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MITTAG-LEFFLER-HYERS-ULAM STABILITY OF EULER-CAUCHY DIFFERENTIAL EQUATION USING LAPLACE TRANSFORM

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ABSTRACT. In this paper, we study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the Euler-Cauchy differential equation using Laplace transform. Basically, for the first time, the Mittag-Leffler-Hyers-Ulam stability of second order differential equation with variable coefficients has been studied through the Laplace transforms. We develop this approach to identify the necessary conditions that the eigenvalue Sturm-Liouville equation is Mittag-Leffler-Hyers-Ulam stable.

Keywords: Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, Euler-Cauchy equation, variable coefficients, Laplace transform, Sturm-Liouville equation.

AMS Subject Classification: 34K20, 26D10, 44A10, 39B82, 34A40, 39A30.

1. Introduction

The classical concept of Ulam stability for in theory of functional equation was first posed by M. Ulam (see [28]) in terms:"what are the conditions for a function which approximately satisfies a functional equation f must be close to an exact solution of f?". He presented many insolvable questions for example: when is it true that a mapping (isomorphism) that approximately satisfies a functional equation must be close to an exact solution of the equation?. If the answer is affirmative, then we could state that the functional equation is stable. One year later, Hyers answered the Ulam's outstanding question (see [9]) in the case of approximately additive mappings on Banach spaces. The Ulam stability theorem was generalized by Rassias (see [24], [25]). Then many authors studied Ulam stability for different problems of functional equations by different methods (see [9], [24] [11], [15], [8]). That is, there exists an exact unique solution of the Euler-Cauchy differential equation near its approximate solution. Where the exact solution of initial value problems for differential equations can be investigated using the Laplace transform method, Fourier transform method, Green's function approach, and other approximation methods which are consider to be valuable and essential techniques.(see [3], [4], [5], [6]).

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Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equations

$$x^{'}(t) + lx(t) = 0, \;\; x^{'}(t) + lx(t) = r(t),$$

$$x^{''}(t) + lx^{'}(t) + mx(t) = 0, \;\; x^{''}(t) + lx^{'}(t) + mx(t) = r(t),$$

were studied using Fourier transforms in (see [26]). By means of Fourier transform and convolution principle (see [19]), the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the homogeneous first-order differential equation and higher-order differential equations

$$H^{'}(x) + aH(x) = 0, \; H^{''}(x) + aH^{'}(x) + bH(x) = 0, \;\; \lim_{|x|\to\infty} = 0,$$

$$H^{''}(x) + \sum_{j=0}^{n-1} a_j H^j(x) = 0, \;\; \lim_{|x|\to\infty} = 0.$$

Moreover, the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the first-order differential equations

$$u^{'}(t) + lu(t) = 0, \;\; u^{'}(t) + lu(t) = r(t),$$

were studied by using Laplace transforms in (see [21]). Also by using Aboodh transforms, the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the second-order differential equations

$$u^{''}(t) + \mu^2 u(t) = 0, \;\; u^{''}(t) + \mu^2 u(t) = q(t),$$

were established (see [22]).

The Hyers-Ulam method was used to demonstrate the Mittag-Leffler stability problems for impulsive fractional difference equations (see [32]), investigate the stability of linear fractional delay difference equations with impulse (see [33]), and demonstrate the stability of fractional discrete-time neural networks using the fixed point method (see [34]).

Mittag-Leffler (see [18]) provided the Mittag-Leffler function $E_\alpha(z^\alpha)$ in relation to the divergent series approach. In [29], [2], [10], [14] the generalisation and characteristics of $E(z)$ were investigated. Some significant results to the Mittag-Leffler-Hyers-Ulam-Rassias stability have been studied in [13], [16], [17], [23], [1], [12], [27], [30], [7], [31], [20]. Especially, in [13] the authors investigated the Mittag-Leffler-Hyers-Ulam-Rassias stabilityfor the second-order differential equation

$$y^{''}(t) + \alpha y^{'}(t) + \beta y(t) = 0.$$

In this paper, we present a new concept regarding the stability of a second order differential equation with variable coefficients of type Euler-Cauchy equation in the sense of Mittag-Leffler-Hyers-Ulam by the Laplace transform method. Also, we apply our new approach results to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the eigenvalue Sturm-Liouville equation.

2. Problem statement and auxiliary assertions

Let $\mathbb{E}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{E}$ is said to be of exponential order c if there exist constants c, $M > 0$, $T > 0$ such that $|f(t)| \leq Me^{ct}$, $\forall t > T$. Essentially, in order for $f(t)$ to have Laplace transform provided

that $\lim_{t\to\infty} \dfrac{f(t)}{e^{ct}} = 0$.
The Laplace transform $\mathcal{L}(f) = F(s)$, and

$$F(s) = \int_0^\infty f(t)e^{-st}dt,$$

where there is $\delta \in \Re$ such that if $\Re(s) > \delta$, then this integral converges and diverges if $\Re(s) < \delta$, where $\Re(s)$ is the real part of s. Moreover, $|F(s)| \to 0$ as $\Re(s) \to \infty$.
The derivative of transform identity,

$$\frac{d}{ds}F(s) = -\int_0^\infty tf(t)e^{-st}dt = -\mathcal{L}(tf(t)), \quad (\Re(s) > \delta).$$

We will often use this identity to compute the following transforms:

$$\mathcal{L}(tf^{'}(t)) = -F(s) - s\frac{d}{ds}F(s),$$

$$\mathcal{L}(t^2f^{''}(t)) = s^2\frac{d^2}{ds^2}F(s) + 4s\frac{d}{ds}F(s) + 2F(s).$$

Since the integral Laplace transform is one-to-one mapping, then the inverse Laplace transform has the formula

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\xi)t}F(\alpha+i\xi)d\xi, \quad (\alpha > \delta).$$

The convolution of two non-zero Lebesgue measurable functions $f, z : (0,\infty) \to \mathbb{E}$ is given by

$$(f * z)(t) = \int_0^t f(t-\tau)z(\tau)d\tau,$$

that is,

$$\mathcal{L}(f * z)(t) = \mathcal{L}(f(t))\mathcal{L}(z(t)).$$

The aim of this paper is to investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability for the nonhomogeneous Euler-Cauchy differential equation

$$t^2u^{''}(t) + atu^{'}(t) + bu(t) = f(t), \quad t \in (0,\infty), \tag{1}$$

where a, b are constants in $\mathbb{E}$.
First, we may provide the following basic definitions concerning the Mittag-Leffler-Hyers-Ulam stability of the Euler-Cauchy differential equation (1):

Definition 2.1. *The Mittag-Leffler function of one parameter is defined as*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)}z^k, \quad Re(\alpha) > 0,\ z, \alpha \in \mathbb{C}.$$

Definition 2.2. *The two-parameter Mittag-Leffler function is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)}z^k, \quad Re(\alpha) > 0,\ z, \alpha \in \mathbb{C}.$$

If $\alpha = \beta = 1$, then

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)}z^k = e^z.$$

Definition 2.3. *We say that the Euler-Cauchy DE* (1) *has the Mittag-Leffler-Hyers-Ulam stability if there exists a constant $K > 0$ satisfying the following condition: If for every $\varepsilon > 0$, there exists a twice continuously differentiable function $u \in C^2(0,\infty)$ satisfying the inequality*

$$|t^2u^{''}(t) + atu^{'}(t) + bu(t) - f(t)| \leq \varepsilon E_\alpha(t),$$

for all $t \in (0,\infty)$, then there exists a solution $u_c(t)$ satisfying equation (1) *such that*

$$|u(t) - u_c(t)| \leq K\varepsilon E_\alpha(t),$$

for all $t \in (0,\infty)$, where K is the Mittag-Leffler-Hyers-Ulam stability constant of (1).

Definition 2.4. *We say that the Euler-Cauchy DE* (1) *has the Mittag-Leffler-Hyers-Ulam-Rassias stability with respect to a continuous $\varphi(t) : (0,\infty) \to (0,\infty)$ if there exists a constant $K_\varphi > 0$ such that for every $\varepsilon > 0$ and for each solution $u \in C^2(0,\infty)$ satisfying the inequality*

$$|t^2u^{''}(t) + atu^{'}(t) + bu(t) - f(t)| \leq \varepsilon\varphi(t)E_\alpha(t),$$

for all $t \in (0,\infty)$, then there exists a solution $u_c(t)$ satisfying equation (1) *such that*

$$|u(t) - u_c(t)| \leq K_\varphi\varepsilon\varphi(t)E_\alpha(t),$$

for all $t \in (0,\infty)$, where K_φ is the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for (1).

Theorem 2.1. *For any $z, \alpha \in \mathbb{C}, Re(\alpha) > 0$, the Laplace transform of Mittag-leffler function is*

$$\mathcal{L}(E_\alpha(t)) = \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\alpha k+1)} s^{-(k+1)}.$$

Proof. The Laplace transform of the Mittag-Leffler function is

$$\mathcal{L}(E_\alpha(t)) = \int_0^\infty E_\alpha(t)e^{st}dt = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \int_0^\infty t^k e^{st}dt.$$

Substitute $st = -z$, $sdt = -dz$, we get

$$\mathcal{L}(E_\alpha(t)) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \int_0^\infty \left(\frac{z}{s}\right)^k e^{-z}\frac{dz}{s} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} s^{-(k+1)} \int_0^\infty z^k e^{-z}dz.$$

Since $\int_0^\infty z^k e^{-z}dz = \Gamma(k+1) = k!$, we get

$$\mathcal{L}(E_\alpha(t)) = \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\alpha k+1)} s^{-(k+1)}.$$

□

3. Main results

3.1. Mittag-Leffler-Hyers-Ulam Stability for the Nonhomogeneous Euler-Cauchy Differential Equation.

In this section, we prove the Mittag-Leffler-Hyers-Ulam stability and generalized Hyers-Ulam stability of the Euler-Cauchy differential equation by using the Laplace transform.

Theorem 3.1. *The Euler-Cauchy differential equation* (1) *is Mittag-Leffler-Hyers-Ulam stable.*

Proof. For a given $\varepsilon > 0$, suppose that a twice continuously differentiable function $u : (0,\infty) \to \mathbb{E}$ satisfying the inequality

$$|t^2 u^{''}(t) + aty^{'}(t) + bu(t) - f(t)| \leq \varepsilon E_\alpha(t), \quad \forall t > 0. \tag{2}$$

Then we will prove that there exists a constant $K > 0$ independent of $\varepsilon, u(t)$ such that

$$|u(t) - u_c(t)| \leq K\varepsilon E_\alpha(t),$$

for some $u_c(t) \in C^2(0,\infty)$ satisfies the Euler-Cauchy DE (1) for all $t > 0$.
Define a function $p(t) : (0,\infty) \to \mathbb{E}$ such that

$$p(t) := t^2 u^{''}(t) + atu^{'}(t) + bu(t) - f(t)$$

for all $t > 0$, Taking Laplace transform to $p(t)$, yields

$$\mathcal{L}(p(t)) = S^2 \frac{d^2}{ds^2}\mathcal{L}(u(t)) + (4-a)s\frac{d}{ds}\mathcal{L}(u(t)) + (b-a+2)\mathcal{L}(u(t)) - \mathcal{L}(f(t)).$$

Assuming $\dfrac{d}{ds} = m$, then we get

$$\begin{aligned}
\mathcal{L}(p(t)) &= m^2 S^2 \mathcal{L}(u(t)) + (4-a)ms\mathcal{L}(u(t)) + (b-a+2)\mathcal{L}(u(t)) - \mathcal{L}(f(t)) \\
&= \big(m^2 s^2 + (4-a)ms + (b-a+2)\big)\mathcal{L}(u(t)) - \mathcal{L}(f(t)) \\
&= m^2\left[\left(s + \frac{4-a}{2m}\right)^2 + \frac{b-a}{m^2} - \left(\frac{a-2}{m}\right)^2\right]\mathcal{L}(u(t)) - \mathcal{L}(f(t)),
\end{aligned}$$

and so

$$\mathcal{L}(u(t)) = \frac{\mathcal{L}(p(t)) + \mathcal{L}(f(t))}{m^2\left[\left(s + \frac{4-a}{2m}\right)^2 + \frac{b-a}{m^2} - \left(\frac{a-2}{m}\right)^2\right]}. \tag{3}$$

We set

$$u_c(t) = \frac{1}{m\sqrt{(a-2)^2 + b - a}}\, e^{-\frac{4-a}{2m}t} \sinh\sqrt{\left(\frac{a-2}{m}\right)^2 + \left(\frac{b^2}{m^2}\right)}\, t * f(t).$$

Then $u_c(0) = u(0) = 0$ and

$$\mathcal{L}(u_c(t)) = \frac{\mathcal{L}(f(t))}{m^2\left[\left(s + \frac{4-a}{2m}\right)^2 + \frac{b-a}{m^2} - \left(\frac{a-2}{m}\right)^2\right]}. \tag{4}$$

Hence we get

$$\begin{aligned}
\mathcal{L}(t^2 u_c^{''}(t) + atu_c^{'}(t) + bu_c(t)) &= m^2 S^2 \mathcal{L}(u_c(t)) \\
&\quad + (4-a)ms\mathcal{L}(u_c(t)) + (b-a+2)\mathcal{L}(u_c(t)) = \mathcal{L}(f(t)).
\end{aligned}$$

Since $\mathcal{L}$ is one-to-one and linear, we deduce that

$$t^2 u_c^{''}(t) + atu_c^{'}(t) + bu_c(t) = f(t).$$

Thus $u_c(t)$ is a solution of Euler-Cauchy differential equation (1).
Applying (6) and (7) and considering

$$\mathcal{L}\left(\frac{1}{m\sqrt{(a-2)^2+b-a}}\,e^{-\dfrac{4-a}{2m}t}\sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\,t*p(t)\right)$$
$$=\frac{\mathcal{L}(p(t))}{m^2\left[\left(s+\dfrac{4-a}{2m}\right)^2+\dfrac{b-a}{m^2}-\left(\dfrac{a-2}{m}\right)^2\right]},$$

we obtain

$$\mathcal{L}(u(t))-\mathcal{L}(u_c(t))$$
$$=\mathcal{L}\left(\frac{1}{m\sqrt{(a-2)^2+b-a}}\,e^{-\dfrac{4-a}{2m}t}\sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\,t*p(t)\right),$$

and consequently

$$u(t)-u_c(t)=\frac{1}{m\sqrt{(a-2)^2+b-a}}\,e^{-\dfrac{4-a}{2m}t}\sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\,t*p(t).$$

In view of (1), it holds that $|p(t)|\le\varepsilon E_\alpha(t)$ and it follows from the convolution property that

$$|u(t)-u_c(t)|=\left|\frac{1}{m\sqrt{(a-2)^2+b-a}}\,e^{-\dfrac{4-a}{2m}t}\sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\,t*p(t)\right|.$$

Taking $\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}=\mathcal{B}$, then

$$|u(t)-u_c(t)|=\frac{1}{m\sqrt{(a-2)^2+b-a}}\left|e^{-\dfrac{4-a}{2m}t}\left(\frac{e^{\mathcal{B}t}-e^{-\mathcal{B}t}}{2}\right)*p(t)\right|$$

$$=\frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|\left(e^{-\left(\dfrac{4-a}{2m}-\mathcal{B}\right)t}-e^{-\left(\dfrac{4-a}{2m}+\mathcal{B}\right)t}\right)*p(t)\right|$$
$$=\frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|\int_0^t\left(e^{-\left(\dfrac{4-a}{2m}-\mathcal{B}\right)(t-\tau)}-e^{-\left(\dfrac{4-a}{2m}+\mathcal{B}\right)(t-\tau)}\right)p(\tau)d\tau\right|$$
$$\le\frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|e^{-\left(\dfrac{4-a}{2m}-\mathcal{B}\right)t}\int_0^t e^{\left(\dfrac{4-a}{2m}-\mathcal{B}\right)\tau}p(\tau)d\tau\right|$$
$$+\frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|e^{-\left(\dfrac{4-a}{2m}+\mathcal{B}\right)t}\int_0^t e^{\left(\dfrac{4-a}{2m}+\mathcal{B}\right)\tau}p(\tau)d\tau\right|.$$

Hence $|p(t)| \le |\varepsilon E_\alpha(t)|$, then

$$\begin{aligned}|u(t)-u_c(t)| &\le \frac{\varepsilon E_\alpha(t) e^{-\left(\frac{4-a}{2m}-\mathcal{B}\right)t}}{2m\sqrt{(a-2)^2+b-a}}\int_0^t e^{\left(\frac{4-a}{2m}-\mathcal{B}\right)\tau}d\tau \\ &+ \frac{\varepsilon E_\alpha(t) e^{-\left(\frac{4-a}{2m}+\mathcal{B}\right)t}}{2m\sqrt{(a-2)^2+b-a}}\int_0^t e^{\left(\frac{4-a}{2m}+\mathcal{B}\right)\tau}d\tau \\ &\le K\varepsilon E_\alpha(t),\end{aligned}$$

where the integrals $\int_0^t e^{\left(\frac{4-a}{2m}-\mathcal{B}\right)\tau}d\tau$, $\int_0^t e^{\left(\frac{4-a}{2m}+\mathcal{B}\right)\tau}d\tau$ exist. Then according to Definition 3, the Euler-Cauchy differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability. □

3.2. **Mittag-Leffler-Hyers-Ulam-Rassias Stability for the Nonhomogeneous Euler-Cauchy Differential Equation.**

Now, we come to prove the Mittag-Leffler-Hyers-Ulam-Rassias stability of the Euler-Cauchy differential equation (1). Although the proof follows a similar way to that of Theorem 1, but we include it for completeness.

Theorem 3.2. *The Euler-Cauchy differential equation* (1) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Given $\varepsilon > 0$, we suppose that $u(t) \in C^2(0,\infty)$ and a function $\varphi(t) : (0,\infty) \to (0,\infty)$ satisfies

$$|t^2u^{''}(t) + aty^{'}(t) + bu(t) - f(t)| \le \varphi(t)\varepsilon E_\alpha(t), \quad \forall t > 0. \tag{5}$$

Then we aim to prove that there exists a real number $K > 0$ independent of ε and $u(t)$ such that

$$|u(t) - u_c(t)| \le K\varphi\varepsilon E_\alpha(t),$$

for some $u_c(t) \in C^2(0,\infty)$ satisfies the Euler-Cauchy DE (1) for all $t > 0$.
Define a function $p(t) : (0,\infty) \to \mathbb{E}$ such that

$$p(t) =: t^2u^{''}(t) + atu^{'}(t) + bu(t) - f(t)$$

for all $t > 0$, Taking Laplace transform to $p(t)$, then

$$\mathcal{L}(p(t)) = S^2\frac{d^2}{ds^2}\mathcal{L}(u(t)) + (4-a)s\frac{d}{ds}\mathcal{L}(u(t)) + (b-a+2)\mathcal{L}(u(t)) - \mathcal{L}(f(t)),$$

Taking $\frac{d}{ds} = m$, then

$$\begin{aligned}\mathcal{L}(p(t)) &= m^2S^2\mathcal{L}(u(t)) + (4-a)ms\mathcal{L}(u(t)) + (b-a+2)\mathcal{L}(u(t)) - \mathcal{L}(f(t)) \\ &= \big(m^2s^2 + (4-a)ms + (b-a+2)\big)\mathcal{L}(u(t)) - \mathcal{L}(f(t)) \\ &= m^2\left[\left(s+\frac{4-a}{2m}\right)^2 + \frac{b-a}{m^2} - \left(\frac{a-2}{m}\right)^2\right]\mathcal{L}(u(t)) - \mathcal{L}(f(t)),\end{aligned}$$

hence

$$\mathcal{L}(u(t)) = \frac{\mathcal{L}(p(t)) + \mathcal{L}(f(t))}{m^2\left[\left(s+\frac{4-a}{2m}\right)^2 + \frac{b-a}{m^2} - \left(\frac{a-2}{m}\right)^2\right]}. \tag{6}$$

Let us check that the solution changes continuously with change in data. So if we set

$$u_c(t) = \frac{1}{m\sqrt{(a-2)^2+b-a}}\, e^{-\dfrac{4-a}{2m}t} \sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\, t * f(t).$$

Then $u_c(0) = u(0) = 0$ and

$$\mathcal{L}(u_c(t)) = \frac{\mathcal{L}(f(t))}{m^2\left[\left(s+\dfrac{4-a}{2m}\right)^2+\dfrac{b-a}{m^2}-\left(\dfrac{a-2}{m}\right)^2\right]}. \tag{7}$$

Hence, we obtain

$$\begin{aligned}\mathcal{L}(t^2u_c^{''}(t)+atu_c^{'}(t)+bu_c(t)) &= m^2S^2\mathcal{L}(u_c(t))\\ &\quad + (4-a)ms\mathcal{L}(u_c(t)) + (b-a+2)\mathcal{L}(u_c(t)) = \mathcal{L}(f(t)).\end{aligned}$$

Since the integral Laplace transform $\mathcal{L}$ is one-to-one operator, then

$$t^2u_c^{''}(t)+atu_c^{'}(t)+bu_c(t) = f(t).$$

Thus $u_c(t)$ is a solution of equation (1).
Applying (6) and (7) and considering

$$\begin{aligned}&\mathcal{L}\left(\frac{1}{m\sqrt{(a-2)^2+b-a}}\, e^{-\dfrac{4-a}{2m}t} \sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\, t * p(t)\right)\\ &= \frac{\mathcal{L}(p(t))}{m^2\left[\left(s+\dfrac{4-a}{2m}\right)^2+\dfrac{b-a}{m^2}-\left(\dfrac{a-2}{m}\right)^2\right]},\end{aligned}$$

we have

$$\begin{aligned}&\mathcal{L}(u(t)) - \mathcal{L}(u_c(t))\\ &= \mathcal{L}\left(\frac{1}{m\sqrt{(a-2)^2+b-a}}\, e^{-\dfrac{4-a}{2m}t} \sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\, t * p(t)\right).\end{aligned}$$

Then

$$u(t) - u_c(t) = \frac{1}{m\sqrt{(a-2)^2+b-a}}\, e^{-\dfrac{4-a}{2m}t} \sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\, t * p(t).$$

Regard to equation (1), it holds that $|p(t)| \le \varepsilon\varphi E_\alpha(t)$ and it follows from the definition of the convolution that

$$|u(t) - u_c(t)| = \left|\frac{1}{m\sqrt{(a-2)^2+b-a}}\, e^{-\dfrac{4-a}{2m}t} \sinh\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)}\, t * p(t)\right|.$$

Taking $\sqrt{\left(\frac{a-2}{m}\right)^2+\left(\frac{b^2}{m^2}\right)} = \mathcal{B}$, then

$$|u(t) - u_c(t)| = \frac{1}{m\sqrt{(a-2)^2+b-a}}\left|e^{-\dfrac{4-a}{2m}t}\left(\frac{e^{\mathcal{B}t}-e^{-\mathcal{B}t}}{2}\right) * p(t)\right|$$

$$= \frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|\left(e^{-\left(\frac{4-a}{2m}-\mathcal{B}\right)t} - e^{-\left(\frac{4-a}{2m}+\mathcal{B}\right)t}\right) * p(t)\right|$$

$$= \frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|\int_0^t \left(e^{-\left(\frac{4-a}{2m}-\mathcal{B}\right)(t-\tau)} - e^{-\left(\frac{4-a}{2m}+\mathcal{B}\right)(t-\tau)}\right)p(\tau)d\tau\right|$$

$$\leq \frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|e^{-\left(\frac{4-a}{2m}-\mathcal{B}\right)t}\int_0^t e^{\left(\frac{4-a}{2m}-\mathcal{B}\right)\tau}p(\tau)d\tau\right|$$

$$+ \frac{1}{2m\sqrt{(a-2)^2+b-a}}\left|e^{-\left(\frac{4-a}{2m}+\mathcal{B}\right)t}\int_0^t e^{\left(\frac{4-a}{2m}+\mathcal{B}\right)\tau}p(\tau)d\tau\right|.$$

Hence

$$|p(t)| \leq |\varepsilon\varphi(t)E_\alpha(t)|,$$

$$|u(t) - u_c(t)| \leq \frac{\varepsilon\varphi(t)E_\alpha(t)e^{-\left(\frac{4-a}{2m}-\mathcal{B}\right)t}}{2m\sqrt{(a-2)^2+b-a}}\int_0^t e^{\left(\frac{4-a}{2m}-\mathcal{B}\right)\tau}d\tau$$

$$+ \frac{\varepsilon\varphi(t)E_\alpha(t)e^{-\left(\frac{4-a}{2m}+\mathcal{B}\right)t}}{2m\sqrt{(a-2)^2+b-a}}\int_0^t e^{\left(\frac{4-a}{2m}+\mathcal{B}\right)\tau}d\tau$$

$$\leq K\varepsilon\varphi(t)E_\alpha(t),$$

where the integrals $\int_0^t e^{\left(\frac{4-a}{2m}-\mathcal{B}\right)\tau}d\tau$, $\int_0^t e^{\left(\frac{4-a}{2m}+\mathcal{B}\right)\tau}d\tau$ exist. Then from Definition 4, the nonhomogeneous Euler-Cauchy differential equation (1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

3.3. **Example. Mittag-Leffler-Hyers-Ulam stability of Sturm-Liouville equation.**

Sturm and Liouville published numerous papers on second order linear differential equations between 1836 and 1837. Their articles had such a profound influence that this topic evolved into Sturm-Liouville theory. Since then, many thousands of different papers have been produced. Yet, surprisingly, there is a lot of study being done on this topic right now. Every year, hundreds of articles on Sturm-Liouville problems are published. Our primary interest in this section is to investigate the stability problem of Sturm-Liouville equation. So we provide the following example as applied result to our research. In this example we study the Mittag-Leffler-Hyers-Ulam stability of eigenvalue Sturm-Liouville equation:

$$\left(x^2y'\right)' + \lambda y = f(x), \qquad x \in (1,2), \tag{8}$$

where λ is its eigenvalue. The eigenvalues of Sturm-Liouville equation are thats values for which the nonzero solution exists.
Equation (8) involves the following Euler-Cauchy equation:

$$x^2y'' + 2xy' + \lambda y = f(x), \quad 1 < x < 2, \tag{9}$$

baised on the above results, we can completely prove that the eigenvalue Sturm-Liouville equation (9) is Mittag-Leffler-Hyers-Ulam stable and determined the necessary condition that eigenvalue Sturm-Liouville equation has no trivial solutions. If a twice continuously differentiable function $y : (0,\infty) \to \mathbb{E}$ satisfies the inequality

$$|x^2 y^{''}(x) + 2xy^{'}(x) + \lambda y(x) - f(x)| \leq \varepsilon E_\alpha(t), \quad \forall\ 1 < x < 2, \tag{10}$$

for some $\varepsilon > 0$ and $\lambda \neq 2$, then there exists a solution $y_c(x) : (0,\infty) \to \mathbb{E}$ of the Euler-Cauchy DE (9) such that

$$|y(x) - y_c(x)| \leq \frac{\varepsilon E_\alpha(t) e^{-\left(\frac{1-\lambda}{m}\right)x}}{2m\sqrt{\lambda - 2}} \int_1^x e^{\left(\frac{1-\lambda}{m}\right)\xi} d\xi + \frac{\varepsilon E_\alpha(t) e^{-\left(\frac{1+\lambda}{m}\right)x}}{2m\sqrt{\lambda - 2}} \int_1^x e^{\left(\frac{1+\lambda}{m}\right)\xi} d\xi.$$

Thus the integrals

$$\int_1^x e^{\left(\frac{1-\lambda}{m}\right)\xi} d\xi, \qquad \int_1^x e^{\left(\frac{1+\lambda}{m}\right)\xi} d\xi$$

exist.
Then

$$|y(x) - y_c(x)| \leq K\varepsilon E_\alpha(t)$$

for all $1 < x < 2$.

4. Conclusion

In this study, we developed the Laplace transform method as a new approach to investigate the Mittag-Leffler-Hyers-Ulam stability and the generalized Mittag-Leffler-Hyers-Ulam stability of Euler-Cauchy differential equation (with variable coefficients). We demonstrated the Mittag-Leffler-Hyers-Ulam stability of eigenvalue Sturm-liouville equation as an application.

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