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SOLVING HIGHER ORDER INTUITIONISTIC FUZZY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we provide the existence and uniqueness intuitionistic fuzzy solutions for the second-order intuitionistic fuzzy differential equations satisfying a lipschitz condition. We generalised these results for the nth-order intuitionistic fuzzy differential equations with initial value conditions by using the Banach fixed point theorem. Some examples are given to illustrate our main results.

Keywords: Intuitionistic fuzzy differential equations, fixed point theorems, intuitionistic fuzzy solutions, fixed point.

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1. Introduction

Generalizations of fuzzy set theory, as discussed in [1], constitute one of the concepts within intuitionistic fuzzy set (IFS). Subsequently, Atanassov introduced the concept of a fuzzy set and proposed the notion of an intuitionistic fuzzy set. He explored the principles of fuzzy set theory through the lens of intuitionistic fuzzy set (IFS) theory [2, 3]. IFS have a highly potent tool for modeling imprecision thanks to further development of the intuitionistic fuzzy set theory, intuitionistic fuzzy geometry, intuitionistic fuzzy logic, intuitionistic fuzzy topology, and an intuitionistic fuzzy approach to artificial intelligence. Numerous fields have benefited from the IFSS's useful applications, including [4, 5, 6, 7, 8].

One of the areas of intuitionistic fuzzy set theory that has recently undergone intensive research is the idea of intuitionistic fuzzy differential equations (IFDEs). Intuitionistic fuzzy solutions for these equations were first introduced by the authors of [9]. Intuitionistic fuzzy solutions for differential equations were recently developed by the authors in

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[10, 11, 12, 13], they used several techniques to prove the existence and uniqueness of intuitionistic fuzzy solutions under particular assumptions for these intuitionistic fuzzy differential equations. Applications of numerical algorithms for handling IFDEs can be found [14, 15, 16, 17]. Several innovative concepts were introduced in [18], including intuitionistic fuzzy N_b metric space, intuitionistic fuzzy quasi-S_b-metric space, intuitionistic fuzzy pseudo-b-metric space, intuitionistic fuzzy quasi-N-metric space, and intuitionistic fuzzy pseudo N_b fuzzy metric space and proved decomposition theorem and fixed-point results in new setting. Garbiec [19] introduced the fuzzy version of the Banach fixed-point result. Several aspects, including topological structure and convergence criteria in [20], they provided a proof of a Banach fixed-point theorem within the context of graphical fuzzy metric spaces. A method for constructing a Hausdorff intuitionistic fuzzy metric on the set of nonempty compact subsets of a given intuitionistic fuzzy metric space was presented by the authors in [21]. The concept of intuitionistic extended fuzzy b-metric-like spaces was introduced, accompanied by the establishment of various fixed point theorems within this framework [22]. Moreover, they applied fuzzy sets principles within metric spaces to introduce fuzzy metric spaces (FMSs), using continuous t-norms (CTNs), thereby expanding on the notion of probabilistic metric spaces. Furthermore, the authors in [23, 24, 25] extended the concept of metric spaces to include intuitionistic fuzzy metric spaces, the study focused on intuitionistic fuzzy real Banach spaces, which result from combining fuzzy Banach spaces with intuitionistic fuzzy sets. The approach taken to address this issue, Pexiderized quadratic functional equations defined within these spaces exhibit stability according to the Hyers-Ulam-Rassias criteria, which was based on fixed-point theory [26]. The research has identified the stability akin to the Hyers-Ulam-Rassias theorem concerning the Pexiderized functional equation within intuitionistic fuzzy Banach spaces. Given specific conditions, the study has confirmed the stability of the Pexiderized functional equation in intuitionistic fuzzy Banach spaces following the Hyers-Ulam-Rassias theorem [27].

The works mentioned above served as motivation and inspiration for this paper. We propose a new complete intuitionistic fuzzy metric space to investigate the existence and uniqueness of intuitionistic fuzzy solutions using the Banach fixed-point theorem for the following second-order intuitionistic fuzzy differential equation:

$$\begin{cases} < u,v >'' (t) = f\left(t, < u,v > (t), < u,v >' (t)\right), \quad t \in [t_0, T], \\ < u,v > (t_0) = k_1, < u,v >' (t_0) = k_2, \end{cases} \tag{1}$$

and the following nth-order intuitionistic fuzzy differential equation:

$$\begin{cases} \langle u,v\rangle^{(n)}(t) = f\left(t, \langle u,v\rangle(t), \langle u,v\rangle'(t), \ldots, \langle u,v\rangle^{(n-1)}(t)\right), t \in [t_0, T], \\ \langle u,v\rangle(t_0) = k_1, \langle u,v\rangle'(t_0) = k_2, \ldots, \langle u,v\rangle^{(n-1)}(t_0) = k_n, \end{cases} \tag{2}$$

where $f : [t_0, T] \times (IF_n)^n \to IF_n$ and $k_1, k_2, \ldots, k_n$ are intuitionistic fuzzy numbers. We will prove that there exists a unique solution for these problems, if f is continuous and satisfies a Lipschitz condition that involves all the variables but the temporal one.

The remaining sections of the paper are organized as follows: Some fundamental definitions and results are presented in Section 2. The existence and uniqueness of an intuitionistic fuzzy solution to the second-order and the nth-order intuitionistic fuzzy differential equations are shown in Section 3. Section 4 contains computational examples to illustrate the theory, Section 5 summarizes the results of research this paper, and suggests future research directions.

2. Preliminaries

Let's start with a few definitions that are related to our research.
Throughout this paper, $(\mathbb{R}^n, B(\mathbb{R}^n), \mu)$ denotes a complete finite measure space.

Let use $P_k(\mathbb{R}^n)$ the set of all nonempty compact convex subsets of $\mathbb{R}^n$. We denote by

$$IF_n = \mathrm{IF}(\mathbb{R}^n) = \left\{ \langle u, v\rangle : \mathbb{R}^n \to [0,1]^2 \mid \forall x \in \mathbb{R}^n \ 0 \leq u(x) + v(x) \leq 1 \right\}.$$

An element $\langle u, v\rangle$ of IF_n is said an intuitionistic fuzzy number if it satisfies the following conditions:

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_1) = 1$.
(ii) u is fuzzy convex and v is fuzzy concave.
(iii) u is upper semi-continuous and v is lower semi-continuous.
(iv) $supp\langle u, v\rangle = cl\{x \in \mathbb{R}^n \mid v(x) < 1\}$ is bounded.

So we denote the collection of all intuitionistic fuzzy number by IF_n. For $\alpha \in [0,1]$ and $\langle u, v\rangle \in IF_n$, the upper and lower α-cuts of $\langle u, v\rangle$ are defined by,

$$[\langle u, v\rangle]^\alpha = \{x \in \mathbb{R}^n : v(x) \leq 1 - \alpha\},$$

and,

$$[\langle u, v\rangle]_\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}.$$

Remark 2.1. *If $\langle u, v\rangle \in IF_n$, so we can see $[\langle u, v\rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v\rangle]^\alpha$ as $[1-v]^\alpha$ in the fuzzy case. We define $0_{(1,0)} \in IF_n$ as,*

$$0_{(1,0)}(t) = \begin{cases} (1,0) & t = 0 \\ (0,1) & t \neq 0. \end{cases}$$

Let $\langle u, v\rangle, \langle u', v'\rangle \in IF_n$ and $\lambda \in \mathbb{R}$, we define the following operations by:

$$\left(\langle u, v\rangle + \langle u', v\rangle\right)(z) = \left(\sup_{z=x+y} \min\left(u(x), u'(y)\right); \inf_{z=x+y} \max\left(v(x), v'(y)\right)\right),$$

$$\lambda\langle u, v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & \text{if } \lambda \neq 0 \\ 0_{(1,0)} & \text{if } \lambda = 0. \end{cases}$$

For $\langle u, v\rangle, \langle z, w\rangle \in IF_n$ and $\lambda \in \mathbb{R}$, the addition and scalar-multiplication are defined as follows:

$$[\langle u, v\rangle + \langle z, w\rangle]^\alpha = [\langle u, v\rangle]^\alpha + [\langle z, w\rangle]^\alpha, \qquad [\lambda\langle z, w\rangle]^\alpha = \lambda[\langle z, w\rangle]^\alpha,$$

$$[\langle u, v\rangle + \langle z, w\rangle]_\alpha = [\langle u, v\rangle]_\alpha + [\langle z, w\rangle]_\alpha, \qquad [\lambda\langle z, w\rangle]_\alpha = \lambda[\langle z, w\rangle]_\alpha.$$

Definition 2.1. ([10]) *Let $\langle u, v\rangle$ an element of IF_n and $\alpha \in [0,1]$, we define the following sets:*

$$[\langle u, v\rangle]_l^+(\alpha) = \inf\{x \in \mathbb{R}^n | u(x) \geq \alpha\}, \qquad [\langle u, v\rangle]_r^+(\alpha) = \sup\{x \in \mathbb{R}^n | u(x) \geq \alpha\},$$

$$[\langle u, v\rangle]_l^-(\alpha) = \inf\{x \in \mathbb{R}^n | v(x) \leq 1-\alpha\}, \qquad [\langle u, v\rangle]_r^-(\alpha) = \sup\{x \in \mathbb{R}^n | v(x) \leq 1-\alpha\}.$$

Remark 2.2. ([14])

$$[\langle u, v\rangle]_\alpha = \left[[\langle u, v\rangle]_l^+(\alpha), [\langle u, v\rangle]_r^+(\alpha)\right],$$

$$[\langle u, v\rangle]^\alpha = \left[[\langle u, v\rangle]_l^-(\alpha), [\langle u, v\rangle]_r^-(\alpha)\right].$$

On the space IF_n we will consider the following metric:

$$\begin{aligned} d_\infty^n(\langle u,v\rangle,\langle z,w\rangle) =&\frac{1}{4}\sup_{0<\alpha\le 1}\|[\langle u,v\rangle]_r^+(\alpha)-[\langle z,w\rangle]_r^+(\alpha)\| \\ &+\frac{1}{4}\sup_{0<\alpha\le 1}\|[\langle u,v\rangle]_l^+(\alpha)-[\langle z,w\rangle]_l^+(\alpha)\| \\ &+\frac{1}{4}\sup_{0<\alpha\le 1}\|[\langle u,v\rangle]_r^-(\alpha)-[\langle uz,w\rangle]_r^-(\alpha)\| \\ &+\frac{1}{4}\sup_{0<\alpha\le 1}\|[\langle u,v\rangle]_l^-(\alpha)-[\langle z,w\rangle]_l^-(\alpha)\|. \end{aligned}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 2.1. ([21]) *d_∞^n define a metric on IF_n.*

Theorem 2.2. ([21]) *The metric space (IF_n, d_∞^n) is complete.*

Definition 2.2. ([10]) *A mapping $f : I \to IF_n$ is called continuous at $t_0 \in I$ provided for any arbitrary $\epsilon > 0$,*

$$d_\infty^n\left(f(t), f(t_0)\right) < \epsilon,$$

for all $t \in I$.

Definition 2.3. ([14]) *A mapping $f : I \times IF_n \times IF_n \to IF_n$ is called continuous at point $(t_0, \langle u,v\rangle_0, \langle z,w\rangle_0) \in I \times IF_n \times IF_n$ provided for any arbitrary $\epsilon > 0$, there exists an $\delta(\epsilon)$ such that*

$$d_\infty^n\left(f(t,\langle u,v\rangle,\langle z,w\rangle), f(t_0,\langle u,v\rangle_0,\langle z,w\rangle_0)\right) < \epsilon,$$

whenever $\max|\, t-t_0 \,| < \delta(\epsilon)$ *and* $d_\infty^n(\langle u,v\rangle,\langle u,v\rangle_0) < \delta(\epsilon)$ *and* $d_\infty^n(\langle z,w\rangle,\langle z,w\rangle_0) < \delta(\epsilon)$ *for all $t \in I$ and $\langle u,v\rangle, \langle z,w\rangle, \langle u,v\rangle_0, \langle z,w\rangle_0 \in IF_n$.*

Definition 2.4. *We say that a mapping $f : I \to IF_n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued mappings $f_\alpha : I \to P_k(\mathbb{R}^n)$ defined by $f_\alpha(t) = [f(t)]_\alpha$ and $f^\alpha : I \to P_k(\mathbb{R}^n)$ defined by $f^\alpha(t) = [f(t)]^\alpha$ are (Lebesgue) measurable, when $P_k(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d_H.*

Where d_H is the Hausdorff metric defined in $P_k(\mathbb{R}^n)$ by

$$d_H\left([a,b]\,[c,d]\right) = \max\left\{||a-c||; ||b-d||\right\}.$$

Definition 2.5. ([10]) *$f : I \to IF_n$ is called integrably bounded if there exists an integrable function $h : I \to \mathbb{R}^n$ such that $\|y\| \le h(t)$ holds for any $y \in supp\,(f(t))\,, t \in I$.*

Theorem 2.3. ([10]) *If $f : I \to IF_n$ is strongly measurable and integrably bounded, then f is integrable.*

Definition 2.6. ([10]) *Supposes $f : I \to IF_n$ is integrably bounded and strongly measurable for each $\alpha = (0,1]$ write,*

$$\begin{aligned} \int_I f(t)dt &= \int_I f_\alpha(t)dt \\ &= \left\{\int_I F(t)dt | F : I \to \mathbb{R}^n \textit{is a measurable selection for } f_\alpha\right\}, \end{aligned}$$

$$\begin{aligned} \int_I f(t)dt &= \int_I f^\alpha(t)dt \\ &= \left\{\int_I F(t)dt | F : I \to \mathbb{R}^n \textit{ is a measurable selection for } f^\alpha\right\}. \end{aligned}$$

Remark 2.3. *If there exists $\langle u,v\rangle \in IF_n$ such that*

$$[\langle u,v\rangle]^{\alpha} = \left[\int_I f(t)dt\right]^{\alpha} \text{ and } [\langle u,v\rangle]_{\alpha} = \left[\int_I f(t)dt\right]_{\alpha},\ \forall\alpha\in(0,1].$$

Then f is called integrable on T, write $\langle u,v\rangle = \displaystyle\int_I f(t)dt$.

Definition 2.7. ([?]) *Let $\langle u,v\rangle$ and $\langle u',v'\rangle \in IF_1$, the Hukuhara difference is the intuitionistic fuzzy number $\langle z,w\rangle \in IF_1$, if it exists, such that*

$$\langle u,v\rangle - \langle u',v'\rangle = \langle z,w\rangle \Leftrightarrow \langle u,v\rangle = \langle u',v'\rangle + \langle z,w\rangle.$$

Definition 2.8. ([10]) *A mapping $F : I \to IF_1$ is said to be Hukuhara derivable at t_0 if there exists $F'(t_0) \in IF_1$ such that both limits:*

$$\lim_{\triangle t\to 0^+} \frac{F(t_0+\triangle t)-F(t_0)}{\triangle t}$$

and

$$\lim_{\triangle t\to 0^-} \frac{F(t_0)-F(t_0-\triangle t)}{\triangle t},$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, which is called the Hukuhara derivative of F at t_0.

Definition 2.9. ([10]) *Let $F :\ I \to IF_1$. We define the n-th order differential of F as follows. Let $F :\ I \to IF_1$ and $t_0 \in I$. We say that F is differentiable of the n-th order at t_0, if there exist elements $F^s(t_0) \in IF_1, \forall s = 1,2,\ldots,n$ such that both limits*

$$\lim_{\triangle t\to 0^+} \frac{F^{(s-1)}(t_0+\triangle t)-F^{(s-1)}(t_0)}{\triangle t}$$

and

$$\lim_{\triangle t\to 0^-} \frac{F^{(s-1)}(t_0)-F^{(s-1)}(t_0-\triangle t)}{\triangle t},$$

exist and they are equal to $F^{(s)}(t_0) = \langle u^{(s)}(t_0), v^{(s)}(t_0)\rangle$.

Definition 2.10. ([12]) *Let $\mathcal{C}(I, IF_n)$ be a metric space, and $T : \mathcal{C}(I, IF_n) \to \mathcal{C}(I, IF_n)$. We will say that T is a contraction if there exists some $0 < k < 1$ such that*

$$H(T(\langle \Psi_1,\Psi_2\rangle), T(\langle \Psi_1',\Psi_2'\rangle)) \le kH(\langle \Psi_1,\Psi_2\rangle, \langle \Psi_1',\Psi_2'\rangle),$$

for all $\langle \Psi_1,\Psi_2\rangle, \langle \Psi_1',\Psi_2'\rangle \in \mathcal{C}(I, IF_n)$.

Lemma 2.1. ([12]) *Let $\big(\mathcal{C}(I, IF_n), H\big)$ be a complete metric space and let $T : \mathcal{C}(I, IF_n) \to \mathcal{C}(I, IF_n)$ be a contraction mapping. Then T has a unique fixed point $\langle \Psi_1,\Psi_2\rangle$ such that $T(\langle \Psi_1,\Psi_2\rangle) = \langle \Psi_1,\Psi_2\rangle$.*

3. The main results

In this part of this section, we provide an existence and uniqueness result for the the following intuitionistic fuzzy differential equation:

3.1. **Second-order intuitionistic fuzzy differential equation.** We know that the space $C(I, IF_n)$ of continuous functions $\langle u,v\rangle : I \to IF_n$ is a complete metric space with the distance

$$H(\langle u,v\rangle, \langle z,w\rangle) = \sup_{t\in I} \{\mathrm{d}_\infty^n(\langle u,v\rangle(t), \langle z,w\rangle(t))\},$$

By $C^1(I, IF_n)$, we denote the set of continuous functions $\langle u,v\rangle : I \to IF_n$ such that $\langle u,v\rangle' : I \to IF_n$ exists as a continuous function. For $\langle u,v\rangle, \langle z,w\rangle \in C^1(I, IF_n)$, we define the distance

$$H_1(\langle u,v\rangle, \langle z,w\rangle) = H(\langle u,v\rangle, \langle z,w\rangle) + H\left(\langle u,v\rangle', \langle z,w\rangle'\right).$$

Lemma 3.1. $\left(C^1(I, IF_n), H_1\right)$ *is a complete metric space.*

Proof. Let $\{\langle u,v\rangle_p\}_{p=1}^{\infty} \subset C^1(I, IF_n)$ be a Cauchy sequence in $\left(C^1(I, IF_n), H_1\right)$, that is,

$$H_1(\langle u,v\rangle_p, \langle u,v\rangle_q) = H(\langle u,v\rangle_p, \langle u,v\rangle_q) + H\left(\langle u,v\rangle_p', \langle u,v\rangle_q'\right) \to 0 \ ; \ p,q \to +\infty.$$

Then the sequences $\{\langle u,v\rangle_p\}_{p=1}^{\infty}$ and $\{\langle u,v\rangle_p'\}_{p=1}^{\infty}$ are Cauchy sequences in $(C(I, IF_n), H)$, which is complete. Then there exist $\langle u,v\rangle, \langle z,w\rangle \in C(I, IF_n)$ such that $\{\langle u,v\rangle_p\} \to \langle u,v\rangle$ and $\left\{\langle u,v\rangle_p'\right\} \to \langle z,w\rangle$ as $p \to +\infty$. We have to prove that $\langle u,v\rangle \in C^1(I, IF_n)$ and that $\langle u,v\rangle' = \langle z,w\rangle$. In that case,

$$\begin{aligned} H_1(\langle u,v\rangle_p, \langle u,v\rangle) =& H(\langle u,v\rangle_p, \langle u,v\rangle) + H(\langle u,v\rangle_p', \langle u,v\rangle') \\ =& H(\langle u,v\rangle_p, \langle u,v\rangle) + H(\langle u,v\rangle_p', \langle z,w\rangle) \to 0, p \to +\infty, \end{aligned}$$

which proves that $\{\langle u,v\rangle_p\} \to \langle u,v\rangle$ in $\left(C^1(I, IF_n), H_1\right)$ and $C^1(I, IF_n)$ is a complete space. We know that $\langle u,v\rangle(t) = \langle u,v\rangle(t_0) + \int_{t_0}^{t} \langle z,w\rangle(s)\mathrm{d}s$, the continuity of $\langle z,w\rangle$ find $\langle u,v\rangle \in C^1(I,\ IF_n)$ and $\langle u,v\rangle' = \langle z,w\rangle$.

We will use that $\langle u,v\rangle_p(t) = \langle u,v\rangle_p(t_0) + \int_{t_0}^{t} \langle u,v\rangle_p'(s)\mathrm{d}s.$

Then

$$\langle u,v\rangle(t) = \langle u,v\rangle(t_0) + \int_{t_0}^{t} \langle z,w\rangle(s)\mathrm{d}s,\ t \in [t_0, T],$$

and the complete character of $C^1(I, IF_n)$ is achieved. □

Theorem 3.1. *Let*

(1) *A mapping* $f : [t_0, T] \times IF_n \times IF_n \to IF_n$ *is continuous,*
(2) *Suppose that there exist* $M_1, M_2 > 0$ *such that*

$$\begin{aligned} d_\infty^n(f(t, \langle u,v\rangle_1, \langle u,v\rangle_2), f(t, \langle z,w\rangle_1, \langle z,w\rangle_2)) \leq \\ M_1 d_\infty^n(\langle u,v\rangle_1, \langle z,w\rangle_1) + M_2 d_\infty^n(\langle u,v\rangle_2, \langle z,w\rangle_2), \end{aligned} \tag{3}$$

for all $t \in [t_0, T], \langle u,v\rangle_1, \langle u,v\rangle_2, \langle z,w\rangle_1, \langle z,w\rangle_2 \in IF_n$.

Then the initial value problem (1) has a unique solution on $[t_0, T]$.

Proof. Let $I = [t_0, T]$, and consider the complete metric space $\left(C^1(I, IF_n), H_1\right)$. Define the operator

$$\begin{aligned} G : C^1(I, IF_n) &\to C^1(I, IF_n) \\ \langle u,v\rangle &\to G\langle u,v\rangle, \end{aligned}$$

given by,

$$(G\langle u,v\rangle)(t)=k_1+k_2(t-t_0)+\int_{t_0}^{t}\int_{t_0}^{z} f\left(s,\langle u,v\rangle(s),\langle u,v\rangle'(s)\right)\mathrm{d}s\mathrm{d}z,\ t\in I.$$

We note that,

$$(G\langle u,v\rangle)'(t)=k_2+\int_{t_0}^{t} f\left(s,\langle u,v\rangle(s),\langle u,v\rangle'(s)\right)\mathrm{d}s,\ t\in I.$$

Indeed,

$$\begin{aligned}
H_1\left(G\langle u,v\rangle,G\langle z,w\rangle\right)&=H\left(G\langle u,v\rangle,G\langle z,w\rangle\right)+H\left((G\langle u,v\rangle)',(G\langle z,w\rangle)'\right)\\
&=\sup_{t\in I}\left\{\mathrm{d}\left(\int_{t_0}^{t}\int_{t_0}^{z} f\left(s,\langle u,v\rangle(s),\langle u,v\rangle'(s)\right)\mathrm{d}s\mathrm{d}z,\int_{t_0}^{t}\int_{t_0}^{z} f\left(s,\langle z,w\rangle(s),\langle z,w\rangle'(s)\right)\mathrm{d}s\mathrm{d}z\right)\right\}\\
&\quad+\sup_{t\in I}\left\{\mathrm{d}\left(\int_{t_0}^{t} f\left(s,\langle u,v\rangle(s),\langle u,v\rangle'(s)\right)\mathrm{d}s,\int_{t_0}^{t} f\left(s,\langle z,w\rangle(s),\langle z,w\rangle'(s)\right)\mathrm{d}s\right)\right\}\\
&\le\sup_{t\in I}\left\{\int_{t_0}^{t}\int_{t_0}^{z}\mathrm{d}\left(f\left(s,\langle u,v\rangle(s),\langle u,v\rangle'(s)\right),f\left(s,\langle z,w\rangle(s),\langle z,w\rangle'(s)\right)\right)\mathrm{d}s\mathrm{d}z\right\}\\
&\le\sup_{t\in I}\left\{\int_{t_0}^{t}\int_{t_0}^{z}\left[M_1\mathrm{d}\left(\langle u,v\rangle(s),\langle z,w\rangle(s)\right)+M_2\mathrm{d}\left(\langle u,v\rangle'(s),\langle z,w\rangle'(s)\right)\right]\mathrm{d}s\mathrm{d}z\right\}\\
&\quad+\sup_{t\in I}\left\{\int_{t_0}^{t}\left[M_1\mathrm{d}\left(\langle u,v\rangle(s),\langle z,w\rangle(s)\right)+M_2\mathrm{d}\left(\langle u,v\rangle'(s),\langle z,w\rangle'(s)\right)\right]\mathrm{d}s\right\}\\
&\le\sup_{t\in I}\left\{\int_{t_0}^{t}\int_{t_0}^{z}\left[M_1H(\langle u,v\rangle,\langle z,w\rangle)+M_2H\left(\langle u,v\rangle',\langle z,w\rangle'\right)\right]\mathrm{d}s\mathrm{d}z\right\}\\
&\quad+\sup_{t\in I}\left\{\int_{t_0}^{t}\left[M_1H(\langle u,v\rangle,\langle z,w\rangle)+M_2H(\langle u,v\rangle',\langle z,w\rangle')\right]\mathrm{d}s\right\}\\
&=\left[M_1H\left(\langle u,v\rangle,\langle z,w\rangle\right)+M_2H\left(\langle u,v\rangle',\langle z,w\rangle\right)'\right]\left(\sup_{t\in I}\left\{\int_{t_0}^{t}\int_{t_0}^{z}\mathrm{d}s\mathrm{d}z\right\}+\sup_{t\in I}\left\{\int_{t_0}^{t}\mathrm{d}s\right\}\right)\\
&\le\max\left\{M_1,M_2\right\}H_1\left(\langle u,v\rangle,\langle z,w\rangle\right)\left(\sup_{t\in I}\left\{\int_{t_0}^{T}\int_{t_0}^{T}\mathrm{d}s\mathrm{d}z\right\}+\sup_{t\in I}\left\{\int_{t_0}^{T}\mathrm{d}s\right\}\right)\\
&\le\max\left\{M_1,M_2\right\}H_1(\langle u,v\rangle,\langle z,w\rangle)\left((T-t_0)^2+(T-t_0)\right)
\end{aligned}$$

Therefore,

$$H_1\left(G\langle u,v\rangle,G\langle z,w\rangle\right)\le\max\{M_1,M_2\}\left((T-t_0)^2+(T-t_0)\right)H_1(\langle u,v\rangle,\langle z,w\rangle).$$

We can choose,

$$\max\left\{M_1,M_2\right\}\left((T-t_0)^2+(T-t_0)\right)<1,$$

and G is a contractive mapping. We note that the unique fixed point of G is in the space $C^1(I,IF_n)$. Using that $G\langle u,v\rangle$ is the integral of a continuous function that we have actually in the space $C^2\left(I,IF_n\right)$. □

3.2. **Higher-order intuitionistic fuzzy differential equations.** The results obtained in the previous subsection admit a generalization for the case of nth-order intutionistic fuzzy initial value problems of the type:

$$\begin{cases}\langle u,v\rangle^{(n)}(t)=f\left(t,\langle u,v\rangle(t),\langle u,v\rangle'(t),\ldots,\langle u,v\rangle^{(n-1)}(t)\right),\ t\in[t_0,T],\\ \langle u,v\rangle(t_0)=k_1,\ \langle u,v\rangle'(t_0)=k_2,\ \ldots\ ,\ \langle u,v\rangle^{(n-1)}(t_0)=k_n,\end{cases}\tag{4}$$

where $f\ :\ [t_0,T]\times(IF_n)^n\rightarrow IF_n, k_1,k_2,\ldots,k_n$ are intuitionistic fuzzy numbers, and $\langle u,v\rangle^{(i)}$ represents the ith-derivative of $\langle u,v\rangle$ in the sense of Hukuhara.

Lemma 3.2. *Consider the space,*

$$C^n\left(I,IF_n\right)=\left\{\langle u,v\rangle\in C\left(I,IF_n\right):\exists\langle u,v\rangle',\ldots,\langle u,v\rangle^{(n)}\in C\left(I,IF_n\right)\right\}$$

furnished with the distance,

$$\begin{aligned} H_n\left(\langle u,v\rangle,\langle z,w\rangle\right) &= H\left(\langle u,v\rangle,\langle z,w\rangle\right)+H\left(\langle u,v\rangle',\langle z,w\rangle'\right)+\cdots+H\left(\langle u,v\rangle^{(n)},\langle z,w\rangle^{(n)}\right)\\ &=\sum_{i=0}^{n} H\left(\langle u,v\rangle^{(i)},\langle z,w\rangle^{(i)}\right), \end{aligned}$$

where of course, $\langle u,v\rangle^{(0)}=\langle u,v\rangle$. *Then, for every* $n\in\mathbb{N}, n\geq 0, (C^n(I,IF_n),H_n)$ *is a complete metric space.*

Theorem 3.2. *The function* $\langle u,v\rangle\in C^n(I,IF_n)$ *is a solution to problem* (4) *if and only if* $\langle u,v\rangle$ *satisfies the following integral equation, for all* $z_1\in[t_0,T]$,

$$\begin{aligned} \langle u,v\rangle(z_1) =& k_1+k_2(z_1-t_0)+k_3\int_{t_0}^{z_1}(z_2-t_0)\,\mathrm{d}z_2+k_4\int_{t_0}^{z_1}\int_{t_0}^{z_2}(z_3-t_0)\mathrm{d}_{z_3}\mathrm{d}_{z_2}\\ &+\cdots+k_n\int_{t_0}^{z_1}\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-2}}(z_{n-1}-t_0)\,\mathrm{d}z_{n-1}\ .\ .\ .\ \mathrm{d}_{z_3}\mathrm{d}_{z_2}\\ &+\int_{t_0}^{z_1}\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s,\langle u,v\rangle(s),\ldots,\langle u,v\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}_{z_3}\mathrm{d}_{z_2}. \end{aligned}$$

Proof. For $z_n\in[t_0,T]$, we have

$$\langle u,v\rangle^{(n-1)}(z_n)=k_n+\int_{t_0}^{z_n} f\left(s,\langle u,v\rangle(s),\ldots,\langle u,v\rangle^{(n-1)}(s)\right)\mathrm{d}s,$$

and, for $z_{n-1}\in[t_0,T]$,

$$\langle u,v\rangle^{(n-2)}(z_{n-1})=k_{n-1}+k_n(z_{n-1}-t_0)+\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s,\langle u,v\rangle(s),\ldots,\langle u,v\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n.$$

By recurrence, for $z_2\in[t_0,T]$, we get

$$\begin{aligned} \langle u,v\rangle'(z_2) =& k_2+k_3(z_2-t_0)+k_4\int_{t_0}^{z_2}(z_3-t_0)\,\mathrm{d}z_3\\ &+\cdots+k_n\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-2}}(z_{n-1}-t_0)\,\mathrm{d}s\mathrm{d}z_{n-1}\ .\ .\ .\ \mathrm{d}_{z_3}\\ &+\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s,\langle u,v\rangle(s),\ldots,\langle u,v\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n\ .\ .\ .\ \mathrm{d}_{z_3}. \end{aligned}$$

The expression is obtained integrating last equality from t_0 to $z_1\in[t_0,T]$ with respect to z_2. □

Theorem 3.3. *Let*

(1) $f:[t_0,T]\times(IF_n)^n\to IF_n$ *is continuous,*
(2) *Suppose that there exist* $M_1,M_2,\ldots,M_n>0$ *such that*

$$\begin{aligned} & d_\infty^n\left(f\left(t,\langle u,v\rangle_1,\langle u,v\rangle_2\ldots,\langle u,v\rangle_n\right),f\left(t,\langle z,w\rangle_1,\langle z,w\rangle_2,\ldots,\langle z,w\rangle_n\right)\right)\\ &\qquad\leq\sum_{i=1}^{n} M_i\mathrm{d}_\infty^{\mathrm{n}}\left(\langle u,v\rangle_i,\langle z,w\rangle_i\right), \end{aligned} \tag{5}$$

for all $t\in[t_0,T],\langle u,v\rangle_1,\langle u,v\rangle_2,\ldots,\langle u,v\rangle_n,\langle z,w\rangle_1,\langle z,w\rangle_2,\ldots,\langle z,w\rangle_n\in IF_n$.

Then the initial value problem (6) *has a unique solution on* $[t_0,T]$.

Proof. Let $I = [t_0, T]$, consider the complete metric space $\left(C^{n-1}(I, IF_n), H_{n-1}\right)$, and define the operator,

$$G_{n-1} : C^{n-1}(I, IF_n) \to C^{n-1}(I, IF_n)$$
$$\langle u, v\rangle \to G_{n-1}\langle u, v\rangle,$$

given by the right-hand side in the integral expression obtained in Theorem 3.2, that is, for $z_1 \in [t_0, T]$,

$$\begin{aligned}(G_{n-1}\langle u, v\rangle)(z_1) =& k_1 + k_2(z_1 - t_0) + k_3 \int_{t_0}^{z_1} (z_2 - t_0)\mathrm{d}_{z_2} + k_4 \int_{t_0}^{z_1}\int_{t_0}^{z_2} (z_3 - t_0)\mathrm{d}_{z_3}\mathrm{d}_{z_2} \\ &+ \cdots + k_n \int_{t_0}^{z_1}\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-2}} (z_{n-1} - t_0)\mathrm{d}z_{n-1} \ . \ . \ . \ \mathrm{d}_{z_3}\mathrm{d}_{z_2} \\ &+ \int_{t_0}^{z_1}\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f(s, \langle u, v\rangle(s), \ldots, \langle u, v\rangle^{(n-1)}(s))\mathrm{d}s\mathrm{d}z_n \ldots \mathrm{d}_{z_3}\mathrm{d}_{z_2}.\end{aligned}$$

We note that $G_{n-1}\langle u, v\rangle \in C^n(I, IF_n)$ and the ith-derivative, for every $i = 1, \ldots, n-1$, is expressed in the proof of Theorem 3.2. Following the procedure of the proof of Theorem 3.1, we prove that G_{n-1} is a contractive mapping.
Indeed, we choose,

$$\max\{M_1, \ldots, M_n\} \sum_{j=0}^{n-1} \frac{1}{j!}(T - t_0)^{j+1} < 1,$$

Then, $H_{n-1}\left(G_{n-1}\langle u, v\rangle, G_{n-1}\langle z, w\rangle\right)$

$$\begin{aligned}&= \sum_{i=0}^{n-1} H\left((G_{n-1}\langle u, v\rangle)^{(i)}, (G_{n-1}\langle z, w\rangle)^{(i)}\right) \\ &= \sup_{z_1 \in I}\Bigg\{\mathrm{d}_\infty^{\mathrm{n}}\Bigg(\int_{t_0}^{z_1}\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s, \langle u, v\rangle(s), \ldots, \langle u, v\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n \ldots \mathrm{d}_{z_3}\mathrm{d}_{z_2}, \\ &\quad \int_{t_0}^{z_1}\int_{t_0}^{z_1} \ . \ . \ . \int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s, \langle z, w\rangle(s), ..., \langle z, w\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n \ . \ . \ . \ \mathrm{d}_{z_3}\mathrm{d}_{z_2}\Bigg)\Bigg\} \\ &\quad \int_{t_0}^{z_2} \ . \ . \ . \int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s, \langle z, w\rangle(s), ..., \langle z, w\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n \ . \ . \ . \ \mathrm{d}z_3\Bigg)\Bigg\} \\ &\quad + \cdots + \sup_{z_{n-2} \in I}\Bigg\{\mathrm{d}_\infty^{\mathrm{n}}\Bigg(\int_{t_0}^{z_{n-2}}\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s, \langle u, v\rangle(s), \ldots, \langle u, v\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n\mathrm{d}z_{n-1}, \\ &\quad \int_{t_0}^{z_{n-2}}\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s, \langle z, w\rangle(s), ...\langle z, w\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n\mathrm{d}z_{n-1}\Bigg)\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&+\sup_{z_{n-1}\in I}\left\{\mathrm{d}_\infty^{\mathrm{n}}\left(\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s,\langle u,v\rangle(s),\ldots,\langle u,v\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n,\right.\right.\\
&\left.\left.\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n} f\left(s,\langle z,w\rangle(s),\ldots,\langle z,w\rangle^{(n-1)}(s)\right)\mathrm{d}s\mathrm{d}z_n\right)\right\}\\
&+\sup_{z_n\in I}\left\{\mathrm{d}_\infty^{\mathrm{n}}\left(\int_{t_0}^{z_n} f\left(s,\langle u,v\rangle(s),\ldots,\langle u,v\rangle^{(n-1)}(s)\right)\mathrm{d}s,\int_{t_0}^{z_n} f\left(s,\langle z,w\rangle(s),\ldots,\langle z,w\rangle^{(n-1)}(s)\right)\mathrm{d}s\right)\right\}\\
\le&\sup_{z_1\in I}\left\{\int_{t_0}^{z_1}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\sum_{i=0}^{n-1}M_{i+1}\mathrm{d}_\infty^{\mathrm{n}}\left(\langle u,v\rangle^{(i)}(s),\langle z,w\rangle^{(i)}(s)\right)\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}_{z_2}\right\}\\
&+\sup_{z_2\in I}\left\{\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\sum_{i=0}^{n-1}M_{i+1}\mathrm{d}_\infty^{\mathrm{n}}\left(\langle u,v\rangle^{(i)}(s),\langle z,w\rangle^{(i)}(s)\right)\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}_{z_3}\right\}\\
&+\cdots+\sup_{z_{n-2}\in I}\left\{\int_{t_0}^{z_{n-2}}\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\sum_{i=0}^{n-1}M_{i+1}\mathrm{d}_\infty^{\mathrm{n}}\left(\langle u,v\rangle^{(i)}(s),\langle z,w\rangle^{(i)}(s)\right)\mathrm{d}s\mathrm{d}z_n\mathrm{d}z_{n-1}\right\}\\
&+\sup_{z_{n-1}\in I}\left\{\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\sum_{i=0}^{n-1}M_{i+1}\mathrm{d}_\infty^{\mathrm{n}}\left(\langle u,v\rangle^{(i)}(s),\langle z,w\rangle^{(i)}(s)\right)\mathrm{d}s\mathrm{d}z_n\right\}\\
&+\sup_{z_n\in I}\left\{\int_{t_0}^{z_n}\sum_{i=0}^{n-1}M_{i+1}\mathrm{d}_\infty^{\mathrm{n}}\left(\langle u,v\rangle^{(i)}(s),\langle z,w\rangle^{(i)}(s)\right)\mathrm{d}s\right\}\\
\le&\sum_{i=0}^{n-1}M_{i+1}H\left(\langle u,v\rangle^{(i)},\langle z,w\rangle^{(i)}\right)\left[\sup_{z_1\in I}\left\{\int_{t_0}^{z_1}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}_{z_2}\right\}\right.\\
&+\sup_{z_2\in I}\left\{\int_{t_0}^{z_2}\cdots\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}_{z_3}\right\}\\
&\left.+\cdots+\sup_{z_{n-2}\in I}\left\{\int_{t_0}^{z_{n-2}}\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\mathrm{d}s\mathrm{d}z_n\mathrm{d}z_{n-1}\right\}+\sup_{z_{n-1}\in I}\left\{\int_{t_0}^{z_{n-1}}\int_{t_0}^{z_n}\mathrm{d}s\mathrm{d}z_n\right\}+\sup_{z_n\in I}\left\{\int_{t_0}^{z_n}\mathrm{d}s\right\}\right]\\
\le&\max\{M_1,\ldots,M_n\}H_{n-1}(\langle u,v\rangle,\langle z,w\rangle)\sum_{j=0}^{n-1}\sup_{z_{n-j}\in I}\{\Delta_j(z_{n-j})\}\,.
\end{aligned}
$$

Where,

$$
\Delta_j(z_{n-j})=\int_{t_0}^{z_{n-j}}\int_{t_0}^{z_{n-(j-1)}}\cdots\int_{t_0}^{z_n}\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}z_{n-(j-2)}\mathrm{d}z_{n-(j-1)}.
$$

Using that, for every F continuous on $[a,b]$,

$$
\int_a^x \mathrm{d}x_n\int_a^{x_n}\mathrm{d}x_{n-1}\cdots\int_a^{x_3}\mathrm{d}x_2\int_a^{x_2}F(x_1)\mathrm{d}x_1=\frac{1}{(n-1)!}\int_a^x(x-t)^{n-1}F(t)\mathrm{d}t,
$$

Then, for the following expressions involving $j+1$ integrals, we get, for $z_{n-j}\in I$,

$$
\begin{aligned}
\Delta_j(z_{n-j})=&\int_{t_0}^{z_{n-j}}\int_{t_0}^{z_{n-j+1}}\cdots\int_{t_0}^{z_n}\mathrm{d}s\mathrm{d}z_n\ldots\mathrm{d}z_{n-(j)}\\
=&\frac{1}{j!}\int_{t_0}^{z_{n-j}}(z_{n-j}-s)^j\mathrm{d}s\ for\quad j=0,1,\ldots,n-1.
\end{aligned}
$$

Then, for $j=0,1,\ldots,n-1$.

$$
\begin{aligned}
\sup_{z_{n-j}\in l}\{\Delta_j(z_{n-j})\}=&\frac{1}{j!}\sup_{z_{n-j}\in l}\left\{\int_{t_0}^{z_{n-j}}(z_{n-j}-s)^j\mathrm{d}s\right\}\\
\le&\frac{1}{j!}\sup_{z_{n-j}\in l}\left\{\int_{t_0}^{z_{n-j}}(T-t_0)^j\mathrm{d}s\right\}=\quad\frac{1}{j!}(T-t_0)^{j+1};
\end{aligned}
$$

and,

$$
H_{n-1}\left(G_{n-1}\langle u,v\rangle,G_{n-1}\langle z,w\rangle\right)\le\max\{M_1,\ldots,M_n\}H_{n-1}\left(\langle u,v\rangle,\langle z,w\rangle\right)\sum_{j=0}^{n-1}\frac{1}{j!}(T-t_0)^{j+1}.
$$

□

4. Applications

To give a clear overview of our study and illustrate the above discussed concept, we consider the following examples.

Example 4.1. *We consider the second order intuitionistic fuzzy differential equation:*

$$\begin{cases} \langle u,v\rangle''(t) = q_1\langle u,v\rangle(t) + q_2\langle u,v\rangle'(t) + A(t), \quad t\in[t_0,T], \\ \langle u,v\rangle(t_0) = k_1, \langle u,v\rangle'(t_0) = k_2, \end{cases} \tag{6}$$

with $A \in C\left([t_0,T], IF_n\right), q_1, q_2 \in \mathbb{R}$ *and* $k_1, k_2 \in IF_n$. *Here*

$$f(t,\langle u,v\rangle_1,\langle u,v\rangle_2) = q_1\langle u,v\rangle_1 + q_2\langle u,v\rangle_2 + A(t) \ ,$$

and hypothesis (3) *holds. Indeed,*

$$\begin{aligned} \mathrm{d}^{\mathrm{n}}_\infty\left(f(t,\langle u,v\rangle_1,\langle u,v\rangle_2), f(t,\langle z,w\rangle_1,\langle z,w\rangle_2)\right) \leq& \mathrm{d}^{\mathrm{n}}_\infty\left(q_1\langle u,v\rangle_1 + q_2\langle u,v\rangle_2 + A(t), q_1\langle z,w\rangle_1 + q_2\langle z,w\rangle_2 + A(t)\right) \\ =& \mathrm{d}^{\mathrm{n}}_\infty\left(q_1\langle u,v\rangle_1 + q_2\langle u,v\rangle_2, q_1\langle z,w\rangle_1 + q_2\langle z,w\rangle_2\right) \\ \leq& \mathrm{d}^{\mathrm{n}}_\infty\left(q_1\langle u,v\rangle_1, q_1\langle z,w\rangle_1\right) + \mathrm{d}^{\mathrm{n}}_\infty\left(q_2\langle u,v\rangle_2, q_2\langle z,w\rangle_2\right) \\ =& |q_1|\mathrm{d}^{\mathrm{n}}_\infty\left(\langle u,v\rangle_1, \langle z,w\rangle_1\right) + |q_2|\mathrm{d}^{\mathrm{n}}_\infty\left(\langle u,v\rangle_2, \langle z,w\rangle_2\right), \end{aligned}$$

for all $t \in [t_0,T], \langle u,v\rangle_1, \langle u,v\rangle_2, \langle z,w\rangle_1, \langle z,w\rangle_2 \in IF_n$. *Then there exists a unique fixed point* $\langle u,v\rangle \in C^1(I,IF_n)$ *of* G, *which is the uniqueness solution* $\langle u,v\rangle \in C^2\left(I,IF_n\right)$ *of* (6).

Example 4.2. *As an application of Theorem 3.3, Consider the nth-order intuitionistic fuzzy differential equation:*

$$\begin{cases} \langle u,v\rangle^{(n)}(t) = q_1\langle u,v\rangle(t) + q_2\langle u,v\rangle'(t) + \cdots + q_n\langle u,v\rangle^{(n-1)}(t) + A(t) \ , \quad t\in[t_0,T], \\ \langle u,v\rangle(t_0) = k_1, \langle u,v\rangle'(t_0) = k_2, \ldots, \langle u,v\rangle^{(n-1)}(t_0) = k_n, \end{cases} \tag{7}$$

with $A \in C\left([t_0,T], IF_n\right), q_1, q_2, \ldots, q_n \in \mathbb{R}$ *and* $k_1, k_2, \ldots, k_n \in IF_n$.
Function f *is given by,*

$$f(t,\langle u,v\rangle_1,\langle u,v\rangle_2,\cdots,\langle u,v\rangle_n) = q_1\langle u,v\rangle_1 + q_2\langle u,v\rangle_2 + \cdots + q_n\langle u,v\rangle_n + A(t),$$

which trivially satisfies (4)

$$d^n_\infty\left(f\left(t,\langle u,v\rangle_1,\langle u,v\rangle_2,\cdots,\langle u,v\rangle_n\right), f\left(t,\langle z,w\rangle_1,\langle z,w\rangle_2,\cdots,\langle z,w\rangle_n\right)\right)$$

$$=\mathrm{d}^{\mathrm{n}}_\infty\left(q_1\langle u,v\rangle_1 + q_2\langle u,v\rangle_2 + \cdots + q_n\langle u,v\rangle_n + A(t), q_1\langle z,w\rangle_1 + q_2\langle z,w\rangle_2 + \cdots + q_n\langle z,w\rangle_n + A(t)\right)$$

$$\leq \sum_{i=1}^n \mathrm{d}^{\mathrm{n}}_\infty\left(q_i\langle u,v\rangle_i, q_i\langle z,w\rangle_i\right) = \sum_{i=1}^n |q_i|\mathrm{d}^{\mathrm{n}}_\infty\left(\langle u,v\rangle_i, \langle z,w\rangle_i\right),$$

for all $t \in [t_0,T], \langle u,v\rangle_1, \langle u,v\rangle_2, \ldots, \langle u,v\rangle_n, \langle z,w\rangle_1, \langle z,w\rangle_2, \ldots, \langle z,w\rangle_n \in IF_n$. *Theorem 3.3 shows that there exists a unique solution to problem (7) in* $C^n(I,IF_n)$.

5. Conclusions

In this paper, we have also derived the existence and uniqueness solutions of second-order intuitionistic fuzzy differential equations using the Banach fixed point theorem. Furthermore, we have generalized these results for nth-order intuitionistic fuzzy differential equations, and examples have been provided to illustrate the findings. These ideas can be applied to investigate non-local nth-order intuitionistic fuzzy differential equations in the next phase of our future research, under weaker hypotheses than the Lipschitz condition.

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