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ON A COUPLED HYBRID SYSTEM OF ORDINARY SECOND-ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this work, the existence of solutions for coupled systems of ordinary second-order hybrid functional differential blueequations (CSHDE) is concerned, due to Dhage's hybrid fixed point theorem. Continuous dependence of the solution of our problem will be proven on delay functions. To demonstrate the produced outcome, an example is provided.

Keywords: Nonlinear functional integral equation, existence of continuous solution, Dhage fixed point Theorem, continuous dependence.

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1. Introduction

The significance of the HDEs that appear in many dynamical systems is of special attention. The treatment of Dhage's research includes HDE [1], and it has been fully addressed in numerous articles on hybrid differential equations of various perturbations [2], [3], [4], [5], and [6], as well as references therein.

The original differential equations are perturbed in various ways in this class of HDE. Many studies on the theory of HDE have been published, and we recommend the papers [7, 8, 9, 10, 11, 12, 13]. The problem of the coupled hybrid fractional differential equations

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(CHFDE) was investigated by Hannabou et al [14].

$$\begin{cases} {}^{C}D^{\alpha}\big(\frac{\mu(t)}{f_1(t,\mu(t),\nu(t))}\big) = \hbar_1(t,\mu(t),\nu(t)),\ a.e.,\ t\in[0,T] \quad 0<\alpha\le 1 \\ {}^{C}D^{\beta}\big(\frac{\nu(t)}{f_2(t,y(t),\mu(t))}\big) = \hbar_2(t,\nu(t),\mu(t)),\ a.e.,\ t\in[0,T] \quad 0<\beta\le 1 \\ a\frac{\mu(0)}{f_1(0,\mu(0),\nu(0))} + b\frac{\mu(T)}{f_1(T,\mu(T),\nu(T))} = c, \\ a\frac{\mu(0)}{f_2(0,\mu(0),\nu(0))} + b\frac{\mu(T)}{f_2(T,\mu(T),\nu(T))} = c, \end{cases}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ respectively indicate the fractional orders α and β of the Caputo derivative.
The corresponding Dirichlet boundary value problem CHFDE was studied by Bashir et al. [15]:

$$\begin{cases} {}^{c}D^{\delta}\big(\frac{\mu(t)}{f_1(t,\mu(t),\nu(t))}\big) = \hbar_1(t,\mu(t),\nu(t)),\ 0<t<1 \quad 1<\alpha\le 2 \\ {}^{c}D^{\omega}\big(\frac{\nu(t)}{f_2(t,\nu(t),\mu(t))}\big) = \hbar_2(t,\nu(t),\mu(t)),\ 0<t<1 \quad 1<\omega\le 2 \\ \mu(0)=\mu(1)=0,\ \nu(0)=\nu(1)=0. \end{cases}$$

Dhage and et al. investigate the existence of coupled solutions to the nonhomogeneous boundary value problem of coupled hybrid integro-differential equations of fractional order in [16].

$$\begin{cases} {}^{c}D^{\omega}\big(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t),y(t))}{f(t,x(t),y(t))}\big) = \phi(t,x(t),y(t)) \quad a.e.\ t\in[0,1], \\ {}^{c}D^{\delta}\big(\frac{x(t)-\sum_{j=1}^{m} I^{\gamma_i}k_j(t,x(t),y(t))}{g(t,x(t),y(t))}\big) = \psi(t,y(t),x(t)) \quad a.e.\ t\in[0,1], \\ x(0)=a,\ x(1)=b,\ y(0)=c,\ y(1)=d, \end{cases} \tag{1}$$

We investigate the following initial value problem of coupled hybrid second-order differential equations (CSHDE), which is driven by current research on coupled systems of HDE.

$$\begin{cases} \frac{d^2}{dt^2}\big(\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\big) = g_1(t,x(t),y(\varphi_3(t))), \quad t\in J=[0,T], \\ \frac{d^2}{dt^2}\big(\frac{y(t)-h_2(t,y(\varphi_1(t)))}{f_2(t,y(\varphi_2(t)))}\big) = g_2(t,y(t),x(\varphi_3(t))), \quad t\in J=[0,T], \\ x(0)=h_1(0,x(0)),\ y(0)=h_2(0,y(0)) \text{ and} \\ x'(0)=\frac{dh_1}{dt}|_{t=0},\ y'(0)=\frac{dh_2}{dt}|_{t=0}, \end{cases} \tag{2}$$

and establish the existence and uniqueness results where $f_i \in \mathcal{C}(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g_i \in \mathcal{C}(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $h_i\in\mathcal{C}(J\times\mathbb{R},\mathbb{R})$, $(i=1,2)$, and $\varphi_i\in\mathcal{C}(J)$ with $\varphi_j(0)=0$, $(j=1,2,3)$. We define the solution of CSHDE (2) to be a pair of functions $(x,y)\in\mathcal{C}(J,\mathbb{R})\times\mathcal{C}(J,\mathbb{R})$ satisfying

(i) the functions $t\to\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}$ and $t\to\frac{y(t)-h_2(t,y(\varphi_1(t)))}{f_2(t,y(\varphi_2(t)))}$, $x,y\in\mathcal{C}(J,\mathbb{R})$ are continuous, and

(ii) the system of equations in (2) is satisfied by (x, y).

To prove our result, we use the standard hybrid fixed point theory developed in [2]-[13]. The unique solution's continuous dependency on the delay functions will be investigated.

The following is a breakdown of the paper's structure: We'll go through some helpful hypotheses and preliminaries in Section 2. In Section 3, we show an additional Theorem connected to the linear variation of the issue (2), and present sufficient criteria that verify our main result for CSHDE (2). In Section 4, we show that solutions are continuously dependent on delay functions, and an illustrated example will be provided to demonstrate our findings.

2. Hypotheses and Auxiliary Results

Here, we present our fundamental result for CSHDE (2), which is based on Dhage's fixed point theorems [1]

Theorem 2.1. [1] *Let S be a nonempty, closed convex and bounded subset of a Banach algebra X and let $A, C : X \to X$ and $B : S \to X$ be three operators such that:*

(a) *A and C are Lipschitzian with Lipschitz constants δ and ρ, respectively,*
(b) *B is compact and continuous,*
(c) *$x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*
(d) *$\delta\ M\ + \rho < r$, for $r > 0$ where $M = \|B(S)\|$.*

Then the operator equation $AxBx + Cx = x$ has a solution in S.

Consider the following assumptions:

(A_1) $f_i : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, and $h_i : J \times \mathbb{R} \to \mathbb{R}$, are continuous and there exist four functions $k_i, L_i \in \mathcal{C}(J, \mathbb{R}_+)$, with norm $\|k_i\|$ and $\|L_i\|$, respectively $(i = 1, 2)$, such that

$$|h_i(t, \mu) - h_i(t, \nu)| \leq k_i(t)|\mu - \nu|,$$
$$|f_i(t, \mu) - f_i(t, \nu)| \ \leq L_i(t)|\mu - \nu|,$$

for all $t \in J$ and $\mu, \nu \in \mathbb{R}$ with $\|k\| = \max\{\|k_1\|, \|k_2\|\}$, and $\|L\| = \max\{\|L_1\|, \|L_2\|\}$.

(A_2) $g_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. There exist functions $p_i \in \mathcal{C}(J, \mathbb{R}_+)$ and four continuous nondecreasing functions $\Phi_i : [0, \infty) \to (0, \infty)$, and $\Psi_i : [0, \infty) \to (0, \infty)$, $(i = 1, 2)$ with

$$|g_i(t, \mu, \nu)| \leq\ p_i(t)\Phi_i(|\mu|)\Psi_i(|\nu|),\ \forall\ (t, \mu, \nu)\ \in J \times \mathbb{R} \times \mathbb{R},$$

with $\|p\| = \max\{\|p_1\|, \|p_2\|\}$, $\Phi(r) = \max\{\Phi_1(|\mu|)|, \Phi_2(|\nu|)\}$,
and $\Psi(r) = \max\{\Psi_1(|\mu|)|, \Psi_2(|\nu|)\}$.

(A_3) The functions $\varphi_j : J \to J$, are continuous, with $\varphi_j(0) = 0,\ j = 1, 2, 3$.

(A_4) There is a real number $r > 0$ for which

$$r \geq \frac{2\ F\ \|p\|\Phi(r)\Psi(r)\ \frac{T^2}{2}\ + H_1 + H_2}{1 - \left(\|p\|\Phi(r)\Psi(r)\ \|L\|\frac{T^2}{2}\ +\ \|k\|\right)}, \tag{3}$$

with $F_i = \sup\limits_{t \in J} |f_i(t, 0)|$, $F = \max\{F_1, F_2\}$, $H_i = \sup_{t \in J} |h_i(t, 0)|$, $(i = 1, 2)$, and

$$\|p\|\Phi(r)\Psi(r)\|L\|\frac{T^2}{2}\ + \|k\| < 1. \tag{4}$$

The following lemma is used in the proof of the main existence result.

Lemma 2.1. *Assume that hypotheses* $(A_1)-(A_4)$ *holds. Then* $x \in \mathcal{C}(J,\mathbb{R})$ *is a solution of the SHDE* (5),

$$\begin{cases} \frac{d^2}{dt^2}\big(\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\big) = g_1(t,x(t),y(\varphi_3(t))) \quad t \in J=[0,T], \\ \\ x(0)=h_1(0,x(0)), \ \ \textit{and } x'(0)=\frac{dh_1}{dt}|_{t=0}, \end{cases} \tag{5}$$

if and only if it is equivalent to the quadratic integral equation

$$x(t) = h_1(t,x(\varphi_1(t))) \ + f_1(t,x(\varphi_2(t))) \int_0^t (t-s) g_1(s,x(s),y(\varphi_3(s))) \ ds. \tag{6}$$

Proof. First step, allowing x to be a solution of the SHDE (5), we obtain by integrating from 0 to t, for two sides of (5)

$$\frac{d}{dt}\left(\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\right) - \frac{d}{dt}\left(\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\right)\Bigg|_{t=0} = \int_0^t g_1(s,x(s),y(\varphi_3(s)))ds.$$

However, since $f_1(0,x(0)) \neq 0$ and $\varphi_j(0)=0$, $j=1,2,3$.

$$\begin{aligned} &\frac{d}{dt}\left(\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\right)\Bigg|_{t=0} \\ &= \frac{f_1(0,x(0))\big(x'(0)-\frac{dh_1}{dt}\big|_{t=0}\big) - \big(x(0)-h_1(0,x(0))\big)\frac{df_1}{dt}\big|_{t=0}}{f_1^2(0,x(0))} = 0. \end{aligned}$$

Using the boundary conditions

$$\begin{aligned} x(0) &= h_1(0,x(0)), \\ x'(0) &= \frac{dh_1}{dt}\big|_{t=0}. \end{aligned}$$

Hence, we obtain

$$\frac{d}{dt}\left(\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\right) = \int_0^t g_1(s,x(s),y(\varphi_3(s))) \ ds. \tag{7}$$

When we integrate two sides of (7) from 0 to t, we get

$$\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))} - \frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\Bigg|_{t=0} = \int_0^t (t-s)g_1(s,x(s),y(\varphi_3(s)))ds, \tag{8}$$

as well as

$$\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))}\Bigg|_{t=0} = \frac{x(0)-h_1(0,x(0))}{f_1(0,x(0))} = 0.$$

As a result, eq.(8) yields

$$\frac{x(t)-h_1(t,x(\varphi_1(t)))}{f_1(t,x(\varphi_2(t)))} = \int_0^t (t-s) \ g_1(s,x(s),y(\varphi_3(s))) \ ds,$$

i.e.,

$$x(t) = h_1(t,x(\varphi_1(t))) \ + f_1(t,x(\varphi_2(t))) \int_0^t (t-s) \ g_1(s,x(s),y(\varphi_3(s))) \ ds.$$

Thus, eq.(6) holds.

Conversely, assume, on the other hand, that x fulfills eq.(6). Then, for both sides of eq.(6), we divide by $f_1(t, x(\varphi_2(t)))$ and make direct differentiation.

$$\frac{d}{dt}\left(\frac{x(t) - h_1(t, x(\varphi_1(t)))}{f_1(t, x(\varphi_2(t)))}\right) = \int_0^t g_1(s, x(s), y(\varphi_3(s)))\ ds.$$

Therefore equation (5) can be verified by direct differentiation.

$$\frac{d^2}{dt^2}\left(\frac{x(t) - h_1(t, x(\varphi_1(t)))}{f_1(t, x(\varphi_2(t)))}\right) = g_1(t, x(t), y(\varphi_3(t))).$$

Thus, inserting $t = 0$ in equation (6), gives

$$\frac{x(0) - h_1(0, x(0))}{f_1(0, x(0))} \to 0 \text{ as } t \to 0,$$

therefore $x(0) = h_1(0, x(0))$, and

$$\frac{d}{dt}\left(\frac{x(t) - h_1(t, x(\varphi_1(t)))}{f_1(t, x(\varphi_2(t)))}\right)\Bigg|_{t=0} = 0$$

$$\frac{f_1(t, x(\varphi_2(t)))(x'(t) - \frac{dh_1}{dt}) - (x(t) - h_1(t, x(t)))\frac{df_1}{dt}}{f_1^2(t, x(\varphi_2(t)))}\Bigg|_{t=0} = 0$$

$$f_1(0, x(0))\big(x'(0) - \frac{dh_1}{dt}|_{t=0}\big) - \big(x(0) - h_1(0, x(0))\big)\frac{df_1}{dt}\Bigg|_{t=0} = 0.$$

As we established that $x(0) = h_1(0, x(0))$, we may deduce that $x'(0) = \frac{dh_1}{dt}|_{t=0}$. □

2.1. **Existence of solution.**

Theorem 2.2. *Assume that the hypotheses $(A_1) - (A_4)$ holds. Then the coupled system (2) has at least one solution defined on $J \times J$.*

Proof. In view of Lemma 2.1, the solutions of the CSHDE (2)

$$x(t) = h_1(t, x(\varphi_1(t)))\ + f_1(t, x(\varphi_2(t))) \int_0^t (t-s)\ g_1(s, x(s), y(\varphi_3(s)))\ ds, \tag{9}$$

$$y(t) = h_2(t, y(\varphi_1(t)))\ + f_2(t, y(\varphi_2(t))) \int_0^t (t-s)\ g_2(s, y(s), x(\varphi_3(s)))\ ds. \tag{10}$$

Construct a subset $\mathcal{S} = (S_1, S_2)$ of the Banach space $E = \mathcal{C}(J, \mathbb{R}) \times \mathcal{C}(J, \mathbb{R})$ by

$$\mathcal{S} := \{(x, y) \in E,\ \|(x, y)\| \leq r\},$$

where r satisfies the inequality given in (3).

Obviously, $\mathcal{S}$ is bounded convex, and a closed subset of the Banach space E. In conjunction with the functions f_i, g_i and h_i, $(i = 1, 2)$, we introduce the three operators $A = (A_1, A_2) : E \to E$, $C = (C_1, C_2) : E \to E$ and $B = (B_1, B_2) : S \to E$ defined by

$$\begin{aligned}
A(x, y) &= (A_1 x, A_2 y) = \big(f_1(t, x(\varphi_2(t))), f_2(t, y(\varphi_2(t)))\big),\\
B(x, y) &= (B_1 x, B_2 y)\\
&= \left(\int_0^t (t-s)\ g_1(s, x(s), y(\varphi_3(s)))ds, \int_0^t (t-s)\ g_2(s, y(s), x(\varphi_3(s)))ds\right),\\
C(x, y) &= (C_1 x, C_2 y) = \big(h_1(t, x(\varphi_1(t))), h_2(t, y(\varphi_1(t)))\big).
\end{aligned}$$

The coupled system of functional integral equations (9) and (10) can thus be expressed as a system of operator equations, as shown below

$$A(x, y)(t) \cdot B(x, y)(t) + C(x, y)(t) = (x, y)(t),\ t \in J, \tag{11}$$

In addition, due to the multiplication of two components in E gives

$$\big(A_1x(t)\cdot B_1y(t)+C_1x(t),A_2y(t)\cdot B_2x(t)+C_2y(t)\big)=(x,y)(t),\ t\in J. \tag{12}$$

Furthermore, this indicates that

$$\begin{cases} A_1x(t)\cdot B_1x(t)+C_1x(t)=x(t),\ t\in J,\\ \\ A_2y(t)\cdot B_2y(t)+C_2y(t)=y(t),\ t\in J.\end{cases} \tag{13}$$

We shall demonstrate that $A,\ B$ and C meet all requirements of B.C. Dhage fixed-point Theorem [17]. This will be accomplished in the steps that follow.

Step 1. We shall demonstrate that $A=(A_1,A_2)$ and $C=(C_1,C_2)$ are Lipschitzian on E. So, we shall demonstrate that $A_i,\ C_i$ are lipschitzian on $\mathcal{C}(J,\mathbb{R}),\ i=1,2$. Let $x,y\in\mathcal{C}(J,\mathbb{R})$. Then by (A_1), we have

$$\begin{aligned} |A_ix(t)-A_iy(t)| &= |f_i(t,x(\varphi_2(t)))-f_i(t,y(\varphi_2(t)))|\\ &\le L_i(t)\ |x(\varphi_2(t))-y(\varphi_2(t))|\le\|L_i\|\ \|x-y\|.\end{aligned}$$

As a result, for any $t\in J$. If we take the supremum over t, we get

$$\|A_ix-A_iy\|\le\|L_i\|\ \|x-y\|.$$

Therefore, A_i is a Lipschitzian on X with Lipschitz constant $\|L_i\|$.
Similarly, for any $x,y\in\mathcal{C}(J,\mathbb{R})$, we get

$$\begin{aligned} |C_ix(t)-C_iy(t)| &= |h_i(t,x(\varphi_1(t)))-h_i(t,y(\varphi_1(t)))|\\ &\le k_i(t)|x(\varphi_1(t))-y(\varphi_1(t))|\ \le\ \|k_i\|\ \|x-y\|.\end{aligned}$$

As a result, for any $t\in J$. If we take the supremum over t, we get

$$\|C_ix-C_iy\|\le\ \|k_i\|\ \|x-y\|.$$

This demonstrates that C_i is a Lipschitz mapping on X with the Lipschitz constant $\|k_i\|$, which yields, for all $u=(x,y),\ v=(\bar{x},\bar{y})\in E$, the operator A is defined as follows:

$$\begin{aligned} Au-Av &= A(x,y)-A(\bar{x},\bar{y})\\ &= (A_1x,A_2y)-(A_1\bar{x},A_2\bar{y})\\ &= (A_1x-A_1\bar{x}\ ,\ A_2y-A_2\bar{y}),\end{aligned}$$

then

$$\begin{aligned} \|Au-Av\| &= \|(A_1x-A_1\bar{x},A_2y-A_2\bar{y})\|\\ &\le \|A_1x-A_1\bar{x}\|+\|A_2y-A_2\bar{y})\|\\ &\le \|L_1\|\ \|x-\bar{x}\|+\|L_2\|\ \|y-\bar{y}\|\\ &\le \|L\|\ \|x-\bar{x}\|+\|L\|\ \|y-\bar{y}\|\\ &\le \|L\|\ \big(\|x-\bar{x}\|+\ \|y-\bar{y}\|\big)\\ &\le \|L\|\ \|u-v\|,\end{aligned}$$

which shows that A is Lipschitzian, as shown by the Lipschitz constant $\|L\|$. Likewise, we have C is Lipschitzian with Lipschitz constant $\|k\|$ as well.

Step 2.To show that $B=(B_1,B_2)$ is completely continuous operator on $\mathcal{S}$ into E.
We begin by demonstrating that B is continuous on E. Assume that the sequence of

points (x_n, y_n) in $\mathcal{S}$ converging to a point $(x, y) \in E$ which satisfies $\{(x_n, y_n)\} \to (x, y)$ as $n \to \infty$. Then, by Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{n\to\infty} B_1 x_n(t) &= \lim_{n\to\infty} \int_0^t (t-s)\ g_1(s, x_n(s), y(\varphi_3(s))\ ds \\ &= \int_0^t (t-s)\ g_1(s, x(s), y(\varphi_3(s))\ ds \\ &= B_1 x(t).\end{aligned}$$

As a result, the operator B_1 is a continuous one. B_2 can also be checked regularly.

$$\lim_{n\to\infty} B_2 y_n(t) = B_2 y(t).$$

Hence

$$\begin{aligned}\lim_{n\to\infty} Bu_n(t) &= (\lim_{n\to\infty} B_1 x_n, \lim_{n\to\infty} B_2 y_n) \\ &= (B_1 x(t), B_2 y(t)) \\ &= B(x, y) \\ &= Bu(t).\end{aligned}$$

Thus, $Bu_n \to Bu$ as $n \to \infty$ uniformly on $\mathbb{R}_+$. As a result, B on $\mathcal{S}$ into $\mathcal{S}$ is a continuous operator. We can now prove that B is a compact operator on $\mathcal{S}$. It is sufficient to demonstrate that $B(\mathcal{S})$ is a uniformly bounded and equi-continuous set in E. Allow $(x, y) \in \mathcal{S}$ be arbitrary to demonstrate this. Then by hypothesis (A_2),

$$\begin{aligned}|B_1 x(t)| &\leq \int_0^t (t-s)\ |g_1(s, x(s), y(\varphi_3(s))|ds \\ &\leq \int_0^t (t-s)\ p_1(t)\ \Phi_1(|x|)\ \Psi_1(|y|)]ds \\ &\leq \|p_1\|\ \Phi_1(|x|)\ \Psi_1(|y|) \int_0^t (t-s)ds \\ &\leq \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}.\end{aligned}$$

If we take the supremum over $t \in J$, we get

$$\|B_1 x(t)\| \leq |p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2},$$

for all $x \in \mathcal{S}$. This demonstrates that B_1 is uniformly bounded on $\mathcal{S}$.
In the same way, one can find that

$$\|B_2 y(t)\| \leq \|p\|\ \Phi(r)\ \Psi(r)\frac{T^2}{2}.$$

We get, for $u = (x, y) \in \mathcal{S}$

$$\begin{aligned}\|Bu\| = \|B(x, y)\| &= \|(B_1 x, B_2 y)\| \\ &= \|B_1 y\| + \|B_2 x\| \\ &\leq 2\ \|p\|\ \Phi(r)\ \Psi(r)\frac{T^2}{2},\end{aligned}$$

for all $u \in \mathcal{S}$, it follows that B is uniformly bounded on $\mathcal{S}$.

Following that, we demonstrate that $B(\mathcal{S})$ is equi-continuous sequence of functions in E. If we choose $t_1,\ t_2 \in J$ with $t_1 < t_2$ and $u = (x, y) \in \mathcal{S}$, then we have

$$
\begin{aligned}
&(B_1x)(t_2) - (B_1x)(t_1)\\
&\le \int_0^{t_2} (t_2 - s) g_1(s, x(s), y(\varphi_3(s)))ds - \int_0^{t_1} (t_1 - s) g_1(s, x(s), y(\varphi_3(s)))ds\\
&\le \int_0^{t_1} (t_2 - s) g_1(s, x(s), y(\varphi_3(s)))ds + \int_{t_1}^{t_2} (t_2 - s) g_1(s, x(s), y(\varphi_3(s)))ds\\
&- \int_0^{t_1} (t_1 - s) g_1(s, x(s), y(\varphi_3(s)))ds\\
&\le \int_0^{t_1} [(t_2 - s) - (t_1 - s)] g_1(s, x(s), y(\varphi_3(s)))ds + \int_{t_1}^{t_2} (t_2 - s) g_1(s, x(s), y(\varphi_3(s)))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&|(B_1x)(t_2) - (B_1x)(t_1)|\\
&\le \int_0^{t_1} (t_2 - t_1)|g_1(s, x(s), y(\varphi_3(s)))|ds + \int_{t_1}^{t_2} (t_2 - s)|g_1(s, x(s), y(\varphi_3(s)))|ds\\
&\le \int_0^{t_1} (t_2 - t_1)\ p_1(t)\ \Phi_1(|x|)\ \Psi_1(|y|)ds + \int_{t_1}^{t_2} (t_2 - s)\ p_1(t)\ \Phi_1(|x|)\ \Psi_1(|y|)ds\\
&\le \|p_1\|\ \Phi(r)\ \Psi(r)\big[T(t_2 - t_1) + \int_{t_1}^{t_2} (t_2 - s)\ ds\big]\\
&\le \|p_1\|\ \Phi(r)\ \Psi(r)\big[T(t_2 - t_1) + \frac{(t_2 - t_1)^2}{2}\big].
\end{aligned}
$$

i.e.,

$$
|(B_1x)(t_2) - (B_1x)(t_1)| \le \|p_1\|\ \Phi(r)\ \Psi(r)[T(t_2 - t_1) + \frac{(t_2 - t_1)^2}{2}],
$$

which is independent of y. Then, for $\epsilon_1 > 0$, there exists a $\delta_1 > 0$, with

$$
|t_2 - t_1| < \delta_1 \implies |(B_1x)(t_2) - (B_1x)(t_1)| < \epsilon_1.
$$

In a similar manner, one can get that for $\epsilon_2 > 0$, there exists a $\delta_2 > 0$, with

$$
|t_2 - t_1| < \delta_2 \implies |(B_2y)(t_2) - (B_2y)(t_1)| < \epsilon_2.
$$

Thus, for $\epsilon > 0$, there exists a $\delta > 0$, with

$$
|t_2 - t_1| < \delta \implies |Bu(t_2) - Bu(t_1)| < \epsilon.
$$

Let $t_2, t_1 \in J$ and for all $u \in \mathcal{S}$. This demonstrates that $B(\mathcal{S})$ is an equi-continuous set in E.

Now, since the set $B(\mathcal{S})$ is a uniformly bounded and equi-continuous set in E, Arzela-Ascoli Theorem states that it is compact. B is thus a complete continuous operator on $\mathcal{S}$.

Step 3. The last condition of Dhage fixed-point Theorem [17] is fulfilled. Assume for $u \in E$,

$$
u = (x, y) = (A_1xB_1x + C_1x, A_2yB_2y + C_2y).
$$

Let $x \in \mathcal{C}(J, \mathbb{R})$ and $y \in \mathcal{S}_{\in}$ be two arbitrary values using the formula
$x = A_1 x B_1 x + C_1 x$. Next, we get

$$\begin{aligned}
|x(t)| &\leq |A_1 x(t)||B_1 x(t)| + |C_1 x(t)| \\
&\leq |f_1(t, x(\varphi_2(t)))| \int_0^t (t-s)\ |g_1(s, x(s), y(\varphi_3(s)))| ds + |h_1(t, x(\varphi_1(t)))| \\
&\leq (|f_1(t, x(\varphi_2(t))) - f_1(t,0)| + |f_1(t,0)|) \int_0^t (t-s)\ p_1(t)\ \Phi_1(|x|)\ \Psi_1(|y|) ds \\
&+ (|h_1(t, x(\varphi_1(t))) - h_1(t,0)| + |h_1(t,0)|) \\
&\leq (\|L_1\| |x(\varphi_2(t))| + F_1)\ \|p_1\| \Phi_1(|x|) \Psi_1(|y|) \int_0^t (t-s)\ ds + \|k_1\| |x(\varphi_1(t))| + H_1 \\
&\leq (\|L_1\|\ \|x\|\ + F_1)\ \|p_1\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k_1\|\ \|x\|\ + H_1.
\end{aligned}$$

Consequently,

$$|x(t)| \leq (\|L\|\ \|x\| + F)\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k\|\ \|x\| + H_1.$$

If we take the supremum over $t \in J$, we get

$$\|x\| \leq (\|L\|\ \|x\| + F)\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k\|\ \|x\| + H_1. \tag{14}$$

Likewise, using similar reasoning, if we let $y \in \mathcal{C}(J, \mathbb{R})$ and $x \in \mathcal{S}_1$ be random components in the sense that $y = A_2 y\ B_2 y + C_2 y$. After that, we get

$$|y(t)| \leq (\|L_2\|\ \|y\|\ + F_2)\ \|p_2\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k_2\|\ \|y\|\ + H_2.$$

Consequently,

$$|y(t)| \leq (\|L\|\ \|y\|\ + F)\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k\|\ \|y\|\ + H_2.$$

If we take the supremum over $t \in J$, we get

$$\|y\| \leq (\|L\|\ \|y\|\ + F)\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k\|\ \|y\|\ + H_2. \tag{15}$$

By condition (c), we can deduce that $u = (x, y) \in \mathcal{S}$.
Adding the inequalities (14) and (15), we get

$$\begin{aligned}
&\|x\| + \|y\| \\
\leq\ & (\|L\|\ \|x\| + F)\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k\|\ \|x\| + H_1 \\
+\ & (\|L\|\ \|y\|\ + F)\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}\ +\ \|k\|\ \|y\|\ + H_2 \\
\leq\ & \big(\|p\|\ \Phi(r)\ \Psi(r) \|L\| \frac{T^2}{2} + \|k\|\big)\ [\|x\| + \|y\|] + \big(2\ F\ \|p\|\ \Phi(r)\ \Psi(r)\ \big) \frac{T^2}{2}\ + H_1 + H_2.
\end{aligned}$$

As $\|(x, y)\| = \|x\| + \|y\|$, we have that $\|(x, y)\| \leq r$.

Step 4. Lastly, we demonstrate that $\delta M + \rho < 1$,

$$\begin{aligned} M &= \|B(\mathcal{S})\| \\ &= \sup\{\|B(x,y)\| : (x,y)\in\mathcal{S}\} \\ &= \sup\{\|B_1(x)\| + \|B_2(y)\| : (x,y)\in\mathcal{S}\} \\ &\leq 2\ \|p\|\ \Phi(r)\ \Psi(r)\ \frac{T^2}{2}, \end{aligned}$$

as well as (A_4) we get

$$\|L\|\ \frac{M}{2}\ + \|k\|\ < 1,$$

with $\delta = \frac{\|L\|}{2}$ and $\rho = \|k\|$.

As a result, all requirements of Dhage fixed-point Theorem [17] are fulfilled. As a consequence, the operator equation $(x, y) = A(x, y)B(x, y) + C(x, y)$ has a solution in $\mathcal{S}$. Therefore, a CSHDE (2) has a solution defined on $J \times J$. □

3. Particular cases, and example

As specific cases, we can derive existence results for certain CSHDE (when $h_i = 0,\ f_i = 1,\ i = 1, 2$)

- Taking $g_i(t, x, y) = -\lambda^2 x(t),\ \lambda \in \mathbb{R}^+$, then we obtain a system of simple harmonic oscillators

$$\begin{cases} \frac{d^2x(t)}{dt^2} = -\lambda^2 x(t) & t \in J, \\ \frac{d^2y(t)}{dt^2} = -\lambda^2 y(t) & t \in J, \\ x(0) = 0,\ y(0) = 0 \text{ and} \\ x'(0) = 0,\ y'(0) = 0. \end{cases} \tag{16}$$

- Taking $g_i(t, x, y) = \left(\frac{t^2-k}{t^2}\right) x + q_i(y),\ k \in \mathbb{R}$ where $q_i(y)$ are continuous functions, then we obtain a coupled system of Riccati differential equations

$$\begin{cases} t^2\frac{d^2x(t)}{dt^2} - (t^2-k)x(t) = t^2 q_1(y(\varphi_3(t))) & t \in J, \\ t^2\frac{d^2y(t)}{dt^2} - (t^2-k)y(t) = t^2 q_2(x(\varphi_3(t))) & t \in J, \\ x(0) = 0,\ y(0) = 0 \text{ and} \\ x'(0) = 0,\ y'(0) = 0, \end{cases} \tag{17}$$

- Taking $g_i(t, x, y) = -(t^2 - 2lt - k)x + q_i(y),\ k \in \mathbb{R}$ where $q_i(y)$ are continuous functions and l is fixed, then we obtain a coupled system of Coulomb wave differential equations

$$\begin{cases} \frac{d^2x(t)}{dt^2} + (t^2 - 2lt - k)x = q_1(y(\varphi_3(t))) & t \in J = [0, T], \\ \frac{d^2y(t)}{dt^2} + (t^2 - 2lt - k)y = q_2(x(\varphi_3(t))) & t \in J, \\ x(0) = 0, \ y(0) = 0 \text{ and} \\ x'(0) = 0, \ y'(0) = 0. \end{cases} \tag{18}$$

- Taking $g_i(t, x, y) = \left(\frac{-8\pi^2 m}{\hbar^2}\right)(Ex - \frac{kt^2}{2}x) + q_i(y), \ k \in \mathbb{R}$ where $q_i(y)$ are continuous functions and $\hbar$ is the Planket's constant and $E, \ k$ are positive real numbers, then we obtain a coupled system of Schrödinger wave differential equations for simple harmonic oscillators

$$\begin{cases} \frac{d^2x(t)}{dt^2} = \left(\frac{-8\pi^2 m}{\hbar^2}\right)(Ex(t) - \frac{kt^2}{2}x(t)) + q_1(y(\varphi_3(t))) & t \in J = [0, T], \\ \frac{d^2y(t)}{dt^2} = \left(\frac{-8\pi^2 m}{\hbar^2}\right)(Ey(t) - \frac{kt^2}{2}y(t)) + q_2(x(\varphi_3(t))) & t \in J, \\ x(0) = 0, \ y(0) = 0 \text{ and} \\ x'(0) = 0, \ y'(0) = 0. \end{cases} \tag{19}$$

Remark:
The existence results for the coupled system (2) can be proved, if we relax assumption (A_2) and replace (A_2) and (A_4) by the conditions that follow

(A_2') $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, fulfills Carathéodory requirement i.e., g is measurable in t for any $(\mu, \nu) \in \mathbb{R} \times \mathbb{R}$ and continuous in $\mu, \ \nu$ for almost all $t \in J$.
Functions $t \to a(t)$, $t \to b_1(t), \ b_2(t)$ are exists with

$$|g(t, \mu, \nu)| \leq \ a(t) + \ b_1(t)|\mu| + \ b_2(t)|\nu|, \ \forall \ (t, \mu, \nu) \ \in J \times \mathbb{R} \times \mathbb{R},$$

(A_4') There is a number $r > 0$ for which

$$r \leq \frac{||L||.||a||T^2 + G||b||T^2 + ||k||}{2||b_1||.||L||T^2 + 2||b_2||.||L||T^2}, \tag{20}$$

where $G = \sup_{t \in J} |f(t, 0)|$, and $(||L||||a|| + G||b_1|| + +G||b_2||)T^2 + ||k|| < 1$.

Example:
Consider CSHDE that follows:

$$\begin{cases} \frac{d^2}{dt^2}\left(\frac{x(t)-\frac{cos\pi t+2t^2}{1+5t^2}|x(t)|}{\left(\frac{|x(t)|+1}{|x(t)|+2}\right)\frac{7+\ln(t+1)}{2\sqrt{25+t^2}}+\frac{2-t}{10}}\right)=\frac{(t-1)^2+3}{70(13-t^2)}(7|x(t)|+15)(|y(t)|+1),\ t\in J=[0,1], \\ \frac{d^2}{dt^2}\left(\frac{x(t)-\frac{3(cos\pi t+2t)}{5(2+10t)^2}\left(\frac{x^2(t)+5|x(t)|}{|x(t)|+3}\right)}{\frac{t}{2(50+t)}\left(|x(t)|+\tan^{-1}x(t)\right)\frac{t+1}{3}}\right)=\frac{(x+1)(y+1)e^{-(|x|+|y|)}\sin^2 t}{5+2t+t^2},\ t\in J=[0,1], \\ x(0)=h_1(0,x(0)),\ y(0)=h_2(0,y(0)) \text{ and} \\ x'(0)=\frac{dh_1}{dt}|_{t=0},\ y'(0)=\frac{dh_2}{dt}|_{t=0}, \end{cases} \tag{21}$$

where

$$f_1(t,x(t))=\left(\frac{|x(t)|+1}{|x(t)|+2}\right)\frac{7+\ln(t+1)}{2\sqrt{25+t^2}}+\frac{2-t}{10},$$
$$f_2(t,x(t))=\frac{t}{2(50+t)}\left(|x(t)|+\tan^{-1}x(t)\right)\frac{t+1}{3}.$$

Take $x,y\in R$ and $t\in J$, then

$$|f_1(t,x(t))-f_1(t,y(t))|\le\left(\frac{7+\ln(t+1)}{2\sqrt{25+t^2}}\right)|x-y|,$$
$$|f_2(t,x(t))-f_2(t,y(t))|\le\frac{t}{50+t}|x-y|,$$

$$h_1(t,x(t))=\frac{cos\pi t+2t^2}{1+5t^2}|x(t)|,\ h_2(t,x(t))=\frac{(cos\pi t+11t)}{(2+10t)^2}|x(t)|,$$

$$|h_1(t,x(t))-h_1(t,y(t))|\le\left(\frac{1+2t^2}{1+5t^2}\right)|x-y|,\ |h_2(t,x(t))-h_2(t,y(t))|\le\left(\frac{1+2t}{(2+10t)^2}\right)|x-y|,$$

and

$$|g_1(t,x(t))|=\Big|\frac{((t-1)^2+3)\cos^2 t}{35(13-t^2)}(7|x(t)|+15)\Big|\le\left(\frac{(t-1)^2+3}{13-t^2}\right)\left(\frac{|x|}{5}+\frac{3}{7}\right),$$

$$|g_2(t,x(t))|=\Big|\frac{(x+1)e^{-|x|}\sin^2 t}{5+2t+t^2}\Big|\le\frac{|x|+1}{5+2t+t^2}.$$

We conclude that

$p_1(t)=\frac{(t-1)^2+3}{13-t^2}$, $\Psi_1(x)=\frac{|x|}{5}+\frac{3}{7}$, and $\Phi_1(y)=\frac{|y|}{2}+\frac{1}{2}$.
$p_2(t)=\frac{1}{5+2t+t^2}$, $\Psi_2(|x|)=(|x|+1)e^{-x}$, and $\Phi_2(|y|)=(|y|+1)e^{-y}$.
We can readily confirm this. $x(0)=h_i(0,x(0)),\ x'(0)=\frac{dh_i}{dt}|_{t=0},\ i=1,2.$
$||k_1||=1/2, ||k_2||=1/12,\ ||p_1||=4/13$ and $\|p_2\|=1/5$, , $\|L_1\|=7/10$,$\|L_2\|=1/51$ and $F_1=0.9,\ F_2=2/3,\ H_1=1,\ H_2=0.252$
For condition $\|L\|\|p\|\Phi(r)\Psi(r)\frac{T^2}{2}+\|k\|=0.7149<1$ is hold, r should be taken $1.2987<\ r$.
Furthermore, all additional criteria for Theorem 2.2 can be simply checked. According to Theorem 2.2, the a solution of CSHDE 2.2 exist.

4. Continuous dependence

4.1. **Uniqueness of the solution.** Take into account the following hypotheses

(A_2^*) $g_i : J \times \mathbb{R} \to \mathbb{R},\ i = 1, 2.$, functions that are continuous fulfilling the Lipschitz criterion, there exists functions $m_i \in C(J, \mathbb{R}_+)$ with $\|m_i\| = \sup_{t\in J} |m_i(t)|$ such that

$$|g_i(t, \mu_1, \nu_1) - g_i(t, \mu_2, \nu_2)| \leq m_i(t)(|\mu_1 - \mu_2| + |\nu_1 - \nu_2|),$$

where $G_{0i} = \sup_{t\in J} |g_i(t, 0, 0)|$,
with $G_0 = \max\{G_{01}, G_{02}\}$ and $\|m\| = \max\{\|m_1\|, \|m_2\|\}$.

Theorem 4.1. *Assume that Theorem 2.2 is valid, with Change hypotheses (A_2) by (A_2^*). If*

$$\|k\| \ + \left[2\ \|L\|\ \|m\|\ r + \|L\|G_0\ + \|m\|\ F\right]\frac{T^2}{2} < 1,$$

then the solution of CSHDE (2) unique. .

Proof. Let $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ be two solutions of CSHDE (2). Then for $t \in J$, we get

$$
\begin{aligned}
&|x_1(t) - x_2(t)|\\
&\leq |h_1(t, x_1(\varphi_1(t))) - h_1(t, x_2(\varphi_1(t)))| + |f_1(t, x_1(\varphi_2(t))) \int_0^t (t-s) g_1(s, x_1(s), y_1(\varphi_3(s)))ds\\
&- f_1(t, x_2(\varphi_2(t))) \int_0^t (t-s) g_1(s, x_2(s), y_2(\varphi_3(s)))ds|\\
&\leq\ k_1(t)|x_1(\varphi_1(t)) - x_2(\varphi_1(t))|\\
&+ |f_1(t, x_1(\varphi_2(t))) - f_1(t, x_2(\varphi_2(t)))| \int_0^t (t-s)|g_1(s, x_1(s), y_1(\varphi_3(s)))|\ ds\\
&+ |f_1(t, x_1(\varphi_2(t)))| \int_0^t (t-s)|g_1(s, x_1(s), y_1(\varphi_3(s))) - g_1(s, x_2(s), y_2(\varphi_3(s)))|\ ds\\
&\leq k_1(t)|x_1(\varphi_1(t)) - x_2(\varphi_1(t))|\\
&+ L_1(t)|x_1(\varphi_2(t)) - x_2(\varphi_2(t))| \int_0^t (t-s)|g_1(s, x_1(s), y_1(\varphi_3(s)))|ds\\
&+ |f_1(t, x_1(\varphi_2(t)))| \int_0^t (t-s)\ m_1(t)\ [|x_1(s) - x_2(s)| + |y_1(\varphi_3(s)) - y_2(\varphi_3(s))|]\ ds\\
&\leq\ |k_1(t)|\ |x_1(\varphi_1(t)) - x_2(\varphi_1(t))|\\
&+ |L_1(t)|x_1(\varphi_2(t)) - x_2(\varphi_2(t))| \int_0^t (t-s)[|g_1(s, x_1(s), y_1(\varphi_3(s))) - g_1(s, 0, 0)|\\
&+ |g_1(s, 0, 0)|]\ ds + |f_1(t, x_1(\varphi_2(t))) - f_1(t, 0)|\\
&+ |f_1(t, 0)|\ |m_1(t)| \int_0^t (t-s)[|x_1(s) - x_2(s)| + |y_1(\varphi_3(s)) - y_2(\varphi_3(s))|]ds
\end{aligned}
$$

$$\leq \ |k_1(t)| \ |x_1(\varphi_1(t)) - x_2(\varphi_1(t))|$$
$$+ |L_1(t)|x_1(\varphi_2(t)) - x_2(\varphi_2(t))| \int_0^t (t-s)\big(|m_1(t)|[|x_1(s)| + |y_1(\varphi_3(s))|] + |g_1(s,0)|\big) ds$$
$$+ (|L_1(t)||x_1(\varphi_2(t))| + |f_1(t,0)|)|m_1(t)|$$
$$\times \int_0^t (t-s)[|x_1(s) - x_2(s)| + |y_1(\varphi_3(s)) - y_2(\varphi_3(s))|] ds.$$

If we take the supremum over $t \in J$, we get

$$\|x_1 - x_2\| \leq \ \|k_1\| \ \|x_1 - x_2\| + \ \|L_1\| \ \|x_1 - x_2\| \ (\|m_1\| \ [\|x_1\| + \|y_1\|] + G_{01}) \ \int_0^t (t-s) \ ds$$
$$+ (\|L_1\| \ \|x_1\| + F_1) \ \|m_1\| \ [\|x_1 - x_2\| + \|y_1 - y_2\|] \int_0^t (t-s) \ ds$$
$$\|x_1 - x_2\| \leq \ \|k_1\| \ \|x_1 - x_2\| + \left[\ \|L_1\| \ \|x_1 - x_2\| \ (\|m_1\| \ [\|x_1\| + \|y_1\|] + G_{01}) \ \right] \frac{T^2}{2}$$
$$+ \left[(\|L_1\| \ \|x_1\| + F_1) \ \|m_1\| \ [\|x_1 - x_2\| + \|y_1 - y_2\|]\right] \frac{T^2}{2}.$$

In a similar manner, one can derive that

$$\|y_1 - y_2\| \leq \ \|k_2\| \ \|y_1 - y_2\| + \left[\ \|L_2\| \ \|y_1 - y_2\| \ (\|m_2\| \ [\|x_1\| + \|y_1\|] + G_{02}) \ \right] \frac{T^2}{2}$$
$$+ \left[(\|L_2\| \ \|y_1\| + F_2) \ \|m_2\| \ [\|x_1 - x_2\| + \|y_1 - y_2\|]\right] \frac{T^2}{2}.$$

Now, consider

$$\|u_1 - u_2\|$$
$$= \|(x_1, y_1) - (x_2, y_2)\| = \|x_1 - x_2\| + \|y_1 - y_2\|$$
$$\leq \ \|k_1\| \ \|x_1 - x_2\| + \left[\ \|L_1\| \ \|x_1 - x_2\| \ (\|m_1\| \ [\|x_1\| + \|y_1\|] + G_{01}) \ \right] \frac{T^2}{2}$$
$$+ \left[(\|L_1\| \ \|x_1\| + F_1) \ \|m_1\| \ [\|x_1 - x_2\| + \|y_1 - y_2\|]\right] \frac{T^2}{2}$$
$$+ \ \|k_2\| \ \|y_1 - y_2\| + \left[\ \|L_2\| \ \|y_1 - y_2\| \ (\|m_2\| \ [\|x_1\| + \|y_1\|] + G_{01}) \ \right] \frac{T^2}{2}$$
$$+ \left[(\|L_2\| \ \|y_1\| + F_2) \ \|m_2\| \ [\|x_1 - x_2\| + \|y_1 - y_2\|]\right] \frac{T^2}{2}$$
$$\leq \ \|k\| \ [\|x_1 - x_2\| + \|y_1 - y_2\|] + \left[\ \|L\| \ (\|m\| \ \|u_1\| + G_0) \ \right] \frac{T^2}{2} [\|x_1 - x_2\| + \|y_1 - y_2\|]$$
$$+ \left[(\|L\| \ \|u_1\| + F) \ \|m\| \ \right] \frac{T^2}{2} [\|x_1 - x_2\| + \|y_1 - y_2\|]$$
$$\leq \left(\ \|k\| \ + \left[\ \|L\| \ (\|m\| \ r + G_0) \ \right] \frac{T^2}{2} + \left[(\|L\| \ r + F) \ \|m\| \ \right] \frac{T^2}{2} \right) [\|x_1 - x_2\| + \|y_1 - y_2\|],$$

we have

$$\|u_1 - u_2\| \leq \left(\|k\| + \left[2 \|L\| \|m\| r + \|L\|G_0 + \|m\| F\right] \frac{T^2}{2}\right)\|u_1 - u_2\|.$$

Then

$$\left(1 - \left(\|k\| + \left[2 \|L\| \|m\| r + \|L\|G_0 + \|m\| F\right] \frac{T^2}{2}\right)\right) \|u_1 - u_2\| \leq 0,$$

as well as that

$$\|u_1 - u_2\| = 0 \implies u_1 = u_2$$

□

4.2. **Continuous dependence on the delay functions.** We now demonstrate that solution for coupled system (2) is continuously dependent on $\varphi_i(t)$

Definition 4.1. *The solution of the initial value problem (2) depends continuously on the delay functions $\varphi_i(t)$ if $\forall\ \epsilon > 0$, $\exists\ \delta > 0$, such that*

$$|\varphi_i(t) - \varphi_i^*(t)| \leq \delta \implies ||u - u^*|| \leq \epsilon.$$

Theorem 4.2. *Assume that assumptions for Theorem 4.1 hold, The CSHDE (2) solution is then continuously dependent on $\varphi_i(t)$.*

Proof. Let $u = (x, y), u^* = (x^*, y^*)$ is being solutions of CSHDE (2). Then for given $\delta > 0$ with $|\varphi_i(t) - \varphi_i^*(t)| \leq \delta$, $\forall\ t \geq 0$, we get

$$\begin{aligned}
&|x(t) - x^*(t)| \\
&\leq |h_1(t, x(\varphi_1(t))) - h_1(t, x^*(\varphi_1^*(t)))| \\
&+ |f_1(t, x(\varphi_2(t))) \int_0^t (t-s) g_1(s, x(s), y(\varphi_3(s)))ds \\
&- f_1(t, x^*(\varphi_2(t))) \int_0^t (t-s) g_1(s, x^*(s), y^*(\varphi_3(s)))ds| \\
&\leq |h_1(t, x(\varphi_1(t))) - h_1(t, x^*(\varphi_1^*(t)))| \\
&+ |f_1(t, x(\varphi_2(t))) \int_0^t (t-s) g_1(s, x(s), y(\varphi_3(s)))ds \\
&- f_1(t, x^*(\varphi_2(t))) \int_0^t (t-s) g_1(s, x(s), y(\varphi_3(s)))ds| \\
&+ |f_1(t, x^*(\varphi_2(t))) \int_0^t (t-s) g_1(s, x(s), y(\varphi_3(s)))ds \\
&- f_1(t, x^*(\varphi_2(t))) \int_0^t (t-s) g_1(s, x^*(s), y^*(\varphi_3(s)))ds| \\
&\leq |h_1(t, x(\varphi_1(t))) - h_1(t, x^*(\varphi_1^*(t)))| \\
&+ |f_1(t, x(\varphi_2(t))) - f_1(t, x^*(\varphi_2(t)))| \int_0^t (t-s)|g_1(s, x(s), y(\varphi_3(s)))|\ ds \\
&+ |f_1(t, x^*(\varphi_2(t)))| \int_0^t (t-s)[g_1(s, x(s), y(\varphi_3(s))) - g_1(s, x^*(s), y^*(\varphi_3(s)))]\ ds
\end{aligned}$$

$$\leq\ k_1(t)|x(\varphi_1(t))\ - x^*(\varphi_1^*(t))|$$
$$+ L_1(t)|x(\varphi_2(t)) - x^*(\varphi_2(t))| \int_0^t (t-s)[\ G_{01} + m_1(s)(|x(s)| + |y(\varphi_3(s))|)]\ ds$$
$$+\ (|L_1(t)||x^*(\varphi_2(t))| + F_1)\ \int_0^t (t-s)\ m_1(s)[|x(s)\ - x^*(s)| + |y(\varphi_3(s)) - y^*(\varphi_3(s))|]\ ds$$
$$\leq\ k_1(t)|x(\varphi_1(t))\ - x^*(\varphi_1(t)) + x^*(\varphi_1(t))\ - x^*(\varphi_1^*(t))|$$
$$+ L_1(t)|x(\varphi_2(t)) - x^*(\varphi_2(t))|[\ G_{01} + \|m_1\|(\|x\| + \|y\|)] \int_0^t (t-s)\ ds$$
$$+\ (\|L_1\|\|x^*\| + F_1)\ \int_0^t (t-s)|m_1(s)|[|x(s)\ - x^*(s)| + |y(\varphi_3(s)) - y^*(\varphi_3(s))|]\ ds$$
$$\leq\ \|k_1\|[|x(\varphi_1(t))\ - x^*(\varphi_1(t))| + |x^*(\varphi_1(t))\ - x^*(\varphi_1^*(t))|]$$
$$+ \|L_1\||x(\varphi_2(t)) - x^*(\varphi_2(t))|\ [\ G_{01} + \|m_1\|(\|x\| + \|y\|)]\frac{T^2}{2}$$
$$+ (\|L_1\|\|x^*\| + F_1)\|m_1\| \int_0^t (t-s)[|x(s)\ - x^*(s)| + |y(\varphi_3(s)) - y^*(\varphi_3(s))|]ds.$$

If we take the supremum over $t \in J$, we get

$$\|x - x^*\| \leq \|k\|[\|x - x^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))|]$$
$$+ \|L\|\|x - x^*\|[\ G_0 + \|m\|(\|x\| + \|y\|)]\frac{T^2}{2}$$
$$+ (\|L\|\|x^*\| + F)\|m\|[\|x - x^*\| + \|y - y^*\|]\frac{T^2}{2}.$$

In a similar manner, one can drive that

$$\|y - y^*\| \leq \|k\|[\|y - y^*\| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))|]$$
$$+ \|L\|\|y - y^*\|[\ G_0 + \|m\|\|x\|]\frac{T^2}{2} + (\|L\|\|y^*\| + F)\|m\|\|x - x^*\|\frac{T^2}{2}.$$

Hence

$$\|u - u^*\| = \|x - x^*\| + \|y - y^*\|$$
$$\leq \|k\|\big[\|x - x^*\| + \|y - y^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))|\big]$$
$$+ \|L\|\big[\|x - x^*\| + \|y - y^*\|\big][\ G_0 + \|m\|[\|x\| + \|y\|]\frac{T^2}{2}$$
$$+ (\|L\|[\|x^*\| + \|y^*\|] + F)\|m\|\big[\|x - x^*\| + \|y - y^*\|\frac{T^2}{2}$$
$$\leq \|k\|\big[\|u - u^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))|\big]$$
$$+ \|L\|\|u - u^*\|[\ G_0 + \|m\|\|u\|]\frac{T^2}{2} + (\|L\|\|u^*\| + F)\|m\|\|u - u^*\|\frac{T^2}{2}$$
$$\leq \|k\|\big[\|u - u^*\| + |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))|\big]$$
$$+ \|L\|\|u - u^*\|[\ G_0 + \|m\|r]\frac{T^2}{2} + (\|L\|r + F)\|m\|\|u - u^*\|\frac{T^2}{2}$$
$$\leq \big[\|k\| + \big(\|L\|G_0 + F\|m\| + 2\|L\|\|m\|\big)\frac{T^2}{2}\big]\|u - u^*\|$$
$$+ \|k\||x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| + |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))|\big].$$

However, based on the continuity of solutions x^* and y^*, we get

$$|\varphi_1(t) - \varphi_1^*(t)| \le \delta \implies |x^*(\varphi_1(t)) - x^*(\varphi_1^*(t))| \le \epsilon_1,$$
$$\implies |y^*(\varphi_1(t)) - y^*(\varphi_1^*(t))| \le \epsilon_2.$$

Then

$$\|u - u^*\| \le \frac{\|k\|[\epsilon_1 + \epsilon_2]}{1 - \left[\|k\| + \left(\|L\|G_0 + F\|m\| + 2\|L\|\|m\|\right)\frac{T^2}{2}\right]} \le \epsilon.$$

This implies that the solution of the CSHDE (2) continuously dependent on φ_1. □

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References

[1] Dhage B.C., (2004), A fixed point theorem in Banach algebras involving three operators with applications, Kyungpook Mathematical Journal, 44(1), pp.145–155.

[2] Dhage B.C., (2010), Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, Differ. Equ. Appl, 2(4), pp.465–486.

[3] Herzallah M.A.E., Baleanu D., (2014), On fractional order hybrid differential equations, Abstract and Applied Analysis, 2014.

[4] Melliani S., Hilal K., and Hannabou M., (2018), Existence results in the theory of hybrid fractional integro-differential equations, Journal of Universal Mathematics, 1(2), pp.166–179.

[5] Zhao Y., Sun S., Han Z., and Li Q., (2011), Theory of fractional hybrid differential equations, Computers & Mathematics with Applications, 62(3), pp.1312–1324.

[6] Zheng L., Zhang X., (2017), Modeling and analysis of modern fluid problems, Elsevier/Academic Press,.

[7] El-Sayed A.M.A., Hashem H.H.G., Al-Issa, Sh. M, (2021), Characteristics of Solutions of Fractional Hybrid Integro-Differential Equations in Banach Algebra, Sahand Communications in Mathematical Analysis, 18(3), pp.109–131. doi: 10.22130/scma.2021.120867.876

[8] El-Sayed A.M.A., Hashem H.H.G., and Al-Issa Sh. M, (2020), Existence of solutions for an ordinary second-order hybrid functional differential equation. Adv Differ Equ 2020, 296.

[9] El-Sayed A.M.A., Hashem H.H.G., Al-Issa Sh. M, (2020), An Implicit Hybrid Delay Functional Integral Equation: Existence of Integrable Solutions and Continuous Dependence, Mathematics, 9(24), 3234.

[10] Srivastava H.M., El-Sayed A.M.A., Hashem H.H.G., Al-Issa Sh. M, (2022), Analytical investigation of nonlinear hybrid implicit functional differential inclusions of arbitrary fractional orders, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 116(1), pp.1-19.

[11] Hashem H.H.G., El-Sayed A.M.A., Al-Issa S.M., (2023) Investigating asymptotic stability for hybrid cubic integral inclusion with fractal feedback control on the real half-axis. Fractal Fract. 2023, 7, 449.

[12] Al-Issa S.M., El-Sayed A.M.A., Hashem H.H.G., (2023) An Outlook on hybrid fractional modeling of a heat controller with multi-valued feedback Control. Fractal Fract. 2023, 7, 759.

[13] Al-Issa S.M., El-Sayed A.M.A., Hashem H.H.G., (2023) An Outlook on hybrid fractional modeling of a heat controller with multi-valued feedback Control. Fractal Fract. 2023, 7, 759.

[14] Hannabou M., Hilal K., (2020), Existence results for a system of coupled hybrid differential equations with fractional order, International Journal of Differential Equations, 2020, Article ID 3038427, 8 pages.

[15] Ahmad B., Ntouyas S.K., Alsaedi A., (2014), Existence results for a system of coupled hybrid fractional differential equations. The Scientific World Journal, 2014(1), Article ID 426438, 6 pages.

[16] Dhage B.C., Dhage S.B., Bhuvaneshwari K., (2019), Existence of mild solutions of nonlinear boundary value problems of coupled hybrid fractional integro differential equations, Journal of Fractional Calculus and Applications, 10(2), pp. 191–206.

[17] Dhage B.C., (1988), On some variants of Schauder's fixed point principle and applications to nonlinear integral equations, J. Math. Phys. Sci, 25, pp.603–611.

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