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SELF CENTERED INTERVAL-VALUED PYTHAGOREAN FUZZY GRAPH STRUCTURE WITH AN APPLICATION

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ABSTRACT. The Interval-valued Pythagorean Fuzzy Set (IVPFS), an extension of the Pythagorean Fuzzy Set (PFS), offers a more accurate description of uncertainty than traditional fuzzy sets. It helps us to discourse about the inestimable values of human life. In this paper, we introduce the notion of Interval-Valued Pythagorean Fuzzy graph Structure (IVPFGS), further we introduce the definition of $\lambda_J - strength$, $\lambda_J - length$, $\lambda_J - distance$, $\lambda_J - eccentricity$, $\lambda_J - radius$, $\lambda_J - diameter$, $\lambda_J - centered$, $\lambda_J - self$ centered, $\lambda_J - path$ cover, $\lambda_J - edge$ cover. Moreover we investigate some properties of $\lambda_J - Self$ centered IVPFGS with illustration and we have discussed application in IPFGS

Keywords: $\lambda_J - Self$ centered IVPFGS, $\lambda_J - path$ cover, $\lambda_J - edge$ cover, application. (Four or five keywords)

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1. INTRODUCTION

Fuzzy Sets (FSs), which expressed the uncertainty of the FSs, were initially introduced by L.A. Zadeh [27] in 1965. In these conditions, identifying and resolving uncertainties in science and medicine as well as ambiguities in our daily lives are important roles played by the business community. Fuzzy graph theory was created by Rosenfeld [2] in 1975 as a derivation of the Euler graph, whose core idea was previously presented by Kauffmann (1973). In order to extend the idea of a FS, Zadeh [26] invented the concept of an interval valued fuzzy subset of a set in 1975. In this concept, membership degrees are represented by intervals of numbers rather than by real numbers. The author first developed this concept in FSs by Turksen [21] in 1986. Then

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Interval-Valued Fuzzy Sets (IVFSs) were introduced. It contains the values of intervals of numbers to accommodate uncertainty rather than utilising numbers as the membership function. It is usually represented by the symbol $[\mu_{AL}^{-}(x), \mu_{AU}^{+}]$ Use the formula $0 \leq \mu_{AL}^{-}(x) + \mu_{AU}^{+}(x) \leq 1$ to characterise the degree of membership of the FS A. In 2009, Hongmei and Lianhua [3] defined Interval-Valued Fuzzy Graphs (IVFGs). Muhammad Akram and Wieslaw A. Dudek [4] presented some operations on IVFSs in 2011. FGs with irregular interval values were studied by Rashmanlou [[10], [12], [13], [14]]. Furthermore, they provided definitions for balanced, antipodal, and IVFGs as well as a few characteristics of very irregular IVFGs. The idea was initially put up in 2011 by M.G. Karunambigai and O.K. Kalaivani [11] as a self-centered intuitionistic FG. Muhammad Akram, M. Murtaza Yousaf [5], and Self-centered IVFG initially developed the concept in 2015. The research papers Applications of graph's total degree with bipolar fuzzy information and Estimation of most effected cycles and busiest network route based on complexity function of graph in fuzzy environment in 2022 by Soumitra Poulik and Ganesh Ghorai [[6], [17], [18]] is worth being referred to for more information. Also, in 2021 proposed the idea of Determining the order of journeys based on a graph's Wiener absolute index using bipolar fuzzy information. In (2019), K. Ullah, T. Mahmood, Z. Ali, and N. Jan [22] describe the notion of various distance measurements of complicated Pythagorean FSs and its applications in pattern recognition. Pythagorean fuzzy subsets are a concept introduced by R.R. Yager[[23], [24], [25]] in 2013. He also introduces the concepts of Pythagorean membership grades, complex numbers, and decision-making, as well as Pythagorean membership grades in multi-criteria decision-making. A graph structure is a generalisation of an undirected graph, which was first described by Sampathkumar [15]. It is quite useful for analysing different structures, including graphs, signed graphs, and graphs with labelled or coloured edges.With the use of a graph structure, the different relations and the corresponding edges may be examined simultaneously. T. Dinesh and T. V. Ramakrishnan [1] first proposed the concept of a fuzzy graph structure in 2011. Some similarity of rough interval Pythagorean fuzzy sets was recently proposed by V. S. Subha and P. Dhanalakshmi [[19], [20]], and An Introduction to Pythagorean Fuzzy Hyper Ideals in Hypersemigroups contains some helpful information about these kinds of structures. Muhammad Akram recently introduced the concepts of certain operations on bipolar fuzzy graph structures and intuitionistic FGSs [[6], [7], [8], [9]]. Introduce further the ideas of Simplified IVPFGs and A Novel Decision-Making Approach under Complex Pythagorean Fuzzy Environment with Applications. In the future, the ideas of Complex Pythagorean Fuzzy Planar Graphs and Simplified IVPFGs with Application were developed. In this study, we present the idea of an IVPFGS. We also give a definition $\lambda_J - strength$, $\lambda_J - length$, $\lambda_J - distance$, $\lambda_J - eccentricity$, $\lambda_J - radius$, $\lambda_J - diameter$, $\lambda_J - centered$, $\lambda_J - self$ centered, $\lambda_J - path$ cover, $\lambda_J - edge$ cover. Moreover, using an illustration, we investigate certain characteristics of the $\lambda J - Self$ centered IVPFGS.

2. Preliminaries

The development of the main results will be helped by the review of some fundamental definitions and properties in this section.[[4], [5], [11]]
A graph is an ordered pair $G^* = (Q,R)$, where Q is the set of vertices of G^*. Two vertices a and b in a graph G^* are said to be adjacent in G^* if $\{a,b\}$ is in an edge of G^*. we write $ab \in R$ to mean $\{a,b\} \in R$. A simple graph is considered complete if an edge connects each pair of unique vertices in it. Path $P : a_1a_2....a_{n+1}(n > 0)$in G^*has a length of n. If $a_1 = a_{n+1}$and $n \geq 3$, a path $P : a_1a_2....a_{n+1}$ in $G^*$is referred to as a cycle. Recall the path graph. By eliminating any edge from the cycle graph, C_n, P_n, which contains $n-1$ edges, can be obtained. If there is a path between every pair of unique vertices in an undirected graph G^*, the graph is said to be linked. If there is a path between every pair of different vertices in a connected graph G^*, then for a pair of vertices a,b, the distance $d(a,b)$ between a and b is equal to the length of the shortest path

linking a and b. The eccentricity $e(a) = \max\{d(a,b)/a \in Q\}$. The formula for the radius of a connected graph $G^*$is $r(G) = \min\{e(a)/a \in Q\}$. The definition of a connected graph's diameter is $d(G) = \max\{e(a)/a \in Q\}$. The eccentric set S of a graph is its set of eccentricities. A graph's $C(G^*)$centre is made up of the collection of vertices with the least amount of eccentricity. A graph is said to be self-centered if all of its nodes are located in the middle. Because every vertex has the same eccentricity, there is only one element in the eccentric set of a self-centered graph. A self-centered graph is equivalently defined as one with a diameter equal to its radius.

Definition 2.1 (7)**.** A Pythagorean fuzzy relation in Y is described as a PFS X in $Y \times Y$ and is characterised by

$$X = \{(rs, \mu_X(rs), \lambda_X(rs)) | rs \in Y \times Y\}$$

where the membership and non-membership functions of X are represented by $\mu_X : Y \times Y \to [0,1]$ and $\lambda_X : Y \times Y \to [0,1]$, respectively, such that $0 \leq \mu_X^2(rs) + \lambda_X^2(rs) \leq 1$ for all $rs \in Y \times Y$.

Definition 2.2 (7)**.** On a non-empty set X, a Pythagorean fuzzy graph is a pair $G = (A,B)$, where A and B are Pythagorean fuzzy sets on X and a Pythagorean fuzzy relation on X, respectively, such that:

$$\begin{aligned} \mu_B(rs) &\leq \min\{\mu_A(r), \mu_A(s)\} \\ \lambda_B(rs) &\leq \max\{\lambda_A(r), \lambda_A(s)\} \end{aligned}$$

$0 \leq \mu_B^2(rs) + \lambda_B^2(rs) \leq 1$ for all $r,s \in X$. We call A and B the Pythagorean fuzzy vertex set and the Pythagorean fuzzy edge set of G, respectively.

Definition 2.3 (15)**.** A graph structure $G^* = (Q, R_{11}, R_{12}, ..., R_{1k})$, where V is a non-empty set together with mutually disjoint, irreflexive and symmetric relations $R_{11}, R_{12}, ...,$
R_{1k} on Q.

3. Interval Valued Pythagorean Fuzzy Graph Structure

This section defines IVPFGS and goes on to define useful tools that are used to create the major results. We also discuss some of the properties of the $\lambda_J - self$ centered IVPFGS using illustrations.

Definition 3.1. Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is referred to as an IVPFGS of graph structure (GS) $G^* = \{Q, R_1, R_2, ..., R_k\}$ if $\mu = (\mu_1, \mu_2) = ([\mu_1^-, \mu_1^+], [\mu_2^-, \mu_2^+])$ is an IVPFS on Q and $\lambda_J = (\lambda_{1J}, \lambda_{2J})$ $= ([\lambda_{1J}^-, \lambda_{1J}^+], [\lambda_{2J}^-, \lambda_{2J}^+])$ are IVPFSs on Q and R_J such that
(i) $\lambda_{1J}^-(a,b) \leq \min\{\mu_1^-(a), \mu_1^-(b)\}$ and $\lambda_{1J}^+(a,b) \leq \min\{\mu_1^+(a), \mu_1^+(b)\}$
(ii) $\lambda_{2J}^-(a,b) \leq \max\{\mu_2^-(a), \mu_2^-(b)\}$ and
$\lambda_{2J}^+(a,b) \leq \max\{\mu_2^+(a), \mu_2^+(b)\}$ and $(\lambda_{1J}^+(a,b))^2 + (\lambda_{2J}^+(a,b))^2 \leq 1$, $\forall\, ab \in R_J, J = 1,2,...,k$.
Note: λ_{1J}^-, λ_{1J}^+, λ_{2J}^- and λ_{2J}^+ are function from R_J to [0,1] such that $\lambda_{1J}^-(a,b) \leq \lambda_{1J}^+(a,b)$ and $\lambda_{2J}^-(a,b) \leq \lambda_{2J}^+(a,b)$ for all $(a,b) \in R_J, J = 1,2,...,k$.

Definition 3.2. An IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ of a GS $G^* = \{Q, R_1, R_2, ..., R_k\}$ is called a complete if (i) $\lambda_{1J}^-(a,b) = \min\{\mu_1^-(a), \mu_1^-(b)\}$ and $\lambda_{1J}^+(a,b) = \min\{\mu_1^+(a), \mu_1^+(b)\}$,
$(ii) \lambda_{2J}^-(a,b) = \max\{\mu_2^-(a), \mu_2^-(b)\}$ and $\lambda_{2J}^+(a,b) = \max\{\mu_2^+(a), \mu_2^+(b)\}, \forall\, ab \in R_J, J = 1,2,...,k$.

Example 3.3. Let $V = \{u_1, u_2, u_3, u_4\}$. Let $R_1 = \{u_1u_2, u_2, u_4, u_3u_4\}, R_2 = \{u_2u_3, u_4u_1\}$ be two disjoint symmetric relation on V. Then $G^* = (Q, R_1, R_2)$ graph structure. Let $\tilde{G} = (\mu, \lambda_1, \lambda_2)$ is a IVPFGS of GS G^* such that, $\mu = \{u_1([0.4,0.5][0.3,0.4]),$
$u_2([0.3,0.4][0.2,0.4]), u_3([0.3,0.4][0.3,0.5]), u_4([0.2,0.3][0.4,0.6])\}$ as shown in Figure-1.

FIGURE 1. $\tilde{G} = (\mu, \lambda_1, \lambda_2)$ is a complete IVPFGS of GS G^*

Definition 3.4. Let $P : u_1, u_2, ..., u_n$ be a path in IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$. The $\lambda_{1J}^s - strength$ of the paths are only selected the $\lambda_{1J}^s - edge$ in the paths is defined as $\max(\lambda_{1J}^s(u_i, u_j))$ for all $(u_i, u_j) \in R_J, s = -, +$. The $\lambda_{2J}^s - strength$ of the paths are only selected the $\lambda_{2J}^s - edge$ in the paths is defined as $\max(\lambda_{2J}^s(u_i, u_j))$ for all $(u_i, u_j) \in R_J, s = -, +$. The strength of the strongest path in P is indicated by $S_{\lambda_J} = ([\lambda_{1J}^-(u_i, u_j), \lambda_{1J}^+(u_i, u_j)]$ $[\lambda_{2J}^-(u_i, u_j), \lambda_{2J}^+(u_i, u_j)])$ for all $J = 1, 2, ..., k$. if the same $\lambda_J - edge$ possesses both the $\lambda_{1J}^s - strength$ and $\lambda_{2J}^s - strength$ values.

Example 3.5. Consider a IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2\}$ as shown in Figure-1 in Example-3.3. Here, u_1, u_2 is a path of length 1 and the $\lambda_1 - strength$ is $([0.3, 0.4], [0.3, 0.4])$, u_1, u_2, u_3 is a path of length 2 and the $\lambda_1 - strength$ is $([0.3, 0.4], [0.3, 0.4])$ and $\lambda_2 - strength$ is $([0.3, 0.4], [0.3, 0.5])$, u_1, u_2, u_4, is a path of length 2 and the $\lambda_1 - strength$ is $([0.3, 0.4], [0.4, 0.6])$, u_1, u_2, u_3, u_4 is a path of length 3 and the $\lambda_1 - strength$ is $([0.3, 0.4], [0.4, 0.6])$ and $\lambda_2 - strength$ is $([0.3, 0.4], [0.4, 0.6])$.

Definition 3.6. Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$. The $\lambda_J^s - length$ of a path $P : u_1 u_2 ... u_n$ in G^*, $l_{\lambda_{1J}^s}(p)$, is defined as $l_{\lambda_{1J}^s}(p) = \sum_{(u_i, u_j) \in R_J} \lambda_{1J}^s(u_i, u_j)$. The $\lambda_{2J}^s - length$ of a path $P : u_1 u_2 ... u_n$ in G^*, $l_{\lambda_{2J}^s}(p)$, is defined as $l_{\lambda_{2J}^s}(p) = \sum_{(u_i, u_j) \in R_J} \lambda_{2J}^s(u_i, u_j)$. The $\lambda_{1J}\lambda_{2J} - length$ of a path $P : u_1 u_2 ... u_n$ in G^*, $l_{\lambda_{1J}\lambda_{2J}}(p)$, is defined as $l_{\lambda_{1J}\lambda_{2J}}(p) = ([l_{\lambda_{1J}^-}, l_{\lambda_{1J}^+}], [l_{\lambda_{2J}^-}, l_{\lambda_{2J}^+}])$ for all $J = 1, 2, ...k, s = -, +$.

Example 3.7. Consider a connected IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2\}$ as shown in Figure-1 in Example-3.3. Here, $u_1 u_4$ is a path of $\lambda_2 - length$ and $l_{\lambda_{12}\lambda_{22}} = ([0.2, 0.3], [0.4, 0.6])$, $u_1 u_2$ is a path of $\lambda_1 - length$ and $l_{\lambda_{11}\lambda_{21}} = ([0.3, 0.4], [0.3, 0.4])$, $u_1 u_2 u_3$ is a path of $\lambda_1 - length$ and $l_{\lambda_{11}\lambda_{21}} = ([0.3, 0.4], [0.3, 0.4])$ and $u_1 u_2 u_3$ is a path of $\lambda_2 - length$ and $l_{\lambda_{12}\lambda_{22}} = ([0.3, 0.4], [0.3, 0.5])$, $u_1 u_2 u_4$ is a path of $\lambda_1 - length$ and $l_{\lambda_{11}\lambda_{12}} = ([0.5, 0.7], [0.7, 1.0])$, $u_1 u_2 u_3 u_4$ is a path of $\lambda_1 - length$ and $l_{\lambda_{11}\lambda_{21}} = ([0.5, 0.7], [0.7, 1.0])$ and $u_1 u_2 u_3 u_4$ is a path of $\lambda_2 - length$ and $l_{\lambda_{12}\lambda_{22}} = ([0.3, 0.4], [0.3, 0.5])$.

Definition 3.8. Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$. The $\lambda_{1J}^s - distance$, $\delta_{\lambda_{1J}^s}(u_i, u_j)$, is the smallest $\lambda_{1J}^s - length$ of any $u_i - u_j$ path P in $\tilde{G}$, where $u_i u_j \in R_J$. That is, $\delta_{\lambda_{1J}^s}(u_i, u_j) = \min(l_{\lambda_{1J}^-}(p))$. The $\lambda_{2J}^s - distance$, $\delta_{\lambda_{2J}^s}(u_i, u_j)$, is the smallest $\lambda_{2J}^s - length$ of any $u_i - u_j$ path P in $\tilde{G}$, where $u_i u_j \in R_J$. That is, $\delta_{\lambda_{2J}^+}(u_i, u_j) = \min(l_{\lambda_{2J}^s}(p))$ for all $s = -, +$. The distance, $\delta(u_i, u_j)$, is defined as $\delta_{\lambda_J}(u_i, u_j) = ([\delta_{\lambda_{1J}^-}(u_i, u_j), \delta_{\lambda_{1J}^+}(u_i, u_j)]$, $[\delta_{\lambda_{2J}^-}(u_i, u_j), \delta_{\lambda_{2J}^+}(u_i, u_j)])$ for all $J = 1, 2, ..., k$.

Example 3.9. Consider a connected IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2\}$ as shown in Figure-1 in Example-3.3. Here, $\delta_{\lambda_{12}^-}(u_1, u_4) = 0.2, \delta_{\lambda_{12}^+}(u_1, u_4) = 0.4, \delta_{\lambda_{22}^-}(u_1, u_4) = 0.4, \delta_{\lambda_{22}^+}(u_1, u_4) = 0.6$. That is $\delta_{\lambda_2}(u_1, u_4) = ([0.2, 0.4], [0.4, 0.6])$. Similarly, we calculate $\delta_{\lambda_1}(u_1, u_2) = ([0.3, 0.4], [0.3, 0.4])$, $\delta_{\lambda_1}(u_1, u_3) = ([0.3, 0.4], [0.3, 0.4])$ and $\delta_{\lambda_2}(u_1, u_3) = ([0.3, 0.4], [0.3, 0.5])$, $\delta_{\lambda_2}(u_2, u_3) = ([0.3, 0.4], [0.3, 0.5])$, $\delta_{\lambda_1}(u_2, u_4) = ([0.2, 0.3], [0.4, 0.6])$, $\delta_{\lambda_2}(u_2, u_4) = ([0.3, 0.4], [0.3, 0.5])$, $\delta_{\lambda_1}(u_3, u_4) = ([0.2, 0.3], [0.4, 0.6])$.

Definition 3.10. Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$. For each $u_i \in Q$, the $\lambda_{1J}^s - eccentricity$ of u_i, denoted by $e_{\lambda_{1J}^s}(u_i)$, is defined as $e_{\lambda_{1J}^s}(u_i) = \max\{\delta_{\lambda_{1J}^s}(u_i, u_j) / u_i \in Q, (u_i, u_j) \in R_J\}$ for all $s = -, +$. For each $u_i \in Q$, the $\lambda_{2J}^s - eccentricity$ of u_i, denoted by $e_{\lambda_{2J}^s}(u_i)$, is defined as $e_{\lambda_{2J}^s}(u_i) = \max\{\delta_{\lambda_{2J}^s}(u_i, u_j) / u_i \in Q, (u_i, u_j) \in R_J\}$ for all $s = -, +$. For each $u_i \in Q$, the $\lambda_J - eccentricity$ of u_i, denoted by $e_{\lambda_J}(u_i)$, is defined as $e_{\lambda_J}(u_i) = ([e_{\lambda_{1J}^-}(u_i), e_{\lambda_{1J}^+}(u_i)], [e_{\lambda_{2J}^-}(u_i), e_{\lambda_{2J}^+}(u_i)])$ for all $J = 1, 2, ..., k$.

Definition 3.11. Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$. The $\lambda_{1J}^s - radius$ of $\tilde{G}$ is denoted by $r_{\lambda_{1J}^s}(G)$ and is defined as $r_{\lambda_{1J}^s}(G) = \min\{e_{\lambda_{1J}^s}(u_i)/u_i \in Q\}$ for all $s = -,+$. The $\lambda_{2J}^s - radius$ of $\tilde{G}$ is denoted by $r_{\lambda_{2J}^s}(G)$ and is defined as $r_{\lambda_{2J}^s}(G) = \min\{e_{\lambda_{2J}^s}(u_i)/u_i \in Q\}$ for all $s = -,+$. The $\lambda_J - radius$ of $\tilde{G}$ is denoted by $r_{\lambda_J}(G)$ and is defined as $r_{\lambda_J}(G) = ([r_{\lambda_{1J}^-}(G), r_{\lambda_{1J}^+}(G)], [r_{\lambda_{2J}^-}(G), r_{\lambda_{2J}^+}(G)])$ for all $J = 1, 2, ..., k$.

Definition 3.12. Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS of graph structure $G^* = \{Q, R_1, R_2, ..., R_k\}$. The $\lambda_{1J}^s - diameter$ of $\tilde{G}$ is denoted by $d_{\lambda_{1J}^s}(G)$ and is defined as $d_{\lambda_{1J}^s}(G) = \max\{e_{\lambda_{1J}^s}(u_i)/u_i \in Q\}$ for all $s = -,+$. The $\lambda_{2J}^s - diameter$ of $\tilde{G}$ is denoted by $d_{\lambda_{2J}^s}(G)$ and is defined as $d_{\lambda_{2J}^s}(G) = \max\{e_{\lambda_{2J}^s}(u_i)/u_i \in Q\}$ for all $s = -,+$. The $\lambda_J - diameter$ of $\tilde{G}$ is denoted by $d_{\lambda_J}(G)$ and is defined as $d_{\lambda_J}(G) = ([d_{\lambda_{1J}^-}(G), d_{\lambda_{1J}^+}(G)], [d_{\lambda_{2J}^-}(G), d_{\lambda_{2J}^+}(G)])$ for all $J = 1, 2, ..., k$.

Example 3.13. From the above Examples-3.3, 3.7, 3.9. By routine computations, it is easy to see that: $e_{\lambda_J^-} - eccentricity$ and $e_{\lambda_J^+} - eccentricity$ of each vertex is
$e_{\lambda_{11}^-}(u_1) = 0.3,\ e_{\lambda_{12}^-}(u_1) = 0.3,\ e_{\lambda_{11}^-}(u_2) = 0.3,\ e_{\lambda_{12}^-}(u_2) = 0.3,$
$e_{\lambda_{11}^-}(u_3) = 0.3,\ e_{\lambda_{12}^-}(u_3) = 0.3,\ e_{\lambda_{11}^-}(u_4) = 0.2,\ e_{\lambda_{12}^-}(u_4) = 0.3,$
$e_{\lambda_{11}^+}(u_1) = 0.4,\ e_{\lambda_{12}^+}(u_1) = 0.4,\ e_{\lambda_{11}^+}(u_2) = 0.4,\ e_{\lambda_{12}^+}(u_2) = 0.3,$
$e_{\lambda_{11}^+}(u_3) = 0.4,\ e_{\lambda_{12}^+}(u_3) = 0.4,\ e_{\lambda_{11}^+}(u_4) = 0.3,\ e_{\lambda_{12}^+}(u_4) = 0.4$
$e_{\lambda_{21}^-}(u_1) = 0.3,\ e_{\lambda_{22}^-}(u_1) = 0.4,\ e_{\lambda_{21}^-}(u_2) = 0.4,\ e_{\lambda_{22}^-}(u_2) = 0.3,$
$e_{\lambda_{21}^-}(u_3) = 0.4,\ e_{\lambda_{22}^-}(u_3) = 0.3,\ e_{\lambda_{21}^-}(u_4) = 0.4,\ e_{\lambda_{22}^-}(u_4) = 0.4,$
$e_{\lambda_{21}^+}(u_1) = 0.4,\ e_{\lambda_{22}^+}(u_1) = 0.6,\ e_{\lambda_{21}^+}(u_2) = 0.6,\ e_{\lambda_{22}^+}(u_2) = 0.5,$
$e_{\lambda_{21}^+}(u_3) = 0.6,\ e_{\lambda_{22}^+}(u_3) = 0.5,\ e_{\lambda_{21}^+}(u_4) = 0.6,\ e_{\lambda_{22}^+}(u_4) = 0.6$
$\lambda_J - eccentricity$ of each vertex is
$e_{\lambda_1}(u_1) = ([0.3, 0.4], [0.3, 0.4])\ e_{\lambda_2}(u_1) = ([0.3, 0.4], [0.4, 0.6]),$
$e_{\lambda_1}(u_2) = ([0.3, 0.4], [0.4, 0.6]),\ e_{\lambda_2}(u_2) = ([0.3, 0.3], [0.3, 0.5]),$
$e_{\lambda_1}(u_3) = ([0.3, 0.4], [0.4, 0.6]),\ e_{\lambda_2}(u_3) = ([0.3, 0.4], [0.3, 0.5]),$
$e_{\lambda_1}(u_4) = ([0.2, 0.3], [0.4, 0.6]),\ e_{\lambda_2}(u_4) = ([0.3, 0.4], [0.4, 0.6]),$
$\lambda_1 - radius$ of $\tilde{G}$ is $r_{\lambda_1}(G) = ([0.2, 0.3], [0.3, 0.4])$ and
$\lambda_2 - radius$ of $\tilde{G}$ is $r_{\lambda_2}(G) = ([0.3, 0.3], [0.3, 0.5])$
$\lambda_1 - diameter$ of $\tilde{G}$ is $d_{\lambda_1}(G) = ([0.3, 0.4], [0.4, 0.6])$ and
$\lambda_2 - diameter$ of $\tilde{G}$ is $d_{\lambda_2}(G) = ([0.3, 0.4], [0.4, 0.6])$ for all $J = 1, 2$.

Definition 3.14. A vertex $u_i \in Q$ is called a $\lambda_J - central$ vertex of a connected IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$, if $r_{\lambda_{1J}^s}(G) = e_{\lambda_{1J}^s}(u_i)$ and $r_{\lambda_{2J}^s}(G) = e_{\lambda_{2J}^s}(u_i)$ for all $s = -,+$ and the set of all $\lambda_J - central$ vertices of an IVPFGS is denoted by $C(\tilde{G})$.

Definition 3.15. A connected IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is a $\lambda_J - self$ centered GS, if every vertex of $\tilde{G}$ is a $\lambda_J - central$ vertex, that is $r_{\lambda_{1J}^s}(G) = e_{\lambda_{1J}^s}(u_i)$ and $r_{\lambda_{11}^s}(G) = r_{\lambda_{12}^s}(G) = ... = r_{\lambda_{1k}^s}(G)$ and $r_{\lambda_{2J}^s}(G) = e_{\lambda_{2J}^s}(u_i)$ and $r_{\lambda_{21}^s}(G) = r_{\lambda_{22}^s}(G) = ... = r_{\lambda_{2k}^s}(G)\ \forall\ u_i \in Q,\ J = 1, 2, ..., k,\ s = -,+$.

Example 3.16. Consider a connected IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2, \lambda_3\}$ of GS $G^* = \{Q, R_1, R_2, R_3\}$ such that $\mu = \{u_1([0.3, 0.4], [0.4, 0.5]), u_2([0.4, 0.5], [0.3, 0.4]), u_3([0.3, 0.5], [0.4, 0.6])\}$ as shown in Figure-2. By routine computations, it is easy to see that:

FIGURE 2. $\tilde{G} = (\mu, \lambda_1, \lambda_2, \lambda_3)$ is $\lambda_J - self$ centered IVPFGS of G^*

(i) $\lambda_J - distance$
$\delta_{\lambda_1}(u_1, u_2) = ([0.3, 0.4], [0.4, 0.5]),\ \delta_{\lambda_2}(u_2, u_3) = ([0.3, 0.4], [0.4, 0.5]),$

$\delta_{\lambda_3}(u_1,u_3) = ([0.3,0.4],[0.4,0.5])$
(ii) $\lambda_J - eccentricity$ of each vertex is
$e_{\lambda_1}(u_1) = ([0.3,0.4],[0.4,0.5]),\ e_{\lambda_3}(u_1) = ([0.3,0.4],[0.4,0.5]),$
$e_{\lambda_1}(u_2) = ([0.3,0.4],[0.4,0.5]),\ e_{\lambda_2}(u_3) = ([0.3,0.4],[0.4,0.5]),$
$e_{\lambda_3}(u_3) = ([0.3,0.4],[0.4,0.5])$
(iii)$\lambda_J - radius$ of $\tilde{G}$ is $r_{\lambda_J}(G) = ([0.3,0.4],[0.4,0.5])$ for all $J = 1,2,3$.
Hence, $\tilde{G}$ is $\lambda - self$ centered interval-valued fuzzy graph structure.

Definition 3.17. A $\lambda_J - path$ cover of an IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ of GS $G^* = \{Q,R_1,R_2,...,R_k\}$ is a set P of $\lambda_J - paths$ such that every vertex of $\tilde{G}$ is incident to some $\lambda_J - path$ of P for all $J = 1,2,...,k$.

Example 3.18. Consider a connected IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2\}$ of GS $G^* = \{Q,R_1,R_2\}$ such that $\mu = \{u_1([0.1,0.3],[0.4,0.6]),u_2([0.3,0.4],[0.5,0.7]),u_3([0.2,0.3],[0.3,0.5]),$ $u_4([0.1,0.2],[0.2,0.3]),u_5([0.4,0.5],[0.3,0.6]),u_6([0.2,0.4],[0.4,0.8])\}$ as shown in Figure-3. In

FIGURE 3. $\tilde{G} = (\mu,\lambda_1,\lambda_2)$ is IVPFGS of G^*

this example, the some $\lambda_1 - path$ covers of an interval-valued fuzzy graph structure $\tilde{G} = (\mu,\lambda_1,\lambda_2)$ are $M^1_{\lambda_1} = \{u_1u_2u_3,u_4u_5,u_5u_6\},\ M^2_{\lambda_1} = \{u_1u_2u_3,u_4u_5u_6\},\ M^3_{\lambda_1} = \{u_1u_2,u_2u_3,u_4u_5u_6\},\ M^4_{\lambda_1} = \{u_5u_6,u_5u_4,u_1u_2,u_2u_3\}$. Similarly, we can calculate $\lambda_2 - path$ covers of $\tilde{G}$.

Definition 3.19. An $\lambda_J - edge$ covers of an IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ of GS $G^* = \{Q,R_1,R_2,...,R_k\}$ is a set E_{λ_J} of $\lambda_J - edge$ such that every vertex of $\tilde{G}$ is incident to some $\lambda_J - edge$ of E for all $J = 1,2,...,k$.

Example 3.20. In above example-3.18, as shown in figure-3. The some of the $\lambda_J - edge$ covers of an IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2\}$ are $E_{\lambda_1} = \{(u_1,u_2),(u_2,u_3),(u_4,u_5),(u_5,u_6)\}$,
$E_{\lambda_2} = \{(u_1,u_6),(u_2,u_4),(u_3,u_5)\}$

Theorem 3.21. *Every complete IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ of GS $G^* = \{Q,R_1,R_2,...,R_k\}$ is a $\lambda - self$ centered IVPFGS and $r_{\lambda^-_{1J}}(G) = \frac{1}{\mu^-_1}$, $r_{\lambda^+_{1J}} = \frac{1}{\mu^+_1}$ and $r_{\lambda^-_{2J}}(G) = \frac{1}{\mu^-_2}$, $r_{\lambda^+_{2J}} = \frac{1}{\mu^+_2}$, where μ^-_1 is the least vertex membership, μ^+_1 is the greatest vertex membership and μ^-_2 is the least vertex membership, μ^+_2 is the greatest vertex membership for all $J = 1,2,...,k$.*

Proof. Let $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ be a complete IVPFGS. To prove that $\tilde{G}$ is a $\lambda_{1J} - self$ centered IVPFGS. That is we have to show that every vertex is a $\lambda_{1J} - central$ vertex. First we claim that $\tilde{G}$ is a λ_{1J}-self centered IVPFG. Then $r_{\lambda^-_{1J}}(G) = \frac{1}{\mu^-_1(u_i)}$ and $r_{\lambda^+_{1J}}(G) = \frac{1}{\mu^+_1(u_i)}$, where $\mu^-_1(u_i)$ is the least and $\mu^+_1(u_i)$ is the greatest. Now fix a vertex $u_i \in Q$ such that $\mu^-_1(u_i)$ is least vertex membership value of $\tilde{G}$ and $\mu^+_1(u_i)$ is greatest vertex membership value of $\tilde{G}$.
Case1: Consider all the $u_i - u_j$ paths P of length n in $\tilde{G}$, $\forall\ u_j \in Q$.
(i) If $n = 1$, then $\lambda^-_{1J}(u_i) = \min\{\mu^-_1(u_i),\mu^-_1(u_j)\}$. Therefore, $\lambda^-_{1J} - length$ of $P = l_{\lambda^-_{1J}}(P) = \frac{1}{\mu^-_1(u_i)}$ and $\lambda^+_{1J}(u_i,u_j) = \min\{\mu^+_1(u_i),\mu^+_1(u_j)\}$. Therefore, $\lambda^+_{1J} - length$ of $P = l_{\lambda^+_{1J}}(P) = \frac{1}{\mu^+_1(u_i)}$.
(ii) If $n > 1$, then one of the edges of P possesses the $\lambda^-_{1J} - strength$ of $\mu^-_1(u_i)$ and hence, $\lambda^-_{1J} - length$ of a $u_i - u_j$ path will exceed $\frac{1}{\mu^-_1(u_i)}$. So that, $\lambda^-_{1J} - length$ of $P = l_{\lambda^-_{1J}}(P) > \frac{1}{\mu^-_1(u_i)}$.

$$\text{Hence, } \delta_{\lambda^-_{1J}}(u_i,u_j) = \min(l_{\lambda^-_{1J}}(p)) = \frac{1}{\mu^-_1(u_i)},\ \forall u_j \in Q. \tag{3.1}$$

Also one of the edges of P possesses the $\lambda_{1J}^{+}-strength$ of $\mu_1^{+}(u_i)$ and hence, $\lambda_{1J}^{+}-length$ of P will exceed $\frac{1}{\mu_1^{+}(u_i)}$. that is, $\lambda_{1J}^{+}-length$ of $P=l_{\lambda_{1J}^{+}}(P)>\frac{1}{\mu_1^{+}(u_i)}$.

$$\text{Hence, } \delta_{\lambda_1^{+}(u_i)}(u_i,u_j)=\min(l_{\lambda_{1J}^{+}}(p))=\frac{1}{\mu_1^{+}(u_i)},\ \forall u_j\in Q \tag{3.2}$$

Case 2: Let $u_k\neq u_i\in Q$. Consider all u_k-u_j paths X of length n in $\tilde{G}$, $\forall\ u_j\in Q$.
(i) If $n=1$, then $\lambda_{1J}^{-}(u_k,u_j)=\min\{\mu_1^{-}(u_k),\mu_1^{-}(u_j)\}\geq\mu_1^{-}(u_i)$, since $\mu_1^{-}(u_i)$ is the least. Hence, $\lambda_{1J}^{-}-length\ (Q)=l_{\lambda_{1J}^{-}}(X)=\frac{1}{\lambda_{1J}^{-}(u_k,u_j)}\leq\frac{1}{\mu_1^{-}(u_i)}$.
Also $\lambda_{1J}^{+}(u_k,u_j)=\min\{\mu_1^{+}(u_k),\mu_1^{+}(u_j)\}\leq\mu_1^{+}(u_i)$, since $\mu_1^{+}(u_i)$ is the greatest. Hence, $\lambda_{1J}^{+}-length\ (Q)=l_{\lambda_{1J}^{+}}(X)=\frac{1}{\lambda_{1J}^{+}(u_k,u_j)}\geq\frac{1}{\mu_1^{+}(u_i)}$.
(ii) If $n=2$, then $l_{\lambda_{1J}^{-}}(X)=\frac{1}{\lambda_{1J}^{-}(u_k,u_{k+1})}+\frac{1}{\lambda_{1J}^{-}(u_{k+1},u_j)}\leq\frac{2}{\mu_1^{-}(u_i)}$, Since, $\mu_1^{-}(u_i)$ is the least.
Also $l_{\lambda_{1J}^{+}}(X)=\frac{1}{\lambda_{1J}^{+}(u_k,u_{k+1})}+\frac{1}{\lambda_{1J}^{+}(u_{k+1},u_j)}\geq\frac{2}{\mu_1^{+}(u_i)}$, Since, $\mu_1^{+}(u_i)$ is the greatest.
(iii) If $n>2$, then $l_{\lambda_{1J}^{-}}(X)\leq\frac{n}{\mu_1^{-}(u_i)}$, since $\mu_1^{-}(u_i)$ is the least.
Also $l_{\lambda_{1J}^{+}}(X)\geq\frac{n}{\mu_1^{+}(u_i)}$, since $\mu_1^{+}(u_i)$ is the greatest.
Hence, $\delta_{\lambda_{1J}^{-}}(u_k,u_j)=\min(l_{\lambda_{1J}^{-}}(X))\leq\frac{1}{\mu_1^{-}(u_i)}$, $\forall u_k,u_j\in Q$. and

$$\delta_{\lambda_{1J}^{+}}(u_k,u_j)=\min(l_{\lambda_{1J}^{+}}(X))\geq\frac{1}{\mu_1^{+}(u_i)},\ \forall u_k,u_j\in Q \tag{3.3}$$

From equation 3.1, 3.2 and 3.3, we have,
$e_{\lambda_{1J}^{-}}(u_i)=\min(\delta_{\lambda_{1J}^{-}}(u_i,u_j))=\frac{1}{\mu_1^{-}(u_i)}$, $\forall u_i\in Q$ and

$$e_{\lambda_{1J}^{+}}(u_i)=\min(\delta_{\lambda_{1J}^{+}}(u_i,u_j))=\frac{1}{\mu_1^{+}(u_i)},\ \forall u_i\in Q. \tag{3.4}$$

Hence, $\tilde{G}$ is a λ_{1J}^{-} and λ_{1J}^{+} self centered IVPFG.
Now, $r_{\lambda_{1J}^{-}}(G)=\min(e_{\lambda_{1J}^{-}}(u_i))=\frac{1}{\mu_1^{-}(u_i)}$, since by 3.4 $r_{\lambda_{1J}^{-}}(G)=\frac{1}{\mu_1^{-}(u_i)}$, where $\mu_1^{-}(u_i)$ is the least and $r_{\lambda_{1J}^{+}}(G)=\min(e_{\lambda_{1J}^{+}}(u_i))=\frac{1}{\mu_1^{+}(u_i)}$, since by 3.4 $r_{\lambda_{1J}^{+}}(G)=\frac{1}{\mu_1^{+}(u_i)}$, where $\mu_1^{+}(u_i)$ is the greatest.
Next, we claim that $\tilde{G}$ is a λ_{2J}-self centered IVPFGS. Then $r_{\lambda_{2J}^{-}}(G)=\frac{1}{\mu_2^{-}(u_i)}$ and $r_{\lambda_{2J}^{+}}(G)=\frac{1}{\mu_2^{+}(u_i)}$, where $\mu_2^{-}(u_i)$ is the least and $\mu_2^{+}(u_i)$ is the greatest. Now fix a vertex $u_i\in Q$ such that $\mu_2^{-}(u_i)$ is least vertex membership value of $\tilde{G}$ and $\mu_2^{+}(u_i)$ is greatest vertex membership value of $\tilde{G}$.
Case 1: Consider all the u_i-u_j paths P of length n in $\tilde{G}$, $\forall\ u_j\in Q$.
(i) If $n=1$, then $\lambda_{2J}^{-}(u_i,u_j)=\max\{\mu_2^{-}(u_i),\mu_2^{-}(u_j)\}=\mu_2^{-}(u_i)$. Therefore, $\lambda_2^{-}-length$ of $P=l_{\lambda_2^{-}}(P)=\frac{1}{\mu_2^{-}(u_i)}$ and $\lambda_{2J}^{+}(u_i,u_j)=\max\{\mu_2^{+}(u_i),\mu_2^{+}(u_j)\}=\mu_2^{+}(u_i)$. Therefore, $\lambda_{2J}^{+}-length$ of $P=l_{\lambda_{2J}^{+}}(P)=\frac{1}{\mu_2^{+}(u_i)}$.
(ii) If $n>1$, then one of the edges of P possesses the $\lambda_{2J}^{-}-strength$ of $\mu_2^{-}(u_i)$ and hence, $\lambda_{2J}^{-}-length$ of a u_i-u_j path will exceed $\frac{1}{\mu_2^{-}(u_i)}$. So that, $\lambda_{2J}^{-}-length$ of $P=l_{\lambda_{2J}^{-}}(P)>\frac{1}{\mu_2^{-}(u_i)}$.

$$\text{Hence, } \delta_{\lambda_{2J}^{-}}(u_i,u_j)=\max(l_{\lambda_{2J}^{-}}(p))=\frac{1}{\mu_2^{-}(u_i)},\ \forall u_j\in Q. \tag{3.5}$$

Also one of the edges of P possesses the $\lambda_{2J}^{+}-strength$ of $\mu_2^{+}(u_i)$ and hence, $\lambda_{2J}^{+}-length$ of P will exceed $\frac{1}{\mu_2^{+}(u_i)}$. that is, $\lambda_{2J}^{+}-length$ of $P=l_{\lambda_{2J}^{+}}(P)>\frac{1}{\mu_2^{+}(u_i)}$.

$$\text{Hence, } \delta_{\lambda_{2J}^{+}}(u_i,u_j)=\max(l_{\lambda_{2J}^{+}}(p))=\frac{1}{\mu_2^{+}(u_i)},\ \forall u_j\in Q. \tag{3.6}$$

Case 2: Let $u_k \neq u_i \in Q$. Consider all $u_k - u_j$ paths X of length n in $\tilde{G}$, $\forall\, u_j \in Q$.
(i) If $n = 1$, then $\lambda_{2J}^-(u_k,u_j) = \max\{\mu_2^-(u_k),\mu_2^-(u_j)\} \geq \mu_2^-(u_i)$, since $\mu_2^-(u_i)$ is the least. Hence, $\lambda_{2J}^- - length\ (Q) = l_{\lambda_{2J}^-}(X) = \frac{1}{\lambda_{2J}^-(u_k,u_j)} \leq \frac{1}{\mu_2^-(u_i)}$.
Also $\lambda_{2J}^+(u_k,u_j) = \max\{\mu_2(u_k)^+,\mu_2^+(u_j)\} \leq \mu_2^+(u_i)$, since $\mu_2^+(u_i)$ is the greatest. Hence, $\lambda_{2J}^+ - length\ (Q) = l_{\lambda_{2J}^+}(P) = \frac{1}{\lambda_{2J}^+(u_k,u_j)} \geq \frac{1}{\mu_2^+(u_i)}$.
(ii) If $n = 2$, then $l_{\lambda_{2J}^-}(X) = \frac{1}{\lambda_{2J}^-(u_k,u_{k+1})} + \frac{1}{\lambda_{2J}^-(u_{k+1},u_j)} \leq \frac{2}{\mu_2^-(u_i)}$, Since, $\mu_2^-(u_i)$ is the least.
Also $l_{\lambda_{2J}^+}(X) = \frac{1}{\lambda_{2J}^+(u_k,u_{k+1})} + \frac{1}{\lambda_{2J}^+(u_{k+1},u_j)} \geq \frac{2}{\mu_2^+(u_i)}$, Since, $\mu_2^+(u_i)$ is the greatest.
(iii) If $n > 2$, then $l_{\lambda_{2J}^-}(X) \leq \frac{n}{\mu_2^-(u_i)}$, since $\mu_2^-(u_i)$ is the least.
Also $l_{\lambda_{2J}^+}(X) \geq \frac{n}{\mu_2(u_i)^+}$, since $\mu_2^+(u_i)$ is the greatest.
Hence, $\delta_{\lambda_{2J}^-}(u_k,u_j) = \max(l_{\lambda_2^-}(X)) \leq \frac{1}{\mu_2^-(u_i)}$, $\forall u_k,u_j \in Q$. and

$$\delta_{\lambda_{2J}^+}(u_k,u_j) = \max(l_{\lambda_{2J}^+}(X)) \geq \frac{1}{\mu_2^+(u_i)},\ \forall u_k,u_j \in Q. \tag{3.7}$$

From equation 3.5, 3.6 and 3.7, we have,
$e_{\lambda_{2J}^-}(u_i) = \max(\delta_{\lambda_{2J}^-}(u_i,u_j)) = \frac{1}{\mu_2^-(u_i)}$, $\forall u_i \in Q$ and

$$e_{\lambda_{2J}^+}(u_i) = \max(\delta_{\lambda_{2J}^+}(u_i,u_j)) = \frac{1}{\mu_2^+(u_i)},\ \forall u_i \in Q. \tag{3.8}$$

Hence, $\tilde{G}$ is a λ_{2J}^- and λ_{2J}^+ self centered IVPFG.
Now, $r_{\lambda_{2J}^-}(G) = \min(e_{\lambda_{2J}^-}(u_i)) = \frac{1}{\mu_2^-(u_i)}$, since by (7) $r_{\lambda_{2J}^-}(G) = \frac{1}{\mu_2^-(u_i)}$, where $\mu_2^-(u_i)$ is the least and $r_{\lambda_{2J}^+}(G) = \max(e_{\lambda_{2J}^+}(u_i)) = \frac{1}{\mu_2^+(u_i)}$, since by (8) $r_{\lambda_{2J}^+}(G) = \frac{1}{\mu_2^+(u_i)}$, where $\mu_2^+(u_i)$ is the greatest. From equation 3.4 and 3.8, every vertex of $\tilde{G}$ is a central vertex. Hence $\tilde{G}$ is a self centered IVPFGS. □

Corollary 3.22. *Every complete IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ of GS $G^* = \{Q,R_1,R_2,...,R_k\}$ is a self centered IVPFGS and $r_{\lambda_{1J},\lambda_{2J}}(G) = ([\frac{1}{\mu_1^-(u_i)},\frac{1}{\mu_1^+(u_i)}],[\frac{1}{\mu_2^-(u_i)},\frac{1}{\mu_2^+(u_i)}])$ where, $\mu_1^-(u_i)$ is the least vertex membership and $\mu_1^+(u_i)$ is the greatest vertex membership. $\mu_2^-(u_i)$ is the least vertex membership and $\mu_2^+(u_i)$ is the greatest vertex membership.*

Proof. By above theorem-3.21, every complete IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ of GS $G^* = \{Q,R_1,R_2,...,R_k\}$ is a self centered IVPFGS.
$r_{\lambda_{1J},\lambda_{2J}}(G) = (r_{\lambda_{1J}}(G),r_{\lambda_{2J}}(G)) = ([\min\{r_{\lambda_{1J}^-(G)}\},\min\{r_{\lambda_{1J}^+(G)}\}],[\min\{r_{\lambda_{2J}^-(G)}\},$ $\min\{r_{\lambda_{2J}^+(G)}\}])$. $r_{\lambda_{1J},\lambda_{2J}}(G) = ([\frac{1}{\mu_1^-(u_i)},\frac{1}{\mu_1^+(u_i)}],[\frac{1}{\mu_2^-(u_i)},\frac{1}{\mu_2^+(u_i)}])$, since $\mu_1^-(u_i)$ is the least membership value and $\mu_1^+(u_i)$ is the greatest membership value. $\mu_2^-(u_i)$ is the least membership value and $\mu_2^+(u_i)$ is the greatest membership value. □

Remark 3.23. Converse of the above theorem-3.21 is not true. By Example-3.16. Then $\tilde{G}$ is self centered IVPFG but not complete.

Lemma 3.24. *An IVPFGS $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ of GS $G^* = \{Q,R_1,R_2,...,R_k\}$ is a self centered IVPFGS if and only if $r_{\lambda_{1J}^-}(G) = d_{\lambda_{1J}^-}(G)$, $r_{\lambda_{1J}^+}(G) = d_{\lambda_{1J}^+}(G)$ and $r_{\lambda_{2J}^-}(G) = d_{\lambda_{2J}^-}(G)$, $r_{\lambda_{2J}^+}(G) = d_{\lambda_{2J}^+}(G)$.*

Theorem 3.25. *Let $\tilde{G} = \{\mu,\lambda_1,\lambda_2,...,\lambda_k\}$ is a connected IVPFGS. Then for at least one edge* $\max(\lambda_{1J}^-(u_i,u_j)) = \lambda_{1J}^-(u_i,u_j)$, $\max(\lambda_{1J}^+(u_i,u_j)) = \lambda_{1J}^+(u_i,u_j)$ *and* $\max(\lambda_{2J}^-(u_i,u_j)) = \lambda_{2J}^-(u_i,u_j)$, $\max(\lambda_{2J}^+(u_i,u_j)) = \lambda_{2J}^+(u_i,u_j)$

Proof. If $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS. Consider a vertex u_i whose least membership value $\mu_1^-(u_i)$ and greatest membership value is $\mu_1^+(u_i)$ and least membership value $\mu_2^-(u_i)$ and greatest membership value is $\mu_2^+(u_i)$.

Case 1: Let $\mu_1^-(u_i)$ be the least value and $\mu_1^+(u_i)$ be the greatest value and $\mu_2^-(u_i)$ be the least value and $\mu_2^+(u_i)$ be the greatest value and in the vertex $u_i \in Q$. Let $u_i, u_j \in Q$, then $([\lambda_{1J}^-(u_i,u_j), \lambda_{1J}^+(u_i,u_j)], [\lambda_{2J}^-(u_i,u_j), \lambda_{2J}^+(u_i,u_j)]) = ([\mu_1^-(u_i), \mu_1^+(u_i)], [\mu_2^-(u_i), \mu_2^+(u_i)])$ and $([\max(\lambda_{1J}^-(u_i,u_j)), \max(\lambda_{1J}^+(u_i,u_j))], [\max(\lambda_{2J}^-(u_i,u_j)), \max(\lambda_{2J}^+(u_i,u_j))]) =$ $([\mu_1^-(u_i), \mu_1^+(u_i)], [\mu_2^-(u_i), \mu_{2i}^+(u_i)])$. The strength of all the edges which are incident on the vertex u_i is $([\mu_1^-(u_i), \mu_1^+(u_i)], [\mu_2^-(u_i), \mu_2^+(u_i)])$. Since $\tilde{G}$ is a connected IVPFGS.

Case 2: Let $\mu_1^-(u_k)$ be the least value and $\mu_1^+(u_i)$ be the greatest value and $\mu_2^-(u_k)$ be the least value and $\mu_2^+(u_i)$ be the greatest value in the vertex $u_i, u_k \in Q$. Then $([\lambda_{1J}^-(u_i,u_k), \lambda_{1J}^+(u_i,u_k)],$ $[\lambda_{2J}^-(u_i,u_k), \lambda_{2J}^+(u_i,u_k)]) = ([\mu_1^-(u_k), \mu_1^+(u_i)], [\mu_2^-(u_k), \mu_2^+(u_i)])$. Since, it is a connected IVPFGS, there will be an edge between u_i and u_k, $\max(\lambda_{1J}^-(u_i,u_k)) = \mu_1^-(u_k)$, $\max(\lambda_{1J}^+(u_i,u_k)) = \mu_1^+(u_i)$ and $\max(\lambda_{2J}^-(u_i,u_k)) = \mu_2^-(u_k)$, $\max(\lambda_2^+(u_i,u_k)) = \mu_2^+(u_i)$. □

Theorem 3.26. *Let $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ be a connected IVPFGS of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$ with λ_J − paths covers P_1 and P_2 of $\tilde{G}$. Then the necessary and sufficient condition for an IVPFGS to be self centered IVPFGS is $\delta_{\lambda_{1J}^-(u_i,u_j)} = r_{\lambda_{1J}^-}(G)$, $\forall\ (u_i,u_j) \in P_1$, $\delta_{\lambda_{1J}^+(u_i,u_j)} = d_{\lambda_{1J}^+}(G)$, $\forall\ (u_i,u_j) \in P_2$, and*

$$\delta_{\lambda_{2J}^-(u_i,u_j)} = r_{\lambda_{2J}^-}(G),\ \forall\ (u_i,u_j) \in P_1,\ \delta_{\lambda_{2J}^+(u_i,u_j)} = d_{\lambda_{2J}^+}(G),\ \forall\ (u_i,u_j) \in P_2. \tag{3.9}$$

Proof. **Necessary Condition:** We now assume that $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is a self centered IVPFGS and we have to prove that equation 3.9 holds. Suppose equation 3.9 does not holds. then we have, $\delta_{\lambda_{1J}^-}(u_i,u_j) \neq r_{\lambda_{1J}^-}(G)$, for some $(u_i,u_j) \in P_1$ and $\delta_{\lambda_{1J}^+}(u_i,u_j) \neq d_{\lambda_{1J}^+}(G)$, for some $(u_i,u_j) \in P_2$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \neq r_{\lambda_{2J}^-}(G)$, for some $(u_i,u_j) \in P_1$ and $\delta_{\lambda_{2J}^+}(u_i,u_j) \neq d_{\lambda_{2J}^+}(G)$, for some $(u_i,u_j) \in P_2$. By using Lemma-3.24, the above inequality becomes $\delta_{\lambda_{1J}^-}(u_i,u_j) \neq r_{\lambda_{1J}^-}(G)$, for some $(u_i,u_j) \in P_1$ and $\delta_{\lambda_{1J}^+}(u_i,u_j) \neq d_{\lambda_{1J}^+}(G)$, for some $(u_i,u_j) \in P_2$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \neq r_{\lambda_{2J}^-}(G)$, for some $(u_i,u_j) \in P_1$ and $\delta_{\lambda_{2J}^+}(u_i,u_j) \neq d_{\lambda_{2J}^+}(G)$, for some $(u_i,u_j) \in P_2$. Then $e_{\lambda_{1J}^-}(u_i) \neq r_{\lambda_{1J}^-}(G)$, $e_{\lambda_{1J}^+}(u_i) \neq r_{\lambda_{1J}^+}(G)$ and $e_{\lambda_{2J}^-}(u_i) \neq r_{\lambda_{2J}^-}(G)$, $e_{\lambda_{2J}^+}(u_i) \neq r_{\lambda_{2J}^+}(G)$ for some $u_i \in Q$, which implies $\tilde{G}$ is not self centered IVIFG, which is contradiction. Hence, $\delta_{\lambda_{1J}^-}(u_i,u_j) = r_{\lambda_{1J}^-}(G)$, $\forall\ (u_i,u_j) \in P_1$ and $\delta_{\lambda_{1J}^+}(u_i,u_j) = d_{\lambda_{1J}^+}(G)$, $\forall\ (u_i,u_j) \in P_2$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) = r_{\lambda_{2J}^-}(G)$, $\forall\ (u_i,u_j) \in P_1$ and $\delta_{\lambda_{2J}^+}(u_i,u_j) = d_{\lambda_{2J}^+}(G)$, $\forall\ (u_i,u_j) \in P_2$.

Sufficient Condition: We now assume that equation 3.9 holds and we have to prove that $\tilde{G}$ is a self centered IVPFGS. If equation 3.9 holds, then we've $e_{\lambda_{1J}^-}(u_i) = \delta_{\lambda_{1J}^-}(u_i,u_j)$, for all $(u_i,u_j) \in P_1$, $e_{\lambda_{1J}^+}(u_i) = \delta_{\lambda_{1J}^+}(u_i,u_j)$, for all $(u_i,u_j) \in P_2$ and $e_{\lambda_{2J}^-}(u_i) = \delta_{\lambda_{2J}^-}(u_i,u_j)$, for all $(u_i,u_j) \in P_1$, $e_{\lambda_{2J}^+}(u_i) = \delta_{\lambda_{2J}^+}(u_i,u_j)$, for all $(u_i,u_j) \in P_2$. Which implies $e_{\lambda_{1J}^-}(u_i) = r_{\lambda_{1J}^-}(G)$, $e_{\lambda_{1J}^+}(u_i) = r_{\lambda_{1J}^+}(G)$ and $e_{\lambda_{2J}^-}(u_i) = r_{\lambda_{2J}^-}(G)$, $e_{\lambda_{2J}^+}(u_i) = r_{\lambda_{2J}^+}(G)$ for all $u_i \in Q$, $J = 1, 2, ..., k$. Hence, $\tilde{G}$ is not self centered IVPFGS. □

Corollary 3.27. *If $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is a connected IVPFGS of GS $G^* = \{Q, R_1, R_2, ..., R_k\}$ with an λ_J − edge cover E_{λ_J} of $\tilde{G}$. Then the necessary and sufficient condition for an IVPFGS to be $\lambda_J - self$ centered IVPFGS is $\delta_{\lambda_{1J}^-}(u_i,u_j) = r_{\lambda_{1J}^-}(G)$, $\forall\ (u_i,u_j) \in E_{\lambda_1}$, $\delta_{\lambda_{1J}^+}(u_i,u_j) = d_{\lambda_{1J}^+}(G)$, $\forall\ (u_i,u_j) \in E_{\lambda_2}$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) = r_{\lambda_{2J}^-}(G)$, $\forall\ (u_i,u_j) \in E_{\lambda_1}$,*

$$\delta_{\lambda_{2J}^+}(u_i,u_j) = d_{\lambda_{2J}^+}(G),\ \forall\ (u_i,u_j) \in E_{\lambda_2}. \tag{3.10}$$

Theorem 3.28. ***Embedding Theorem:*** *Let $\tilde{H} = \{\mu', \lambda_1', \lambda_2', ..., \lambda_k'\}$ is a connected $\lambda_J - self$ centered IVPFGS. Then there exist a connected IVPFGS $\tilde{G}$ such that $< C(\tilde{G}) >$ is isomorphic to $\tilde{H}$. Also $d_{\lambda_{1J}^-}(G) = 2r_{\lambda_{1J}^-}(G),\ d_{\lambda_{1J}^+}(G) = 2r_{\lambda_{1J}^+}(G)$ and $d_{\lambda_{2J}^-}(G) = 2r_{\lambda_{2J}^-}(G),\ d_{\lambda_{2J}^+}(G) = 2r_{\lambda_{2J}^+}(G)$.*

Proof. Given that $\tilde{H} = \{\mu', \lambda_1', \lambda_2', ..., \lambda_k'\}$ is a connected $\lambda_J - self$ centered IVPFGS. Let $d_{\lambda_{1J}^-}(H) = p_1,\ d_{\lambda_{1J}^+}(H) = q_1$ and $d_{\lambda_{2J}^-}(H) = p_2,\ d_{\lambda_{2J}^+}(H) = q_2$. Then construct $\tilde{G}$ from $\tilde{H}$ as follows:
Take two vertices $u_i, u_j \in Q$ with $\mu_1^-(u_i) = \mu_1^-(u_j) = \frac{1}{p_1}, \mu_1^+(u_i) = \mu_1^+(u_j) = \frac{1}{2q_1}$ and $\mu_2^-(u_i) = \mu_2^-(u_j) = \frac{1}{p_2}, \mu_2^+(u_i) = \mu_2^+(u_j) = \frac{1}{2q_2}$ and join all the vertices of $\tilde{H}$ to both u_i and u_j with $\lambda_{1J}^-(u_i,u_k) = \lambda_{1J}^-(u_j,u_k) = \frac{1}{p_1},\ \lambda_{1J}^+(u_j,u_k) = \lambda_{1J}^+(u_j,u_k) = \frac{1}{2q_1}$ and $\lambda_{2J}^-(u_j,u_k) = \lambda_{2J}^-(u_j,u_k) = \frac{1}{p_2},\ \lambda_{2J}^+(u_j,u_k) = \lambda_{2J}^+(u_j,u_k) = \frac{1}{2q_2}$ for all $u_k \in Q'$. Put $\mu_1^-(u_i) = (\mu_1^-)'(u_i),\ \mu_1^+(u_i) = (\mu_1^+)'(u_i)$ and $\mu_2^-(u_i) = (\mu_2^-)'(u_i),\ \mu_2^+(u_i) = (\mu_2^+)'(u_i)$ for all vertices in $\tilde{H}$. and $\lambda_{1J}^-(u_i,u_j) = (\lambda_{1J}^-)'(u_i,u_j)$,
$\lambda_{1J}^+(u_i,u_j) = (\lambda_{1J}^+)'(u_i,u_j)$ for all edges in $\tilde{H}$ and $\lambda_{2J}^-(u_i,u_j) = (\lambda_{2J}^-)'(u_i,u_j),\ \lambda_{2J}^+(u_i,u_j) = (\lambda_{2J}^+)'(u_i,u_j)$ for all -edges in $\tilde{H}$.
Claim: $\tilde{G}$ is an IVPFGS. First note that $\mu_1^-(u_i) \leq \mu_1^-(u_k),\ \mu_2^-(u_i) \leq \mu_2^-(u_k)$ for all $u_k \in \tilde{H}$. If possible, let $\mu_1^-(u_i) > \mu_1^-(u_k)$ and $\mu_2^-(u_i) > \mu_2^-(u_k)$ for at least one vertex $u_k \in \tilde{H}$. Then $\frac{1}{p_1} > \mu_1^-(u_k),\ \frac{1}{p_2} > \mu_2^-(u_k)$, that is $p_1 < \frac{1}{\mu_1^-(u_k)} \leq \frac{1}{\lambda_{1J}^-(u_k,u_l)},\ p_2 < \frac{1}{\mu_2^-(u_k)} \leq \frac{1}{\lambda_{2J}^-(u_k,u_l)}$, where the last inequality holds for every $u_l \in Q'$, since $\tilde{H}$ is an IVPFGS. That is $\frac{1}{\lambda_{1J}^-(u_k,u_l)} > p_1,\ \frac{1}{\lambda_{2J}^-(u_k,u_l)} > p_2$ for all $u_k \in \tilde{H}$ which contradicts that $d_{\lambda_{1J}^-}(\tilde{H}) = p_1,\ d_{\lambda_{2J}^-}(\tilde{H}) = p_2$. Therefore $\mu_1^-(u_i) \leq \mu_1^-(u_k), \mu_2^-(u_i) \leq \mu_2^-(u_k)$ for all $u_k \in Q'$ and $\lambda_{1J}^-(u_i,u_k) \leq \min\{\mu_1^-(u_i), \mu_1^-(u_k)\} = \frac{1}{p_1},\ \lambda_{2J}^-(u_i,u_k) \leq \max\{\mu_2^-(u_i), \mu_2^-(u_k)\} = \frac{1}{p_2}$, similarly, $\lambda_{1J}^-(u_j,u_k) \leq \min\{\mu_1^-(u_j), \mu_1^-(u_k)\} = \frac{1}{p_1},\ \lambda_{2J}^-(u_j,u_k) \leq \max\{\mu_2^-(u_j),$
$\mu_2^-(u_k)\} = \frac{1}{p_2}$ for all $u_k \in Q'$. Note that $\mu_1^+(u_i) \leq \mu_1^+(u_k),\ \mu_1^+(u_j) \leq \mu_1^-(u_k)$ and $\mu_2^+(u_i) \leq \mu_2^+(u_k),\ \mu_2^+(u_j) \leq \mu_2^-(u_k)$ for all $u_k \in Q'$, since $d_{\lambda_{1J}^+}(H) = q_{1J}$ and $d_{\lambda_{2J}^+}(H) = q_2$. Therefore $\lambda_{1J}^+(u_i,u_k) \leq \min\{\mu_1^+(u_i), \mu_1^+(u_k)\} = \frac{1}{2q_1},\ \lambda_{2J}^+(u_i,u_k) \leq \max\{\mu_2^+(u_i), \mu_2^+(u_k)\} = \frac{1}{2q_2}$, similarly, $\lambda_{1J}^+(u_j,u_k) \leq \min\{\mu_1^+(u_j), \mu_1^+(u_k)\} = \frac{1}{2q_1}$ and $\lambda_{2J}^+(u_j,u_k) \leq \max\{\mu_2^+(u_j), \mu_2^+(u_k)\}$
$= \frac{1}{2q_2}$. Hence, $\tilde{G}$ is an IVPFGS. Also, $e_{\lambda_{1J}^-}(u_k) = p_1,\ e_{\lambda_{1J}^-}(u_k) = p_2$ for all $u_k \in Q'$ and $e_{\lambda_{1J}^-}(u_i) = e_{\lambda_{1J}^-}(u_j) = \frac{1}{\lambda_{1J}^-(u_i,u_k)} + \frac{1}{\lambda_{1J}^-(u_k,u_l)} = 2p_1,\ r_{\lambda_{1J}^-}(G) = p_1,\ d_{\lambda_{1J}^-}(G) = 2p_1$ and $e_{\lambda_{2J}^-}(u_i) = e_{\lambda_{2J}^-}(u_j) = \frac{1}{\lambda_{2J}^-(u_i,u_k)} + \frac{1}{\lambda_{2J}^-(u_k,u_l)} = 2p_2,\ r_{\lambda_{2J}^-}(G) = p_2,\ d_{\lambda_{2J}^-}(G) = 2p_2$. Next, $e_{\lambda_{1J}^+}(u_k) = q_1,\ e_{\lambda_{2J}^+}(u_k) = q_2$ for all $u_k \in Q'$ and $e_{\lambda_{1J}^+}(u_i) = e_{\lambda_{1J}^+}(u_j) = \frac{1}{\lambda_{1J}^+(u_l,u_k)} = 2q_1,\ e_{\lambda_{2J}^+}(u_i) = e_{\lambda_{2J}^+}(u_j) = \frac{1}{\lambda_{2J}^+(u_l,u_k)} = 2q_2$ for all $u_k \in Q'$. Therefore, $r_{\lambda_{1J}^+}(G) = q_1,\ d_{\lambda_{1J}^+}(G) = 2q_1$ and $r_{\lambda_{2J}^+}(G) = q_2,\ d_{\lambda_{2J}^+}(G) = 2q_2$. Hence, $< C(\tilde{G}) >$ is isomorphic to $\tilde{H}$. □

Theorem 3.29. *An IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is a $\lambda_J - self$ centered if and only if $\delta_{\lambda_{1J}^-}(u_i,u_j) \leq r_{\lambda_{1J}^-}(G),\ \delta_{\lambda_{1J}^+}(u_i,u_j) \geq r_{\lambda_{1J}^+}(G)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \leq r_{\lambda_{2J}^-}(G),\ \delta_{\lambda_{2J}^+}(u_i,u_j) \geq r_{\lambda_{2J}^+}(G)$ for all $u_i, u_j \in Q$, $J = 1,2,...,k$*

Proof. We assume that $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is a $\lambda_J - self$ centered IVPFGS. That is, $e_{\lambda_{1J}^-}(u_i) = e_{\lambda_{1J}^-}(u_j), e_{\lambda_{1J}^+}(u_i) = e_{\lambda_{1J}^+}(u_j)$ and $e_{\lambda_{2J}^-}(u_i) = e_{\lambda_{2J}^-}(u_j),\ e_{\lambda_{2J}^+}(u_i) = e_{\lambda_{2J}^+}(u_j)$ for all $u_i, u_j \in Q$, $r_{\lambda_{1J}^-}(G) = e_{\lambda_{1J}^-}(u_i),\ r_{\lambda_{1J}^+}(G) = e_{\lambda_{1J}^+}(u_i)$ and $r_{\lambda_{2J}^-}(G) = e_{\lambda_{2J}^-}(u_i),\ r_{\lambda_{2J}^+}(G) = e_{\lambda_{2J}^+}(u_i)$ for all $u_i \in Q$. Now we wish to show that $\delta_{\lambda_{1J}^-}(u_i,u_j) \leq r_{\lambda_{1J}^-}(G),\ \delta_{\lambda_{1J}^+}(u_i,u_j) \geq r_{\lambda_{1J}^+}(G)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \leq r_{\lambda_{2J}^-}(G),\ \delta_{\lambda_{2J}^+}(u_i,u_j) \geq r_{\lambda_{2J}^+}(G)$ for all $u_i, u_j \in Q$. By the definition of $\lambda_J - eccentricity$, we obtain, $\delta_{\lambda_{1J}^-}(u_i,u_j) \leq e_{\lambda_{1J}^-}(u_i)$, $\delta_{\lambda_{1J}^+}(u_i,u_j) \geq e_{\lambda_{1J}^+}(u_i)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \leq e_{\lambda_{2J}^-}(u_i),\ \delta_{\lambda_{2J}^+}(u_i,u_j) \geq e_{\lambda_{2J}^+}(u_i)$ for all $u_i, v_i \in Q$. This is possible only when $e_{\lambda_{1J}^-}(u_i) = e_{\lambda_{1J}^-}(u_j),\ e_{\lambda_{1J}^+}(u_i) = e_{\lambda_{1J}^+}(u_j)$ and $e_{\lambda_{2J}^-}(u_i) = e_{\lambda_{2J}^-}(u_j),\ e_{\lambda_{2J}^+}(u_i) =$

$e_{\lambda_{2J}^+}(u_j)$ for all $u_i, u_j \in Q$. Since, $\tilde{G}$ is a $\lambda_J - self$ centered IVPFGS, the above inequality becomes $\delta_{\lambda_{1J}^-}(u_i,u_j) \leq r_{\lambda_{1J}^-}(G), \delta_{\lambda_{1J}^+}(u_i,u_j) \geq r_{\lambda_{1J}^+}(G)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \leq r_{\lambda_{2J}^-}(G)$,
$\delta_{\lambda_{2J}^+}(u_i,u_j) \geq r_{\lambda_{2J}^+}(G)$. Conversely, we now assume that $\delta_{\lambda_{1J}^-}(u_i,u_j) \leq r_{\lambda_{1J}^-}(G)$, $\delta_{\lambda_{1J}^+}(u_i,u_j) \geq r_{\lambda_{1J}^+}(G)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \leq r_{\lambda_{2J}^-}(G)$, $\delta_{\lambda_{2J}^+}(u_i,u_j) \geq r_{\lambda_{2J}^+}(G)$ for all $u_i, u_j \in Q$. Then we have to prove that $\tilde{G}$ is a $\lambda_J - self$ centered IVPFGS. Suppose that $\tilde{G}$ is not $\lambda_J - self$ centered IVPFGS. Then $r_{\lambda_{1J}^-}(G) \neq e_{\lambda_{1J}^-}(u_i), r_{\lambda_{1J}^+}(G) \neq e_{\lambda_{1J}^+}(u_i)$ and $r_{\lambda_{2J}^-}(G) \neq e_{\lambda_{2J}^-}(u_i), r_{\lambda_{2J}^+}(G) \neq e_{\lambda_{2J}^+}(u_i)$ for some $u_i \in Q$. Let us assume that $e_{\lambda_{1J}^-}(u_i)$, $e_{\lambda_{1J}^+}(u_i)$ and $e_{\lambda_{2J}^-}(u_i)$, $e_{\lambda_{2J}^+}(u_i)$ is the least value among all other $\lambda_J - eccentricity$. That is, $r_{\lambda_{1J}^-}(G) = e_{\lambda_{1J}^-}(u_i), r_{\lambda_{1J}^+}(G) = e_{\lambda_{1J}^+}(u_i)$ and $r_{\lambda_{2J}^-}(G) = e_{\lambda_{2J}^-}(u_i), r_{\lambda_{2J}^+}(G) = e_{\lambda_{2J}^+}(u_i)$. (i)
where $e_{\lambda_{1J}^-}(u_i) < e_{\lambda_{1J}^-}(u_j), e_{\lambda_{1J}^+}(u_i) < e_{\lambda_{1J}^+}(u_j)$ and $e_{\lambda_{2J}^-}(u_i) < e_{\lambda_{2J}^-}(u_j), e_{\lambda_{2J}^+}(u_i) < e_{\lambda_{2J}^+}(u_j)$ for some $u_i, u_j \in Q$ and $\delta_{\lambda_{1J}^-}(u_i,u_j) = e_{\lambda_{1J}^-}(u_j) > e_{\lambda_{1J}^-}(u_i)$, $\delta_{\lambda_{1J}^+}(u_i,u_j) = e_{\lambda_{1J}^+}(u_j) > e_{\lambda_{1J}^+}(u_i)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) = e_{\lambda_{2J}^-}(u_j) > e_{\lambda_{2J}^-}(u_i)$, $\delta_{\lambda_{2J}^+}(u_i,u_j) = e_{\lambda_{2J}^+}(u_j) > e_{\lambda_{2J}^+}(u_i)$ for some $u_i, u_j \in Q$. (ii)
Hence, from equation (i) and (ii), we have, $\delta_{\lambda_{1J}^-}(u_i,u_j) > r_{\lambda_{1J}^-}(G)$, $\delta_{\lambda_{1J}^+}(u_i,u_j) > r_{\lambda_{1J}^+}(G)$, and $\delta_{\lambda_{2J}^-}(u_i,u_j) > r_{\lambda_{2J}^-}(G)$, $\delta_{\lambda_{2J}^+}(u_i,u_j) > r_{\lambda_{2J}^+}(G)$, for some $u_i, u_j \in Q$, which is a contradiction to the fact that $\delta_{\lambda_{1J}^-}(u_i,u_j) \leq r_{\lambda_{1J}^-}(G)$, $\delta_{\lambda_{1J}^+}(u_i,u_j) \geq r_{\lambda_{1J}^+}(G)$ and $\delta_{\lambda_{2J}^-}(u_i,u_j) \leq r_{\lambda_{2J}^-}(G)$, $\delta_{\lambda_{2J}^+}(u_i,u_j) \geq r_{\lambda_{2J}^+}(G)$ for all $u_i, u_j \in Q$. Hence, $\tilde{G} = \{\mu, \lambda_1, \lambda_2, ..., \lambda_k\}$ is a $\lambda_J - self$ centered IVPFGS. □

4. APPLICATION

The results of this study can be used in many applications in human life. In this article, we'll use our findings to helf people live in ways that will keep them safe during times of epidemic. Recent events have made us acutely aware of the Covid-19 virus's effects. Take this as an illustration. In these circumstances, we shall make use of our methods to reduce the spread of disease. We need to strengthen security between all states in our country because of the spread of this infectious disease from one person to another and because it can affect many people. We will estimate the proportion of people infected with an infectious disease and the proportion of people who have fully recovered in the cities of our nation by using uncertainty values. Consider Q as the nation, and u_1, u_2, u_3, and u_4 as its cities. By using IVPFGS 3.1, we will evaluate this study. Assume that μ_1^- represents people who are most vulnerable to infectious diseases and assume μ_1^+ to be a representation of those who have recovered from infectious diseases. Assume μ_2^- is a representation of those who are most vulnerable to other ailments, and μ_2^+ is a representation of those who have recovered from other ailments. Let us consider the IVPFGS vertex set $Q = \{u_1, u_2, u_3, u_4\}$. Assume that R_1

TABLE 1.

Q	$[\mu_1^-, \mu_1^+]$	$[\mu_2^-, \mu_2^+]$
u_1	[0.4,0.5]	[0.3,0.4]
u_2	[0.3,0.4]	[0.2,0.4]
u_3	[0.3,0.4]	[0.3,0.5]
u_4	[0.2,0.3]	[0.4,0.6]

is a measurement of the contagion effect between two states. Assume that R_2 measures the rate at

TABLE 2. The R_1 relation is responsible for IVPFGS.

R_1	$[\lambda_{11}^-, \lambda_{11}^+]$	$[\lambda_{21}^-, \lambda_{21}^+]$
u_1u_2	[0.3,0.4]	[0.3,0.4]
u_2u_4	[0.2,0.3]	[0.4,0.6]
u_3u_4	[0.2,0.3]	[0.4,0.6]

which two states may recover from the effects of contagion. Consider IVPFGS $\tilde{G} = \{\mu, \lambda_1, \lambda_2\}$

TABLE 3. The R_2 relation is responsible for IVPFGS.

R_2	$[\lambda_{12}^-, \lambda_{12}^+]$	$[\lambda_{22}^-, \lambda_{22}^+]$
u_1u_4	[0.2,0.3]	[0.4,0.6]
u_2u_3	[0.3,0.4]	[0.3,0.5]

from Example-3.3, which is depicted in Figure-1. The effect of R_1 and R_2 spread between two states in a nation can be ascertained using the definition-3.4. The maximum amount of affect

TABLE 4.

R_1	λ_1-strength
u_1, u_2	([0.3,0.4],[0.3,0.4])
u_1, u_2, u_3	([0.3,0.4],[0.3,0.4])
u_1, u_2, u_4	([0.3,0.4],[0.4,0.6])
u_1, u_2, u_3, u_4	([0.3,0.4],[0.4,0.6])
R_2	λ_2-strength
u_1, u_2, u_3	([0.3,0.4],[0.3,0.5])
u_1, u_2, u_3, u_4	([0.3,0.4],[0.3,0.5])

reflected in the Table-5 is the impact of R_1 and R_2 on the states of the nation. According to

TABLE 5.

$e_{\lambda_1}(u_i)$	$([e_{\lambda_{11}^-}(u_i), e_{\lambda_{11}^+}(u_i)], [e_{\lambda_{21}^-}(u_i), e_{\lambda_{21}^+}(u_i)])$
$e_{\lambda_1}(u_1)$	([0.3,0.4],[0.3,0.4])
$e_{\lambda_1}(u_2)$	([0.3,0.4],[0.4,0.6])
$e_{\lambda_1}(u_3)$	([0.3,0.4],[0.4,0.6])
$e_{\lambda_1}(u_4)$	([0.2,0.3],[0.4,0.6])
$e_{\lambda_2}(u_i)$	$([e_{\lambda_{12}^-}(u_i), e_{\lambda_{12}^+}(u_i)], [e_{\lambda_{22}^-}(u_i), e_{\lambda_{22}^+}(u_i)])$
$e_{\lambda_2}(u_1)$	([0.3,0.4],[0.4,0.6])
$e_{\lambda_2}(u_2)$	([0.3,0.3],[0.3,0.5])
$e_{\lambda_2}(u_3)$	([0.3,0.4],[0.3,0.5])
$e_{\lambda_2}(u_4)$	([0.3,0.4],[0.4,0.6])

the table-4 and table-5 above, $d_{\lambda_1}(G) = ([0.3, 0.4], [0.4, 0.6])$, $d_{\lambda_2}(G) = ([0.3, 0.4], [0.4, 0.6])$ has a higher number of vulnerabilities and the defenses' recover effect for R_1 and R_2, while $r_{\lambda_1} = ([0.2, 0.3], [0.3, 0.4])$, $r_{\lambda_2} = ([0.3, 0.3], [0.3, 0.5])$ have the lowest number of vulnerabilities and defenses' recover effect for R_1 and R_2. These applications have the purpose to improve our nation's defenses more robust to the degree of its vulnerabilities.

5. CONCLUSIONS

IVPFGS is a notion that the researcher has introduced in this study article. Many other fields are affected by it. Most often, certain aspects of a graph-theoretical problem can be hazy or ambiguous. We have introduced the definition of $\lambda_J - strength$, $\lambda_J - length$, $\lambda_J - distance$, $\lambda_J - eccentricity$, $\lambda_J - radius$, $\lambda_J - diameter$, $\lambda_J - centered$, $\lambda_J - self$ centered, $\lambda_J - path$ cover, $\lambda_J - edge$ cover. With examples, we also go over some of the characteristics of the $\lambda_J - self$ centered IVPFGS.

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