta \in \mathbf{C}$ and $\mathfrak{S}_0 = \mathfrak{S}(0)$, then, $\vartheta_1 \preceq \vartheta_2$ implies $|\vartheta_1| \leq |\vartheta_2|$ and $\vartheta_1 \precnsim \vartheta_2$ implies $|\vartheta_1| < |\vartheta_2|$, for all $\vartheta_1, \vartheta_2 \in \mathfrak{S}_0$.

Definition 2.1. *[14] Suppose* $\mathrm{w} : \mathcal{X} \times \mathcal{X} \longrightarrow (\xi_n, \infty)$ *is a function with* $\liminf_{n\to\infty} \xi_n > 1$. *Then* $d_{\mathbf{C}} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbf{C}$ *is said to be a complex-valued metric if the below items are true:*
(i) $\forall \varrho, \varsigma \in \mathcal{X}$, $d_{\mathbf{C}}(\varrho, \varsigma) \succeq 0$ *and* $d_{\mathbf{C}}(\varrho, \varsigma) = 0$ *if and only if* $\varrho = \varsigma$,
(ii) $\forall \varrho, \varsigma \in \mathcal{X}$, $d_{\mathbf{C}}(\varrho, \varsigma) = d_{\mathbf{C}}(\varsigma, \varrho)$,
(iii) $\forall \varrho, \varsigma, \xi \in \mathcal{X}$, $d_{\mathbf{C}}(\varrho, \varsigma) \preceq \mathrm{w}(\varrho, \varsigma)[d_{\mathbf{C}}(\varrho, \xi) + d_{\mathbf{C}}(\xi, \varsigma)]$,
and $(\mathcal{X}, d_{\mathbf{C}})$ *is called* $d_{\mathbf{C}}$*-metric space (*$d_{\mathbf{C}}$*-MS).*

Definition 2.2. *[14] Let* $(\mathcal{X}, d_{\mathbf{C}})$ *be a* $d_{\mathbf{C}}$*-MS and let* $\{\varrho_n\}$ *be a sequence in* $\mathcal{X}$ *and* $\varrho \in \mathcal{X}$. *For every* $\mathbf{c} \in \mathbf{C}$, *with* $\mathbf{c} \succ 0$, *there exists* $N \in \mathbb{N}$ *such that,*
(i) for all $n > N$, $d_{\mathbf{C}}(\varrho_n, \varrho) \prec \mathbf{c}$, *then* $\{\varrho_n\}$ *is said to be convergent to* ϱ *and denoted by* $\lim_{n\to\infty} \varrho_n = \varrho$;
(ii) for $n > N$, *if* $d_{\mathbf{C}}(\varrho_n, \varrho_{n+\varrho}) \prec \mathbf{c}$, *then* $\{\varrho_n\}$ *is said to be a Cauchy sequence;*
(iii) If every Cauchy sequence is convergent, then $(\mathcal{X}, d_{\mathbf{C}})$ *is complete.*

Lemma 2.1. *[14] For* $(\mathcal{X}, d_{\mathbf{C}})$ *if* $\lim_{n\to\infty} |d_{\mathbf{C}}(\varrho_n, \varrho)| = 0$, *then* $\{\varrho_n\}$ *converges to* ϱ. *Also* $\{\varrho_n\}$ *is a Cauchy sequence if* $\lim_{n,p\to\infty} |d_{\mathbf{C}}(\varrho_n, \varrho_{n+p})| = 0$.

Definition 2.3. *[6] Suppose that* $(\mathcal{X}, d_{\mathbf{C}})$ *is a* $d_{\mathbf{C}}$*-MS.*
(i) $\mathcal{A} \subseteq \mathcal{X}$ *is said to be bounded from below if there exists* $s \in \mathcal{X}$ *with* $s \preceq a$ *for* $a \in \mathcal{A}$.
(ii) $\mathcal{F} : \mathcal{X} \to 2^{\mathbf{C}}$ *is said to be bounded from below if*
$\forall\ \varrho \in \mathcal{X}$, *there exists* $x_p \in \mathbf{C}$ *with* $x_p \preceq \mathbf{u}$ *for* $\mathbf{u} \in \mathcal{F}\varrho := \mathcal{F}(\varrho)$.

Let $\mathcal{CB}(\mathcal{X})$ denote all, closed and bounded subsets of $\mathcal{X}$. For $a \in \mathcal{X}$ and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{X})$, we use the following notations:
(i) $\mathfrak{S}(a, \mathcal{B}) = \varrho_{b\in\mathcal{B}}\mathfrak{S}(d_{\mathbf{C}}(a, b)) = \varrho_{b\in\mathcal{B}}\{\vartheta \in \mathbf{C} : d_{\mathbf{C}}(a, b) \preceq \vartheta\}$,
(ii) $\mathfrak{S}(\mathcal{A}, \mathcal{B}) = (\varrho_{a\in\mathcal{A}}\mathfrak{S}(a, \mathcal{B})) \cap (\varrho_{b\in\mathcal{B}}\mathfrak{S}(b, \mathcal{A}))$.

Definition 2.4. *[6] For* $\mathcal{F} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *and* $\varrho, \varsigma \in \mathcal{X}$, *if* $\mathcal{A}_\varrho(\mathcal{F}\varsigma) = \{\vartheta \in \mathbf{C} |\ \vartheta = d_{\mathbf{C}}(\varrho, u) : u \in \mathcal{F}\varsigma\}$, *then* $\mathcal{F}$
(i) has the lower bound (l.b.) property on $(\mathcal{X}, d_{\mathbf{C}})$ *if for* $\varrho, \varsigma \in \mathcal{X}$, $\exists\ \vartheta_0 \in \mathbf{C}$ *with* $\vartheta_0 \preceq \vartheta$ *for* $\vartheta \in \mathcal{A}_\varrho(\mathcal{F}\varsigma)$, *that is said to be a lower bound of* $\mathcal{F}$ *and denoted by* $l_\varrho(\mathcal{F}\varsigma)$.
(ii) has the greatest lower bound (g.l.b.) property if a greatest lower bound of $\mathcal{A}_\varrho(\mathcal{F}\varsigma)$ *exists in* $\mathbf{C}$ *for* $\varrho, \varsigma \in \mathcal{X}$. *Then,* $d_{\mathbf{C}}(\varrho, \mathcal{F}\varsigma) =\in \mathcal{F}\{d_{\mathbf{C}}(\varrho, u) : u \in \mathcal{F}\varsigma\}$.

Definition 2.5. *$W \subseteq \mathcal{X}$, is said to be proximinal in $\mathcal{X}$ if for $s \in \mathcal{X}$, there exists $\varsigma \in W$, $d_{\mathbf{C}}(s,\varsigma) = d_{\mathbf{C}}(s,W)$.*

Definition 2.6. *If $\alpha_{\mathbf{C}} : \mathcal{X} \times \mathcal{X} \to \mathfrak{S}_0$ and $\mathcal{F},\mathcal{G} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$. Then*
(i) $\mathcal{F}$ is $\alpha_{\mathbf{C}}$-admissible if for $\varrho,\varsigma \in \mathcal{X}$, $\alpha_{\mathbf{C}}(\varrho,\varsigma) \succeq 1$ implies $\alpha_{\mathbf{C}}(u,v) \succeq 1$, $u \in \mathcal{F}\varrho$, $v \in \mathcal{F}\varsigma$,
(ii) $(\mathcal{F},\mathcal{G})$ is $\alpha_{\mathbf{C}}$-admissible if $\alpha(\varrho,\varsigma) \succeq 1$ implies $\alpha_{\mathbf{C}}(u,v) \succeq 1$ and $\alpha_{\mathbf{C}}(v,u) \succeq 1$, $u \in \mathcal{F}\varrho$, $v \in \mathcal{F}\varsigma$.

Definition 2.7. *For a $d_{\mathbf{C}}$-MS $\mathcal{X}$, put $\alpha_{\mathbf{C}} : \mathcal{X} \times \mathcal{X} \to \mathfrak{S}_0$. Then $\mathcal{X}$ satisfies the condition (B^*) with respect to $\alpha_{\mathbf{C}}$ if for any $\{\varrho_n\}$ in $\mathcal{X}$ with $\alpha_{\mathbf{C}}(\varrho_n,\varrho_{n+1}) \succeq 1$ for $n \geq 1$ and $\varrho_n \to \varrho$, we have $\alpha_{\mathbf{C}}(\varrho_n,\varrho) \succeq 1$ and $\alpha_{\mathbf{C}}(\varrho,\varrho_n) \succeq 1$.*

Definition 2.8. *If Ψ be a family of nondecreasing continuous functions $\psi : \mathfrak{S}_0 \to \mathfrak{S}_0$ with $\sum_{n=1}^{\infty} |\psi^n(w)| < \infty$, where ψ^n is the nth iterate of ψ, also $\psi(ct) \leq c\psi(r)$, $c > 1$. It is obviously $\psi(w) \prec w$.*

It is essential to note that $\max\{\vartheta_{\mathbf{1}},\vartheta_{\mathbf{2}}\}$ is not necessarily equal to $\vartheta_{\mathbf{1}}$ or $\vartheta_{\mathbf{2}}$ for each pair $\vartheta_{\mathbf{1}},\vartheta_{\mathbf{2}} \in \mathbf{C}$, because

$$\max\{\vartheta_{\mathbf{1}},\vartheta_{\mathbf{2}}\} = \max\{\mathrm{Re}(\vartheta_{\mathbf{1}}),\mathrm{Re}(\vartheta_{\mathbf{2}})\} + i\max\{\mathrm{Im}(\vartheta_{\mathbf{1}}),\mathrm{Im}(\vartheta_{\mathbf{2}})\}.$$

3. Main results

Theorem 3.1. *Let $(\mathcal{X}, d_{\mathbf{C}})$ be complete and $\mathcal{F},\mathcal{G} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be a pair of $\alpha_{\mathbf{C}}$-admissible with g.l.b. property such that $\mathcal{F}\varrho$ and $\mathcal{G}\varrho$ are proximinal for $\varrho \in \mathcal{X}$. Suppose also that there exists $\beta : \mathfrak{S}_0 \to \mathfrak{S}_0$ such that*

$$\frac{1}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma)) \in \mathfrak{S}(\mathcal{F}\varrho,\mathcal{G}\varsigma),\ |\beta(\psi(M(\varrho,\varsigma)))| \leq \frac{1}{\mathrm{w}^2(\varrho,\varsigma)}, \tag{1}$$

for $\varrho,\varsigma \in \mathcal{X}$, where $\psi \in \Psi$ and

$$M(\varrho,\varsigma) = \max\{d_{\mathbf{C}}(\varrho,\varsigma), d_{\mathbf{C}}(\varrho,\mathcal{F}\varrho), d_{\mathbf{C}}(\varsigma,\mathcal{G}\varsigma), \frac{d_{\mathbf{C}}(\varrho,\mathcal{G}\varsigma) + d_{\mathbf{C}}(\varsigma,\mathcal{F}\varrho)}{2\mathrm{w}(\varrho,\varsigma)}\}. \tag{2}$$

Assume that
(i). For sequence $\{\varrho_n\}$ (is convergent), we have

$$\lim_{j\to\infty} \frac{\mathrm{w}(\varrho_{j+1},\varrho_{j+2})}{\mathrm{w}(\varrho_j,\varrho_{j+1})} < 1, \tag{3}$$

in addition, $\lim_{n\to\infty} \mathrm{w}(\varrho,\varrho_n)$ and $\lim_{n\to\infty} \mathrm{w}(\varrho_n,\varrho)$ are finite;
(ii). there exist $\varrho_0 \in \mathcal{X}$ and $\varrho_1 \in \mathcal{F}\varrho_0$ with $\alpha_{\mathbf{C}}(\varrho_0,\varrho_1) \succeq 1$ and $\alpha_{\mathbf{C}}(\varrho_1,\varrho_0) \succeq 1$.
If $\mathcal{X}$ has condition (B^*) respect to $\alpha_{\mathbf{C}}$, then $\mathcal{F}$ and $\mathcal{G}$ have a CFP.

Proof. Put $\varrho_0 \in \mathcal{X}$ and $\varrho_1 \in \mathcal{F}\varrho_0$ such that $\alpha_{\mathbf{C}}(\varrho_0,\varrho_1) \succeq 1$ and $\alpha_{\mathbf{C}}(\varrho_1,\varrho_0) \succeq 1$. By using (1), construct a sequence $\{\varrho_\kappa\}$ in $\mathcal{X}$ such that
$\varrho_{2\kappa+1} \in \mathcal{F}\varrho_{2\kappa},\ \varrho_{2\kappa+2} \in \mathcal{G}\varrho_{2\kappa+1},\ for\ all\ k \geq 0,$

$$d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2}) \preceq \frac{1}{\alpha_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})}\beta(\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})), \tag{4}$$

and

$$d_{\mathbf{C}}(\varrho_{2\kappa+2},\varrho_{2\kappa+3}) \preceq \frac{1}{\alpha_{\mathbf{C}}(\varrho_{2\kappa+2},\varrho_{2\kappa+1})}\beta(\psi(M(\varrho_{2\kappa+2},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa+2},\varrho_{2\kappa+1})), \tag{5}$$

where

$$\begin{aligned} M(\varrho_{2\kappa},\varrho_{2\kappa+1}) \ &= \max\{d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1}), d_{\mathbf{C}}(\varrho_{2\kappa},\mathcal{F}\varrho_{2\kappa}), d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{G}\varrho_{2\kappa+1}), \\ &\quad \tfrac{d_{\mathbf{C}}(\varrho_{2\kappa},\mathcal{G}\varrho_{2\kappa+1}) + d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{F}\varrho_{2\kappa})}{2\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})}\}, \end{aligned} \tag{6}$$

and

$$M(\varrho_{2\kappa+1},\varrho_{2\kappa+2}) = \max\{d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2}), d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{F}\varrho_{2\kappa+1}), d_{\mathbf{C}}(\varrho_{2\kappa+2},\mathcal{G}\varrho_{2\kappa+2}), \frac{d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{G}\varrho_{2\kappa+2})+d_{\mathbf{C}}(\varrho_{2\kappa+2},\mathcal{F}\varrho_{2\kappa+1})}{2\mathrm{w}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})}\}.$$

Since $(\mathcal{F},\mathcal{G})$ is $\alpha_{\mathbf{C}}$-admissible, then $\alpha_{\mathbf{C}}(\varrho_\kappa,\varrho_{\kappa+1}) \succeq 1$ and $\alpha_{\mathbf{C}}(\varrho_{\kappa+1},\varrho_\kappa) \succeq 1$ for all $k \geq 0$. If $\varrho_{2\kappa} = \varrho_{2\kappa+1}$ for some $k \geq 0$, then $\varrho_{2\kappa+1} = \varrho_{2\kappa+2}$. In fact, if $\varrho_{2\kappa+1} \neq \varrho_{2\kappa+2}$, then

$$\begin{aligned} 0 < |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| &\leq |\alpha_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| \\ &\leq |\beta(\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1}))| \\ &< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1}))| \\ &< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|M(\varrho_{2\kappa},\varrho_{2\kappa+1})|, \end{aligned}$$

now, we have the below cases:
Step 1: If $M(\varrho_{2\kappa},\varrho_{2\kappa+1}) = d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})$ thus we have:

$$\begin{aligned} 0 < |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| &\leq |\alpha_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| \\ &\leq |\beta(\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1}))| \quad (7) \\ &< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|M(\varrho_{2\kappa},\varrho_{2\kappa+1})| \\ &= \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|. \end{aligned}$$

(8)

Setting $t = \sup\{\frac{1}{\mathrm{w}(\varrho_\kappa,\varrho_{\kappa+1})}, k = 1,2,3,\ldots\}$,
we get

$$\begin{aligned} 0 < |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| &< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})| \\ &\leq t^2|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})| < t^4|d_{\mathbf{C}}(\varrho_{2\kappa-1},\varrho_{2\kappa})| \\ &< \ldots < t^{4k+2}|d_{\mathbf{C}}(\varrho_0,\varrho_1)|. \end{aligned} \quad (9)$$

Now, as regards $0 < t < 1$, we obtain $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| = 0$, (Similarly, it can be proved that $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_{2\kappa+2},\varrho_{2\kappa+3})| = 0$), and in general it can be proved that $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_\kappa,\varrho_{\kappa+1})| = 0$.
Step 2: If $M(\varrho_{2\kappa},\varrho_{2\kappa+1}) = d_{\mathbf{C}}(\varrho_{2\kappa},\mathcal{F}\varrho_{2\kappa})$, then considering $\varrho_{2\kappa+1} \in \mathcal{F}\varrho_{2\kappa}$, we have:

$$\begin{aligned} 0 < |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| &\leq |\alpha_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| \\ &\leq |\beta(\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1}))| \quad (10) \\ &< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|M(\varrho_{2\kappa},\varrho_{2\kappa+1})| \\ &= \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa},\mathcal{F}\varrho_{2\kappa})| \\ &< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}(\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1}) + d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{F}\varrho_{2\kappa})|) \\ &= \frac{1}{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|. \end{aligned}$$

As in case (1), we obtain $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_\kappa,\varrho_{\kappa+1})| = 0$.

Step 3: If $M(\varrho_{2\kappa},\varrho_{2\kappa+1}) = d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{G}\varrho_{2\kappa+1})$, then by considering $\varrho_{2\kappa+2}\in\mathcal{G}\varrho_{2\kappa+1}$, we have

$$\begin{aligned}
0<|d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| &\leq |\alpha_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})|\\
&\leq |\beta(\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1}))| \qquad (11)\\
&< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|M(\varrho_{2\kappa},\varrho_{2\kappa+1})|\\
&= \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{G}\varrho_{2\kappa+1})|\\
&< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}(\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})|d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})\\
&+ d_{\mathbf{C}}(\varrho_{2\kappa+2},\mathcal{G}\varrho_{2\kappa+1})|)\\
&= \frac{1}{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| < |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})|,
\end{aligned}$$

which is a contradiction. Hence, this case is not possible.

Step 4: If $M(\varrho_{2\kappa},\varrho_{2\kappa+1}) = \frac{d_{\mathbf{C}}(\varrho_{2\kappa},\mathcal{G}\varrho_{2\kappa+1})+d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{F}\varrho_{2\kappa})}{2\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})}$, then we have:

$$\begin{aligned}
0 &< |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})| \leq |\alpha_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})|\\
&\leq |\beta(\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa},\varrho_{2\kappa+1}))| \qquad (12)\\
&< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}|M(\varrho_{2\kappa},\varrho_{2\kappa+1})|\\
&= \frac{1}{\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}\left|\frac{d_{\mathbf{C}}(\varrho_{2\kappa},\mathcal{G}\varrho_{2\kappa+1})+d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{F}\varrho_{2\kappa})}{2\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})}\right|\\
&< \frac{1}{2\mathrm{w}^3(\varrho_{2\kappa},\varrho_{2\kappa+1})}(\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+2})|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})+d_{\mathbf{C}}(\varrho_{2\kappa+1},\mathcal{G}\varrho_{2\kappa+1})|)\\
&< \frac{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})\\
&+ \mathrm{w}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})(d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})+d_{\mathbf{C}}(\varrho_{2\kappa+2},\mathcal{G}\varrho_{2\kappa+1}))|\\
&< \left(\frac{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa},\varrho_{2\kappa+1})}+\frac{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+2})\mathrm{w}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa},\varrho_{2\kappa+1})}\right)\\
&\quad \max\{|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|,|d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})|\}.
\end{aligned}$$

If $\max\{|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|,|d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})|\} = |d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|$, then we get

$$\begin{aligned}
0 &< |d_{\mathbf{C}}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})|\\
&< \left(\frac{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa},\varrho_{2\kappa+1})}+\frac{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+2})\mathrm{w}(\varrho_{2\kappa+1},\varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa},\varrho_{2\kappa+1})}\right)|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|\\
&< (\frac{1}{2\mathrm{w}^2(\varrho_{2\kappa},\varrho_{2\kappa+1})}+\frac{1}{2\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})})|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|\\
&= \frac{1}{\mathrm{w}(\varrho_{2\kappa},\varrho_{2\kappa+1})}|d_{\mathbf{C}}(\varrho_{2\kappa},\varrho_{2\kappa+1})|,
\end{aligned}$$

so, as in case (1), we obtain $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_\kappa, \varrho_{\kappa+1})| = 0$.
If $\max\{|d_{\mathbf{C}}(\varrho_{2\kappa}, \varrho_{2\kappa+1})|, |d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})|\} = |d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})|$, then we have

$$\begin{aligned}
0 &< |d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| \\
&< \left(\frac{\mathrm{w}(\varrho_{2\kappa}, \varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa}, \varrho_{2\kappa+1})} + \frac{\mathrm{w}(\varrho_{2\kappa}, \varrho_{2\kappa+2})\mathrm{w}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})}{2\mathrm{w}^3(\varrho_{2\kappa}, \varrho_{2\kappa+1})}\right)|d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| \\
&< (\frac{1}{2\mathrm{w}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})} + \frac{1}{2\mathrm{w}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})})|d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| \\
&= \frac{1}{\mathrm{w}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})}|d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| < |d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})|,
\end{aligned}$$

which again is a contradiction. Therefore, $\varrho_{2\kappa}$ will be a CFP for $\mathcal{G}$ and $\mathcal{F}$.
(Also, in the same way, we have $\varrho_{2\kappa+2} = \varrho_{2\kappa+3}$ if $\varrho_{2\kappa+1} = \varrho_{2\kappa+2}$ for some $k \geq 0$).
Suppose that $\varrho_\kappa \neq \varrho_{\kappa+1}$ for all $k \geq 0$. From inequalities (4) and (6), we have

$$\begin{aligned}
|d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| &\leq |\alpha_{\mathbf{C}}(\varrho_{2\kappa}, \varrho_{2\kappa+1})d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| \\
&< |\beta(\psi(M(\varrho_{2\kappa}, \varrho_{2\kappa+1})))\psi(M(\varrho_{2\kappa}, \varrho_{2\kappa+1}))| \\
&< \frac{1}{\mathrm{w}^2(\varrho_{2\kappa}, \varrho_{2\kappa+1})}|M(\varrho_{2\kappa}.\varrho_{2\kappa+1})|,
\end{aligned}$$

Thus, as in the above process, $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_\kappa, \varrho_{\kappa+1})| = 0$. We will show next that $\{\varrho_\kappa\}$ is Cauchy. By (9) and condition (iii) from the definition of $d_{\mathbf{C}}$, we have

$$\begin{aligned}
|d_{\mathbf{C}}(\varrho_\kappa, \varrho_{\kappa+\iota})| &\leq \mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})|d_{\mathbf{C}}(\varrho_\kappa, \varrho_{\kappa+1}) + d_{\mathbf{C}}(\varrho_{\kappa+1}, \varrho_{\kappa+\iota})| \\
&\leq \mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})t^\kappa|d_{\mathbf{C}}(\varrho_0, \varrho_1)| \\
&+ |\mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})\mathrm{w}(\varrho_{\kappa+1}, \varrho_{\kappa+\iota})\big(d_{\mathbf{C}}(\varrho_{\kappa+1}, \varrho_{\kappa+2}) \\
&+ d_{\mathbf{C}}(\varrho_{\kappa+2}, \varrho_{\kappa+\iota})\big)| \\
&\leq \mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})t^\kappa|d_{\mathbf{C}}(\varrho_0, \varrho_1)| \\
&+ \mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})\mathrm{w}(\varrho_{\kappa+1}, \varrho_{\kappa+\iota})t^{\kappa+1}|d_{\mathbf{C}}(\varrho_0, \varrho_1)| \\
&+ \mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})\mathrm{w}(\varrho_{\kappa+1}, \varrho_{\kappa+\iota})\mathrm{w}(\varrho_{\kappa+2}, \varrho_{\kappa+\iota})t^{\kappa+2}|d_{\mathbf{C}}(\varrho_0, \varrho_1)| \\
&\vdots \\
&+ \mathrm{w}(\varrho_\kappa, \varrho_{\kappa+\iota})\mathrm{w}(\varrho_{\kappa+1}, \varrho_{\kappa+\iota})\mathrm{w}(\varrho_{\kappa+2}, \varrho_{\kappa+\iota})\dots \\
&\quad \mathrm{w}(\varrho_{\kappa+\iota-1}, \varrho_{\kappa+\iota})t^{\kappa+\iota-1}|d_{\mathbf{C}}(\varrho_0, \varrho_1)| \\
&< nt^{\kappa-1}|d_{\mathbf{C}}(\varrho_0, \varrho_1)|.
\end{aligned}$$

Taking into account, $t^\kappa \to 0$ as $\kappa \to 0$, we get $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_\kappa, \varrho_{\kappa+\iota})| = 0$. Hence, the $\{\varrho_\kappa\}$ is Cauchy. As regards $(\mathcal{X}, d_{\mathbf{C}})$ is complete $d_{\mathbf{C}}$-MS, there exists $\varrho^* \in \mathcal{X}$ with $\lim_{\kappa\to\infty} |d_{\mathbf{C}}(\varrho_\kappa, \varrho^*)| = 0$. Since X has the property (B^*), we have
$\alpha_{\mathbf{C}}(\varrho^*, \varrho_\kappa) \succeq 1$ and $\alpha_{\mathbf{C}}(\varrho_\kappa, \varrho^*) \succeq 1$ for all $k \geq 1$. Moreover,

$$\begin{aligned}
&\frac{\beta(\psi(M(\varrho_{2\kappa}, \varrho^*)))\psi(M(\varrho_{2\kappa}, \varrho^*))}{\alpha_{\mathbf{C}}(\varrho_{2\kappa}, \varrho^*)} \in \mathfrak{S}(\mathcal{F}\varrho_{2\kappa}, \mathcal{G}\varrho^*), \qquad (13) \\
&\frac{\beta(\psi(M(\varrho^*, \varrho_{2\kappa+1})))\psi(M(\varrho^*, \varrho_{2\kappa+1}))}{\alpha_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})} \in \mathfrak{S}(\mathcal{F}\varrho^*, \mathcal{G}\varrho_{2\kappa+1}),
\end{aligned}$$

for all $k \geq 1$, where

$$M(\varrho_{2\kappa}, \varrho^*) = \max\{d_{\mathbf{C}}(\varrho_{2\kappa}, \varrho^*), d_{\mathbf{C}}(\varrho_{2\kappa}, \mathcal{F}\varrho_{2\kappa}), d_{\mathbf{C}}(\varrho^*, \mathcal{G}\varrho^*), \frac{d_{\mathbf{C}}(\varrho_{2\kappa}, \mathcal{G}\varrho^*) + d_{\mathbf{C}}(\varrho^*, \mathcal{F}\varrho_{2\kappa})}{2\mathrm{w}(\varrho_{2\kappa}, \varrho^*)}\}, \tag{14}$$

and

$$M(\varrho^*, \varrho_{2\kappa+1}) = \max\{d_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1}), d_{\mathbf{C}}(\varrho^*, \mathcal{F}\varrho^*), d_{\mathbf{C}}(\varrho_{2\kappa+1}, \mathcal{G}\varrho_{2\kappa+1}), \frac{d_{\mathbf{C}}(\varrho^*, \mathcal{G}\varrho_{2\kappa+1}) + d_{\mathbf{C}}(\varrho_{2\kappa+1}, \mathcal{F}\varrho^*)}{2\mathrm{w}(\varrho^*, \varrho_{2\kappa+1})}\}.$$

Since $\mathfrak{S}(\mathcal{F}\varrho_{2\kappa}, \mathcal{G}\varrho^*) \subseteq \mathfrak{S}(a, \mathcal{G}\varrho^*)$ and $\mathfrak{S}(\mathcal{F}\varrho^*, \mathcal{G}\varrho_{2\kappa+1}) \subseteq \mathfrak{S}(\mathcal{F}\varrho^*, b)$, $a \in \mathcal{F}\varrho_{2\kappa}$, $b \in \mathcal{G}\varrho_{2\kappa+1}$ and $k \geq 0$, then

$$\frac{\beta(\psi(M(\varrho_{2\kappa}, \varrho^*)))\psi(M(\varrho_{2\kappa}, \varrho^*))}{\alpha_{\mathbf{C}}(\varrho_{2\kappa}, \varrho^*)} \in \mathfrak{S}(\varrho_{2\kappa+1}, \mathcal{G}\varrho^*),$$

$$\frac{\beta(\psi(M(\varrho^*, \varrho_{2\kappa+1})))\psi(M(\varrho^*, \varrho_{2\kappa+1}))}{\alpha_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})} \in \mathfrak{S}(\mathcal{F}\varrho^*, \varrho_{2\kappa+2})$$

and so there exist sequences $\{u_\kappa\} \subseteq \mathcal{F}\varrho^*$ and $\{v_\kappa\} \subseteq \mathcal{G}\varrho^*$ such that

$$d_{\mathbf{C}}(\varrho_{2\kappa+1}, v_\kappa) \preceq \frac{1}{\alpha_{\mathbf{C}}(\varrho_{2\kappa}, \varrho^*)}\beta(\psi(M(\varrho_{2\kappa}, \varrho^*)))\psi(M(\varrho_{2\kappa}, \varrho^*)), \tag{15}$$

and

$$d_{\mathbf{C}}(u_\kappa, \varrho_{2\kappa+2}) \preceq \frac{1}{\alpha_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})}\beta(\psi(M(\varrho^*, \varrho_{2\kappa+1})))\psi(M(\varrho^*, \varrho_{2\kappa+1})). \tag{16}$$

Next, we show that $\{u_\kappa\}$ and $\{v_\kappa\}$ converge to ϱ^*. For some ($m > 1$, m is an integer number), we have

$$\begin{aligned}
|d_{\mathbf{C}}(\varrho^*, v_\kappa)| &\leq \mathrm{w}(\varrho^*, v_\kappa)|d_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1}) + d_{\mathbf{C}}(\varrho_{2\kappa+1}, v_\kappa)| \\
&\leq \mathrm{w}(\varrho^*, v_\kappa)[|d_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})| \\
&+ \mathrm{w}(\varrho_{2\kappa+1}, v_\kappa)|d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2}) + d_{\mathbf{C}}(\varrho_{2\kappa+2}, v_\kappa)|] \\
&\leq \mathrm{w}(\varrho^*, v_\kappa)\Bigg[|d_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})| \\
&+ \mathrm{w}(\varrho_{2\kappa+1}, v_\kappa)\big[|d_{\mathbf{C}}(\varrho_{2\kappa+1}, \varrho_{2\kappa+2})| \\
&+ \mathrm{w}(\varrho_{2\kappa+2}, v_\kappa)|d_{\mathbf{C}}(\varrho_{2\kappa+2}, \varrho_{2\kappa+3}) + d_{\mathbf{C}}(\varrho_{2\kappa+3}, v_\kappa)|\big]\Bigg] \qquad (17) \\
&\leq \mathrm{w}(\varrho^*, v_\kappa)[|d_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})| + \sum_{j=1}^{m}\left(\Pi_{i=2\kappa+1}^{2\kappa+j}\mathrm{w}(\varrho_i, v_\kappa)\right) d_{\mathbf{C}}(\varrho_{2\kappa+j}, \varrho_{2\kappa+j+1}) \\
&+ \left(\Pi_{i=2\kappa+1}^{2\kappa+j}\mathrm{w}(\varrho_i, v_\kappa)\right) d_{\mathbf{C}}(\varrho_{2\kappa+m+1}, v_\kappa)] \qquad (18) \\
&\leq \mathrm{w}(\varrho^*, v_\kappa)[|d_{\mathbf{C}}(\varrho^*, \varrho_{2\kappa+1})| + \sum_{j=1}^{m}\left(\Pi_{i=2\kappa+1}^{2\kappa+j}\mathrm{w}(\varrho_i, v_\kappa)\right) t^{2\kappa+j} d_{\mathbf{C}}(\varrho_0, \varrho_1)] \\
&+ \left(\Pi_{i=2\kappa+1}^{2\kappa+j}\mathrm{w}(\varrho_i, v_\kappa)\right) d_{\mathbf{C}}(\varrho_{2\kappa+m+1}, v_\kappa)] \to 0. \qquad (19)
\end{aligned}$$

By definition of t and regarding that $t^{2\kappa} \to 0$, while $k \to \infty$, we get

$$\sum_{j=1}^{m} \left(\Pi_{i=2\kappa+1}^{2\kappa+j} \mathrm{w}(\varrho_i, v_\kappa)\right) t^{2\kappa+j} d_{\mathbf{C}}(\varrho_0, \varrho_1) \to 0.$$

So, $\lim_{\kappa\to\infty} |d(\varrho^*, v_\kappa)| = 0$. It can be shown similarly, $\lim_{\kappa\to\infty} |d(\varrho^*, u_\kappa)| = 0$. Considering that $\mathcal{F}\varrho^*$ and $\mathcal{G}\varrho^*$ are closed, we obtain $\varrho^* \in \mathcal{G}\varrho^*$ and $\varrho^* \in \mathcal{F}\varrho^*$. Thus, $\mathcal{F}$ and $\mathcal{G}$ have a CFP. □

If we take $\mathcal{F} = \mathcal{G}$ in Theorem 3.1, then we get the following corollary.

Corollary 3.1. *Let* $(\mathcal{X}, d_{\mathbf{C}})$ *be complete and* $\mathcal{F} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *a* $\alpha_{\mathbf{C}}$*-admissible with g.l.b. property such that* $\mathcal{F}\varrho$ *be proximinal and*

$$\frac{1}{\alpha_{\mathbf{C}}(\varrho, \varsigma)} \beta(\psi(M(\varrho, \varsigma)))\psi(M(\varrho, \varsigma)) \in \mathfrak{S}(\mathcal{F}\varrho, \mathcal{F}\varsigma), \ |\beta(\psi(M(\varrho, \varsigma)))| \le \frac{1}{\mathrm{w}^2(\varrho, \varsigma)}, \tag{20}$$

for $\varrho, \varsigma \in \mathcal{X}$ *and* $\psi \in \Psi$*, where*
$M(\varrho, \varsigma) = \max\{d_{\mathbf{C}}(\varrho, \varsigma), d_{\mathbf{C}}(\varrho, \mathcal{F}\varrho), d_{\mathbf{C}}(\varsigma, \mathcal{F}\varsigma), \frac{d_{\mathbf{C}}(\varrho, \mathcal{F}\varsigma)+d_{\mathbf{C}}(\varsigma, \mathcal{F}\varrho)}{2\mathrm{w}(\varrho, \varsigma)}\}$. *Assume that*
(i). For each point ϱ *and convergent sequence* $\{\varrho_n\}$*, the following inequality holds;*

$$\lim_{j\to\infty} \frac{\mathrm{w}(\varrho_{j+1}, \varrho_{j+2})}{\mathrm{w}(\varrho_j, \varrho_{j+1})} < 1,$$

additionally, $\lim_{n\to\infty} \mathrm{w}(\varrho, \varrho_n)$ *and* $\lim_{n\to\infty} \mathrm{w}(\varrho_n, \varrho)$ *are finite;*
(ii). There exists $\varrho_0 \in \mathcal{X}$ *and* $\varrho_1 \in \mathcal{F}\varrho_0$ *such that* $\alpha_{\mathbf{C}}(\varrho_0, \varrho_1) \succeq 1$ *and* $\alpha_{\mathbf{C}}(\varrho_1, \varrho_0) \succeq 1$.
If $\mathcal{X}$ *satisfies the condition* (B^*)*, then* $\mathcal{F}$ *has a FP.*

If in Theorem 3.1, we set $M(\varrho, \varsigma) = d(\varrho, \varsigma)$, then we can deduce the following corollary.

Corollary 3.2. *Let* $(\mathcal{X}, d_{\mathbf{C}})$ *be complete and* $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *be a pair of* $\alpha_{\mathbf{C}}$*-admissible with g.l.b. property such that* $\mathcal{F}\varrho$ *and* $\mathcal{G}\varrho$ *be proximinal, and*

$$\frac{1}{\alpha_{\mathbf{C}}(\varrho, \varsigma)} \beta(\psi(d(\varrho, \varsigma)))\psi(d(\varrho, \varsigma)) \in \mathfrak{S}(\mathcal{F}\varrho, \mathcal{G}\varsigma), \ |\beta(\psi(d_{\mathbf{C}}(\varrho, \varsigma)))| \le \frac{1}{\mathrm{w}^2(\varrho, \varsigma)}, \tag{21}$$

for $\varrho, \varsigma \in \mathcal{X}$*, where* $\psi \in \Psi$.
Also assume that
(i). For each point ϱ *and convergent sequence* $\{\varrho_n\}$*,*

$$\lim_{j\to\infty} \frac{\mathrm{w}(\varrho_{j+1}, \varrho_{j+2})}{\mathrm{w}(\varrho_j, \varrho_{j+1})} < 1, \tag{22}$$

in addition, $\lim_{n\to\infty} \mathrm{w}(\varrho, \varrho_n)$ *and* $\lim_{n\to\infty} \mathrm{w}(\varrho_n, \varrho)$ *are finite;*
(ii). There exists $\varrho_0 \in \mathcal{X}$ *and* $\varrho_1 \in \mathcal{F}\varrho_0$ *such that* $\alpha_{\mathbf{C}}(\varrho_0, \varrho_1) \succeq 1$ *and* $\alpha_{\mathbf{C}}(\varrho_1, \varrho_0) \succeq 1$.
If $\mathcal{X}$ *satisfies the condition* (B^*)*, then* $\mathcal{F}$ *and* $\mathcal{G}$ *have CFP.*

Corollary 3.3. *Let* $(\mathcal{X}, d_{\mathbf{C}})$ *be complete and* $\mathcal{F} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *be* $\alpha_{\mathbf{C}}$*-admissible with g.l.b. property such that* $\mathcal{F}\varrho$ *is proximinal and*

$$\frac{1}{\alpha_{\mathbf{C}}(\varrho, \varsigma)} \beta(\psi(M(\varrho, \varsigma)))\psi(M(\varrho, \varsigma)) \in \mathfrak{S}(\mathcal{F}\varrho, \mathcal{F}\varsigma), \ |\beta(\psi(M(\varrho, \varsigma)))| \le \frac{1}{\mathrm{w}^2(\varrho, \varsigma)}, \tag{23}$$

for $\varrho, \varsigma \in \mathcal{X}$ *and* $\psi \in \Psi$*, where*
$M(\varrho, \varsigma) = \mathcal{X}\{d_{\mathbf{C}}(\varrho, \varsigma), d_{\mathbf{C}}(\varrho, \mathcal{F}\varrho), d_{\mathbf{C}}(\varsigma, \mathcal{F}\varsigma), \frac{d_{\mathbf{C}}(\varrho, \mathcal{F}\varsigma)+d_{\mathbf{C}}(\varsigma, \mathcal{F}\varrho)}{2\mathrm{w}(\varrho, \varsigma)}\}$. *Assume that*

(i). For each point ϱ and sequences $\{\varrho_n\}$ and $\{\varsigma_n\}$ (respectively convergent and arbitrary sequence) in $\mathcal{X}$,

$$\lim_{j\to\infty} \frac{\mathrm{w}(\varrho_{j+1}, \varrho_{j+2})}{\mathrm{w}(\varrho_j, \varrho_{j+1})} < 1,$$

additionally, $\lim_{n\to\infty} \mathrm{w}(\varrho, \varrho_n)$ and $\lim_{n\to\infty} \mathrm{w}(\varrho_n, \varrho)$ are finite;
(ii). There exists $\varrho_0 \in \mathcal{X}$ and $\varrho_1 \in \mathcal{F}\varrho_0$ such that $\alpha_{\mathbf{C}}(\varrho_0, \varrho_1) \succeq 1$ and $\alpha_{\mathbf{C}}(\varrho_1, \varrho_0) \succeq 1$.
If $\mathcal{X}$ satisfies the condition (B^*), then $\mathcal{F}$ has a FP.

If $\mathcal{F}$ and $\mathcal{G}$ are self-maps and $\alpha_{\mathbf{C}}(\varrho, \varsigma) = 1$, then we can obtain the following consequence.

Corollary 3.4. *Let $(\mathcal{X}, d_{\mathbf{C}})$ be complete and $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{X}$ be self mappings with g.l.b. property such that $\mathcal{F}\varrho$ and $\mathcal{G}\varrho$ are proximinal, and*

$$d_{\mathbf{C}}(\mathcal{F}\varrho, \mathcal{G}\varsigma) \preceq \beta(\psi(M(\varrho, \varsigma)))\psi(M(\varrho, \varsigma)),\ |\beta(\psi(M(\varrho, \varsigma)))| \leq \frac{1}{\mathrm{w}^2(\varrho, \varsigma)}, \tag{24}$$

for each $\varrho, \varsigma \in \mathcal{X}$, where $\psi \in \Psi$, and
$M(\varrho, \varsigma) = \mathcal{X}\{d_{\mathbf{C}}(\varrho, \varsigma), d_{\mathbf{C}}(\varrho, \mathcal{F}\varrho), d_{\mathbf{C}}(\varsigma, \mathcal{G}\varsigma), \frac{d_{\mathbf{C}}(\varrho, \mathcal{G}\varsigma) + d_{\mathbf{C}}(\varsigma, \mathcal{F}\varrho)}{2\mathrm{w}(\varrho, \varsigma)}\}$.
Also, if (i) of Theorem 3.1 holds, then $\mathcal{F}$ and $\mathcal{G}$ have a CFP.

Example 3.1. *Let $\mathcal{X} = \{\frac{1}{16}, \frac{1}{8}, \frac{1}{4}\}$. Define $\mathrm{w} : \mathcal{X} \times \mathcal{X} \to [1, \infty)$ and $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ as the following*

$$\mathrm{w}(\varrho, \varsigma) = \begin{cases} 1 + \varrho & \varrho > \varsigma, \\ \frac{1}{\varsigma} & \varrho < \varsigma, \\ 1 & \varrho = \varsigma. \end{cases}$$

$$\rho(\varrho, \varsigma) = \begin{cases} 0 & \varrho = \varsigma, \\ \min\{\varrho, \varsigma\} & otherwise. \end{cases}$$

Consider $d_{\mathbf{C}}(\varrho, \varsigma) = (1 + i)\rho(\varrho, \varsigma)$. Clearly, $d_{\mathbf{C}}(\varrho, \varsigma) \succeq 0$, and $d_{\mathbf{C}}(\varrho, \varsigma) = d_{\mathbf{C}}(\varsigma, \varrho)$. Checking the triangular inequality; if $\varrho = \varsigma$, then $\rho(\varrho, \varsigma) = 0$, and so it is true. For $\varrho, \varsigma, \xi \in \mathcal{X}$,

- *if $\varrho < \xi < \varsigma$, then, $\varrho \leq \frac{1}{\varsigma}(\varrho + \xi)$;*
- *if $\varrho < \varsigma < \xi$, then, $\varrho \leq \frac{1}{\varsigma}(\varrho + \varsigma)$;*
- *and if $\xi < \varrho < \varsigma$, then, $\varrho \leq \frac{1}{\varsigma}2\xi$.*

So, $\rho(\varrho, \varsigma) \leq \mathrm{w}(\varrho, \xi)(\rho(\varrho, \xi) + \rho(\xi, \varsigma))$. When $\varsigma < \varrho$, the same results is true. Thus, $d_{\mathbf{C}}(\varrho, \varsigma) \preceq \mathrm{w}(\varrho, \xi)(d_{\mathbf{C}}(\varrho, \xi) + d_{\mathbf{C}}(\xi, \varsigma))$.
Also, the condition (i) from Theorem 3.1 is satisfied.
Consider $\psi(w) = \frac{9}{10}w$, $\beta(\psi(r)) = \frac{1}{\psi^2(r)}$, and

$$\mathcal{F}(\varrho) = \begin{cases} \{\frac{1}{8}\} & \varrho = \frac{1}{16}, \\ \{\frac{1}{16}, \frac{1}{4}\} & \varrho = \frac{1}{8}, \\ \{\frac{1}{4}\} & \varrho = \frac{1}{4}, \end{cases}$$

$$\mathcal{G}(\varsigma) = \begin{cases} \{\frac{1}{16}, \frac{1}{8}\} & \varsigma = \frac{1}{16}, \\ \{\frac{1}{16}\} & \varsigma = \frac{1}{8}, \\ \{\frac{1}{4}\} & \varsigma = \frac{1}{4}, \end{cases}$$

$$\alpha_{\mathbf{C}}(\varrho,\varsigma)=\begin{cases}1+i\ \varrho=\varsigma=\frac{1}{4},\\ \frac{1}{(1+i)^2}\ otherwise.\end{cases}$$

Clearly, $(\mathcal{F},\mathcal{G})$ is $\alpha_{\mathbf{C}}$-admissible and $\mathcal{F}\varrho$ and $\mathcal{G}\varrho$ are proximinal, because $\mathcal{X}$ is finite. Also, $\mathcal{X}$ has condition (B^*) respect to $\alpha_{\mathbf{C}}$. There exist $\varrho_0=\frac{1}{4}$ and $\varrho_1=\frac{1}{4}\in\mathcal{F}\varrho_0=\{\frac{1}{4}\}$ such that $\alpha_{\mathbf{C}}(\varrho_0,\varrho_1)=\alpha_{\mathbf{C}}(\varrho_1,\varrho_0)=1+i\succeq 1$.
Next, we will show that $\frac{\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}\in\mathfrak{S}(\mathcal{F}\varrho,\mathcal{G}\varsigma)$.
We show that for $a\in\mathcal{F}\varrho$, there exists $b\in\mathcal{G}\varsigma$ with $d_{\mathbf{C}}(a,b)\preceq\frac{\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}$, and also for $b\in\mathcal{G}\varsigma$, there exists $a\in\mathcal{F}\varrho$ with $d_{\mathbf{C}}(a,b)\preceq\frac{\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}$.
If $\varrho=\varsigma=\frac{1}{4}$ then $\mathcal{F}\varrho=\mathcal{G}\varsigma=\{\frac{1}{4}\}$, clearly $d_{\mathbf{C}}(a,b)\preceq\frac{\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}$ is established for all $a\in\mathcal{F}\varrho$ and $b\in\mathcal{G}\varsigma$.
Let $\varrho=\varsigma=\frac{1}{8}$. Then, $\mathcal{F}\varrho=\{\frac{1}{16},\frac{1}{4}\}$ and $\mathcal{G}\varsigma=\{\frac{1}{16}\}$, hence

$$\frac{1}{16}\le\frac{(1+i)\frac{1}{16}\frac{9}{10}}{\frac{99}{100}\frac{1}{256}(1+i)^2(1+i)^{-2}},$$

and so $d_{\mathbf{C}}(a,b)\preceq\frac{\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}$. Also, if $a=\frac{1}{4}\in\mathcal{F}(\frac{1}{8})$, then for $b=\frac{1}{16}\in\mathcal{G}(\frac{1}{8})$, the mentioned inequality is established.
Now, if $\varrho=\varsigma=\frac{1}{16}$. Then, for $\mathcal{F}\varrho=\{\frac{1}{8}\}$ and $\mathcal{G}\varsigma=\{\frac{1}{16},\frac{1}{8}\}$ the aforementioned relation is established.
For $a=\frac{1}{8}\in\mathcal{F}(\frac{1}{16})$ and $b=\frac{1}{8}\in\mathcal{G}(\frac{1}{16})$, $d_{\mathbf{C}}(a,b)=0$, and so $d_{\mathbf{C}}(a,b)\preceq\frac{\beta(\psi(M(\varrho,\varsigma)))\psi(M(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)}$. Other cases are checked as above. Therefore, the assumptions of Theorem 3.1 are fulfilled. Here, $\varrho^*=\frac{1}{4}$ is a CFP of $\mathcal{F}$ and $\mathcal{G}$.

4. Application

Here, we investigate the existence of the solution for an inclusion system with derivative of fractional order.
Set $r\in[a,b]\subseteq\mathbb{R}^+$ and $I_a^\iota\tau(r,x(r))=\int_a^r\frac{(r-s)^{\iota-1}}{\Gamma(\iota)}\tau(s,x(s))ds$, $(\tau\in C([a,b]\times\mathbb{R},\mathbb{R})$, $\iota\in\mathbb{R}^+)$.
Consider the following system:

$$\begin{cases}x(r)\in w(r)+\mathcal{W}(r,I_a^\iota(U(r,x(r)))),\\ y(r)\in z(r)+\mathcal{Z}(r,I_a^\iota(V(r,y(r)))),\end{cases}\tag{25}$$

where, $x,y,w,z:[a,b]\to\mathbb{R}$, $U,V,\mathcal{W},\mathcal{Z}:[a,b]\times\mathbb{R}\to\mathbb{R}$ are continuous and two functions $\mathcal{W}$ and $\mathcal{Z}$ are bounded, additionally.
Let $\mathcal{X}=L^1([a,b])$ (Lebesgue integrable functions). Consider $d_{\mathbf{C}}:\mathcal{X}\times\mathcal{X}\to\mathbf{C}$ by:

$$d_{\mathbf{C}}(\mathcal{F},\mathcal{G})=\parallel\mathcal{F}(r)-\mathcal{G}(r)\parallel_1 e^{i\theta}=\left(\int_a^b\mid\mathcal{F}(r)-\mathcal{G}(r)\mid dr\right)e^{i\theta},\tag{26}$$

where $\theta\in[\frac{\pi}{4},\frac{\pi}{2}]$ and function w : $\mathcal{X}\times\mathcal{X}\to[1,+\infty)$ which satisfy in the condition (i) of Corollary 3.2. Clearly, $(\mathcal{X},d_{\mathbf{C}})$ is a complete respect to function w.
Define $\Lambda_w^U:\mathcal{X}\to 2^{\mathcal{X}}$ by

$$\Lambda_w^U\mathcal{F}(r)=\{z(r)+\mathcal{W}(r,I_a^\iota(U(r,\mathcal{F}(r)))),z(r)+\mathcal{Z}(r,I_a^\iota(U(r,\mathcal{F}(r))))\}\quad(\forall r\in[a,b]),\tag{27}$$

with $w\in\mathcal{X}$ (is continuous), $U:[a,b]\times\mathbb{R}\to\mathbb{R}$ and $\mathcal{W},\mathcal{Z}:[a,b]\times\mathbb{R}\to\mathbb{R}$ are bounded continuous. Then, it can be easily shown that $\Lambda_w^U\mathcal{F}\subseteq\mathcal{X}$ is closed and bounded for $\mathcal{F}\in\mathcal{X}$.
Clearly, $\mathcal{F}\in\mathcal{X}$ is a solution of the system (25) whenever it is a CFP of Λ_w^U and Λ_z^V.

The following theorem shows that Corollary 3.2 can be used to check the existence and uniqueness of system (25).

Theorem 4.1. *Let* $(\mathcal{X}, d_{\mathbf{C}})$ *be complete with respect to the metric* ($d_{\mathbf{C}}$ *defined by (26)). Assume that* $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *are defined by:*

$$\mathcal{F}\varrho(r) = \Lambda_w^U \varrho(r), \quad \mathcal{G}\varsigma(r) = \Lambda_z^V \varsigma(r), \ r \in [a, b].$$

Suppose that $\gamma = \| w - z \|_1$, *also, for* $r \in [a, b]$, $\mu_1, \mu_2 \in \mathbb{R}$, *we have;*

$$\begin{cases} |\mathcal{W}(r, \mu_1) - \mathcal{Z}(r, \mu_2)| \leq \phi_1(r) \mid \mu_1 - \mu_2 \mid, \\ |U(r, \varrho(s)) - V(r, \varsigma(s))| \leq \phi_2(r) \left[\frac{16|\varrho(s) - \varsigma(s)|}{4 + {\|\varrho(s) - \varsigma(s)\|_1}^2} - \frac{2\gamma}{\lambda(b-a)} \right], \end{cases} \tag{28}$$

where ϕ_1 *and* ϕ_2 *have upper bounds of* $\Gamma(\iota + 1)/8$ *and* $(b - a)^{-\iota}$ *on* $[a, b]$, *respectively. In this case, the Corollary 3.2 is satisfied for* $\mathcal{F}$ *and* $\mathcal{G}$. *Thus they have a CFP.*

Proof. Define $\psi(w) = \frac{w}{2}$, $\beta(r) = \frac{e^{2i\theta}}{e^{2i\theta} + t^2}$, for all $w \in \mathfrak{S}_0$ and $\alpha_{\mathbf{C}}(\varrho, \varsigma) = \sqrt{2}[\sin\theta_0 + (0.1)i\cos\theta_0]$, for all $\varrho, \varsigma \in \mathcal{X}$, where θ_0 is an arbitrary angel in $[\frac{\pi}{4}, \frac{\pi}{2}]$. Clearly, $\alpha_{\mathbf{C}}(\varrho, \varsigma) \succeq 1$, for each $\varrho, \varsigma \in \mathcal{X}$ and so the pair $(\mathcal{F}, \mathcal{G})$ is $\alpha_{\mathbf{C}}$-admissible. In addition, $\mathcal{X}$ has the property (B^*) and the subsets $\mathcal{F}\varrho$ and $\mathcal{G}\varsigma$ of $\mathcal{X}$ are proximinal. By using 28, for each $O \in \mathcal{F}\varrho$, $R \in \mathcal{G}\varsigma$, and $r \in [a, b]$, we have

$$\begin{aligned}
\| O(r) - R(r) \|_1 &= \int_a^b \mid O(r) - R(r) \mid dr \\
&= \int_a^b |w(r) + \mathcal{W}(r, I_a^\iota(U(r, \varrho(r)))) - z(r) - \mathcal{Z}(r, I_a^\iota(V(r, \varsigma(r))))| dr \\
&\leq \| w(r) - z(r) \|_1 + \int_a^b \phi_1(r) \mid I_a^\iota(U(r, \varrho(r))) - I_a^\iota(V(r, \varsigma(r))) \mid dr \\
&\leq \gamma + \frac{\Gamma(\iota+1)}{8} \int_a^b \mid \int_a^r \frac{(r-\zeta)^{\iota-1}}{\Gamma(\iota)} [U(\zeta, \varrho(\zeta)) - V(\zeta, \varsigma(\zeta))] \, d\zeta \mid dr \\
&\leq \gamma + \frac{\Gamma(\iota+1)}{8} \int_a^b \int_a^r \frac{(r-\zeta)^{\iota-1}}{\Gamma(\iota)} \mid U(\zeta, \varrho(\zeta)) - V(\zeta, \varsigma(\zeta)) \mid d\zeta \, dr \\
&\leq \gamma + \frac{\Gamma(\iota+1)}{8} \int_a^b \int_\zeta^b \frac{(r-\zeta)^{\iota-1}}{\Gamma(\iota)} \mid U(\zeta, \varrho(\zeta)) - V(\zeta, \varsigma(\zeta)) \mid dr \, d\zeta \\
&\leq \gamma + \frac{1}{8} \int_a^b (b-\zeta)^\iota \mid U(\zeta, \varrho(\zeta)) - V(\zeta, \varsigma(\zeta)) \mid d\zeta \\
&\leq \gamma + \frac{1}{8}(b-a)^\iota \int_a^b \mid U(\zeta, \varrho(\zeta)) - V(\zeta, \varsigma(\zeta)) \mid d\zeta \\
&\leq \gamma + \frac{1}{8}(b-a)^\iota \int_a^b \phi_2(\zeta) \left[\frac{16 \mid \varrho(\zeta) - \varsigma(\zeta) \mid}{4 + {\| \varrho(\zeta) - \varsigma(\zeta) \|_1}^2} - \frac{\gamma}{(b-a)} \right] d\zeta \\
&\leq \frac{1}{8} \int_a^b \frac{16 \mid \varrho(\zeta) - \varsigma(\zeta) \mid}{4 + {\| \varrho(\zeta) - \varsigma(\zeta) \|_1}^2} d\zeta \\
&= \frac{2\| \varrho(\zeta) - \varsigma(\zeta) \|_1}{4 + {\| \varrho(\zeta) - \varsigma(\zeta) \|_1}^2}.
\end{aligned}$$

So

$$\| O(r) - R(r) \|_1 e^{i\theta} \preceq \frac{2\| \varrho(\zeta) - \varsigma(\zeta) \|_1 e^{i\theta}}{4 + \| \varrho(\zeta) - \varsigma(\zeta) \|_1{}^2},$$

and then

$$d_{\mathbf{C}}(O, \xi) \preceq \beta(\psi(d_{\mathbf{C}}(\varrho, \varsigma)))\psi(d_{\mathbf{C}}(\varrho, \varsigma)) \preceq \frac{\beta(\psi(d_{\mathbf{C}}(\varrho, \varsigma)))\psi(d_{\mathbf{C}}(\varrho, \varsigma))}{\alpha_{\mathbf{C}}(\varrho, \varsigma)}.$$

Consequently, $\frac{\beta(\psi(d_{\mathbf{C}}(\varrho,\varsigma)))\psi(d_{\mathbf{C}}(\varrho,\varsigma))}{\alpha_{\mathbf{C}}(\varrho,\varsigma)} \in \mathfrak{S}(\mathcal{F}\varrho, \mathcal{G}\varsigma)$. Thus, from Corollary 3.2, $\mathcal{F}$ and $\mathcal{G}$ have a CFP in $\mathcal{X}$. Hence, the system (25) has a solution in $\mathcal{X}$. □

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