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DIAMETRAL DESIGNS ARISING FROM HYPERGRAPH OF SOME CLASSES OF DIAMETER 3 DISTANCE REGULAR GRAPHS

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ABSTRACT. Hypergraph is a graph $H = (V, E)$ where V is the set of vertices and E is the set containing subsets of elements from set V. The elements of set E are called hyperedges and these need not always be of order 2 as in the case of graphs. In this paper, we have considered 3 classes of distance regular graphs (DRGs) of diameter 3 namely, crown graph, Johnson graph and Hamming graph. We have considered hypergraph models of these graphs and obtained the parameters of diametral designs arising from them. We have also obtained a condition when hypergraph H_1 of DRGs with diameter 3 forms a strongly regular graph with parameters $(n, n-2, n-4, n-2)$.

Keywords: PBIB-design, Distance regular graph, Diametral path, Hypergraph.

AMS Subject Classification: 05C12, 92E10, 05C65

1. INTRODUCTION

Hypergraph theory was introduced in 1960s as a generalization of graph theory. Hypergraph theory considers family of sets as a generalized graph. The expository text, 'Graphs and Hypergraphs' by Berge [2] in 1973 introduces the concept lucidly. The generalization of graph problems to hypergraphs brings a number of new perspectives to the field of graph theory. Research into the theories of set systems and hypergraphs provide a valuable basis to various fields of mathematics such as matroids, designs, combinatorial probability and Ramsey theory for infinite sets. Hypergraph theory studies a mathematical structure on a set of elements with a relation, as a recognised discipline in a relatively new era. In recent years, theory of hypergraphs has proved to be of major interest in applications to real world problems. Recent developments in this theory have played a major part in revealing hypergraphs as a prominent mathematical tool in a variety of applications [4]. A lot of applications of hypergraphs have been developed in the fields of engineering, particularly in computer science, software engineering, image processing, molecular biology, and related businesses and industries, chemistry and so on [4], [11], [12], [13].

Hypergraph theory is used to model cellular mobile communication systems, parallel data structure, and has been used in databases in order to model relational database schemes. For complete

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information on this application, readers can refer [14]. Properties of hypergraphs such as hypergraph transversal computation has a large number of applications in many areas of computer science such as distributed systems, databases and artificial intelligence [7]. Hypergraph partitioning is an interesting property which yields results in many areas such as VLSI designs and data mining [1]. Bretto [4] has lucidly explained the application of hypergraphs in image processing. Using hypergraph theory in image modeling seems to be a very natural way to study image processing where in each digital image is associated with a hypergraph. This model includes several features of image such as gray level, neighbourhood and so on which prove to be quite useful in many aspects.

Usually, in chemistry, molecular structures are represented by (simple) graphs, where vertices correspond to atoms and edges correspond to covalent bonds between them. A major drawback of this model is the lack of convenient tools to represent organometalic compounds, benzenoid systems and other molecules that have delocalized polycentric bonds. Graphical representation of such bonds are not illustrative enough to model them which makes structural analysis incomplete. This drawback can be overcome, if hypergraph model is considered for structures containing bonds like polycentric bonds, where such bonds are considered as one of the hyperedges. Thus hypergraph theory plays an important role in chemistry which has been affirmatively explained in [11], [12], [13] to name a few.

Distance regular graphs (DRGs) are a class of regular graphs that have an intersection array $(b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d)$ such that, for any two vertices u and v that are r distance apart, $d(u, v) = r$, b_r is the number of vertices that are adjacent to u and at a distance $r+1$ to v, c_r is the number of vertices that are adjacent to u and at distance of $r-1$ from v. These numbers depend only on distance r and not on choice of pairs of vertices u and v. A monograph titled 'Distance-Regular graphs' by Brouwer et al. [5] speaks volume of growth and depth of the subject. A detailed survey on DRGs with the same title by van Dam et al. [18] gives an up to date information about developments. It is well known that strongly regular graphs are a special class of distance regular graphs with diameter 2.

Combinatorial design theory is a part of combinatorics that deals with existence, construction and properties of systems of finite sets whose arrangements satisfy certain conditions. Balanced incomplete block (BIB)-designs and partially balanced incomplete block (PBIB)-designs are two major subfields finding a wide range of applications in group theory, graph theory, number theory, information theory, combinatorial matrix theory, finite geometries, statistics, computer science, biology, engineering, etc. In [10], Ionin and Shrikhande defined (v, k, λ, μ) designs over a regular graph G with blocks as certain k-subsets of vertices and replication number $r = 2\lambda - \mu$. Motivated by this work, Walikar et al. in [19] introduced (v, β_0, μ) design over regular graphs G where blocks are maximum independent sets of G. Huilgol et al. introduced a new partially balanced incomplete block (PBIB)-designs, called diametral designs with parameters $(v, b, r, diam(G)+1, \lambda, \mu)$ arising from strongly regular graphs of order up to 50 and extended up to 100, where the blocks are vertices of diametral paths of G [9], [8].

In this paper we consider the hypergraph model H_k where $k \geq 1$ of distance regular graphs as given in [11], [13]. In particular we have considered DRGs - Crown graph $K_{n,n} - I$, Johnson graph $J(n, k)$ (when k is 3) and Hamming graph $H(d, q)$ (when d is 3) all of which are having diameter 3. On taking their hypergraph H_1, the diameter of these graphs reduces to 2. Depending upon their structure, $H_1(K_{n,n} - I)$ is a strongly regular graph hence a DRG, $H_1(J(n, 3))$ and $H_1(H(3, q))$ are neither strongly regular nor distance regular except $H_1(J(6, 3))$ which is strongly regular. We have constructed diametral designs arising from these hypergraphs and given composition of their diametral paths by constructing each graph. Also we have given distance based association scheme for the obtained designs.

2. Preliminaries

First we consider some basic definitions. Undefined graph theoretical terms are used in the sense of Buckley and Harary [6].

Definition 2.1. [5] *A connected graph G is called distance regular if there are integers b_i, c_i ($i \geq 0$) such that for any two vertices $u, v \in G$ at distance $i = d(u, v)$, there are precisely c_i neighbours of v in $G_{i-1}(u)$ and b_i neighbours of v in $G_{i+1}(u)$. In particular, G is regular with degree of each vertex $k = b_0$.*

The sequence $\iota(G) = (b_0, b_1, \ldots, b_{d-1}; c_0, c_1, \ldots, c_d)$, where d is diameter of G is called the intersection array of G.

The numbers c_i, b_i and a_i where $a_i = k - b_i - c_i$ ($i = 0, 1, \ldots, d$) is the number of neighbours of v in $G_i(u)$ for $d(u, v) = i$, are called the intersection numbers of G.

Clearly, $b_0 = k$, $b_d = c_0 = 0$,$c_1 = 1$.

Definition 2.2. [5] *A regular graph on v vertices and degree k is called a strongly regular graph with parameters (v, k, λ, μ) if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours and these numbers are independent of the pair of vertices chosen.*

Remark 2.1. [5] *All connected strongly regular graphs have diameter 2.*

Remark 2.2. [6] *If G is a strongly regular graph with parameters (v, k, λ, μ), then $(v-k-1)\mu = k(k-1-\lambda)$.*

Definition 2.3. [15] *A balanced incomplete block (BIB)-design is a set of v elements arranged in b blocks of k elements each in such a way that each element occurs in exactly r blocks and every pair of unordered elements in λ blocks. The combinatorial configuration so obtained is called a (v, b, r, k, λ)-design. A BIB-design satisfies the following conditions.*

(1) $vr = bk$
(2) $\lambda(v-1) = r(k-1)$
(3) $b \geq v$

Definition 2.4. [16] *Given a set $\{1, 2, 3, \ldots, v\}$ of v elements, a relation satisfying the following conditions is said to be an association scheme with m classes.*

- *Any two elements α and β are i^{th} associates for some i with $1 \leq i \leq m$ and this relation of being i^{th} associates is symmetric.*
- *The number of i^{th} associates of each element is n_i.*
- *If α and β are two elements which are i^{th} associates, then the number of elements which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β.*

Definition 2.5. [3] *Consider a set $V = \{1, 2, \ldots, v\}$ and an association scheme with m classes on V. A partially balanced incomplete block (PBIB)-design represented as $(v, b, r, k, \lambda_1, \ldots, \lambda_m)$ is a collection of b subsets of V called blocks, each of them containing k elements ($k < v$) such that every element occurs in r blocks and any two elements α and β which are i^{th} associates occur together in λ_i blocks, numbers λ_i being independent of the choice of pairs α and β.*

The numbers v, b, r, k, λ_i ($i = 1, 2, \ldots, m$) are called parameters of first kind and $n_i's$ and p^i_{jk} are called parameters of second kind.

Definition 2.6. [9] *A $(v, b, r, k, \lambda, \mu)$-design, called a diametral design (in short) over a strongly regular graph $G = (V, E)$ of degree d, diameter diam(G), is an ordered pair $D = (V, \mathcal{B})$, where $V = V(G)$ and $\mathcal{B}$, the set of all diametral paths of G, called blocks, containing vertices belonging to diametral paths, satisfying following conditions:*

(1) *If* $x,\ y \in V$ *and* $(x, y) \in E$*, then there are exactly* λ*-blocks containing* $\{x, y\}$*.*
(2) *If* $x,\ y \in V$ *and* $(x, y) \notin E$*, then there are exactly* μ*-blocks containing* $\{x, y\}$*.*

Definition 2.7. [2] *A hypergraph* $H_k = (V, E)$ *consists of a non-empty set of vertices* $V = \{v_i \mid i = 1, 2, \ldots, p\}$ *and a family* $E = \{he_j \mid j = 1, 2, \ldots, q\}$ *of different sized subsets of the set of vertices. Sets* he_j *are called edges of a hypergraph or hyperedges.*
If $v_i \in he_j$ *then vertex* v_i *is said to be incident to edge* he_j*.*
Cardinality of the set of edges incident to a vertex v_i *is called the degree of* v_i *and is denoted as* d_v*.*
Cardinality of set of vertices incident to an edge he_j *of a hypergraph gives the degree of the edge* he_j *and is denoted as* $deg(he_j)$*.*
Any two vertices in a hypergraph H_k *are adjacent if they belong to the same hyperedge.*

Note: An ordinary graph is a special case of a hypergraph with degrees of all edges equal to two.

Definition 2.8. [17] *Let* $G = (V(G), E(G))$ *be a graph with* n *vertices numbered arbitrarily by numbers* $1, 2, 3, \ldots, n$*, then a hypergraph* $H_k = (V(H_k), E(H_k))$*,* $k \geq 1$ *is such that* $V(H_k) = V(G)$ *and* $E(H_k) = \{e_1, e_2, e_3, \ldots, e_p\}$*,* $e_i =$*{set of vertices* $j : d(i, j) \leq k\}$ *where* $d(i, j)$ *is the distance between vertices* i *and* j *in G. In other words, hyperedge* e_i *is a neighbour of* k^{th} *order of vertex* i $(i = 1, 2, 3, \ldots, n)$*.*

In the next section we see how structure of a hypergraph model varies with the graph.

3. Hypergraphs of DRGs with diameter 3

In this section, we consider hypergraphs H_1 of three different families of distance regular graphs of diameter 3, namely, Crown graph, Johnson graph and Hamming graph. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. From Definition 2.8, it is clear that $H_1 = (V(H_1), E(H_1))$ is such that $V(H_1) = V(G)$ and $E(H_1) = \{e_1, e_2, e_3, \ldots, e_p\}$, $e_i =$\{set of vertices $j : d(i, j) \leq 1\}$ where $d(i, j)$ is the distance between vertices i and j, that is, hyperedges are closed neighbours of vertices of the underlying graph G. Thus from the construction, we see that for a vertex v in G, all vertices which are at distance 1 and 2 be adjacent in $H_1(G)$, and vertices at distance 3 from v in G becomes eccentric vertices of v in $H_1(G)$. Thus $H_1(G)$ are regular graphs of diameter 2.

Remark 3.1. *Suppose* G *is a DRG on* n *vertices having diameter 3 and each vertex has a unique eccentric vertex, then their hypergraph* $H_1(G)$ *is distance regular with intersection array* $(n - 2, 1; 1, n - 2)$ *and is also strongly regular graph having parameters* $(n, n - 2, n - 4, n - 2)$*.*

4. Diametral designs arising from DRGs

4.1. **Crown graph** $K_{n,n} - I$**.** A crown graph on $2n$ vertices is an undirected graph with two sets of vertices $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ and with an edge from u_i to v_j whenever $i \neq j$. It can also be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed.

Example: In the following figure, we depict the Crown graph $K(4, 4) - I$. From the definition of crown graph, vertex set of $K_{4,4} - I$ is partitioned into two independent sets $\{u_1, u_2, u_3, u_4\}$ and $\{v_1, v_2, v_3, v_4\}$. There is an edge from one set to the other set whenever the subscripts of the vertices are not same. For clarity see Figure 1.

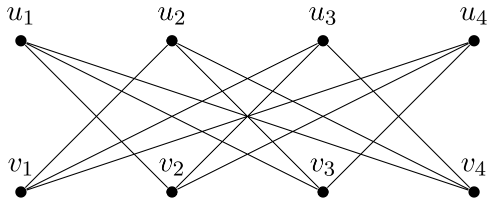

FIGURE 1. $K_{4,4} - I$

Here we give the generalized result obtained for the diametral design arising from hypergraph H_1 of crown graph $K_{n,n} - I$.

Theorem 4.1. *The collection of all diametral paths in hypergraph H_1 of Crown graph forms a PBIB-design having parameters* $(v, b, r, k, \lambda_1, \lambda_2, \mu) = (2n, n(2n-2), (3n-3), 3, 2, 2, 2n-2)$.

Proof. Consider the crown graph $K_{n,n} - I$ where $i \geq 2$. Let the vertex set be partitioned into two independent sets having labels $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$. The edges in $K_{n,n} - I$ are of the form $u_i v_j$ where $i \neq j$ and $j \equiv (i+l) mod(n)$, $1 \leq l \leq n$. Hence there are $n(n-1)$ edges in $K_{n,n} - I$. For a vertex u_i, all those vertices u_k where $k \neq i$ and $1 \leq k \leq n$ (vertices that lie in the same independent set) are at distance 2, and v_i is at distance 3.

Consider the hypergraph $H_1(K_{n,n} - I)$. For a vertex v_i, the vertex u_i is at distance 2 and all the remaining vertices are at distance 1. Hence there are $2n-2$ vertices at distance 1 for any vertex in $H_1(K_{n,n} - I)$ and a unique eccentric vertex. Thus from the structure of hypergraph it is evident that between any pair of eccentric vertices, there are $2n-2$ diametral paths of length 2. Since there are n such distinct pairs of eccentric vertices, there are $n(2n-2)$ distinct diametral paths in $H_1(K_{n,n} - I)$.

There are $(2n-2)$ diametral paths starting with a particular vertex say v_i. Since this vertex is adjacent to all the other vertices except u_i, the remaining $(n-1)$ pairs of eccentric vertices will have a diametral path with v_i as its intermediate vertex. Hence the repetition number is $2n-2+n-1 = 3(n-1)$. Since diametral paths are of length 2, block size k is 3.

Consider a pair of adjacent vertices in $K_{n,n} - I$ say v_i and u_j $(i \neq j)$. Since $H_1(K_{n,n} - I)$ is a unique eccentric vertex graph, we get two distinct diametral paths of the form $v_i - u_j - u_i$ and $u_j - v_i - v_j$ where these vertices appear together, thus giving $\lambda_1 = 2$. Similarly all those vertices which are at distance 2 in $K_{n,n} - I$ from v_i are adjacent in $H_1(K_{n,n} - I)$. Hence due to similar reasoning as above we get $\lambda_2 = 2$. Between every pair of eccentric vertices there are $2n-2$ diametral paths giving $\mu = 2n-2$.

According to the distance between vertices in the graph $K_{n,n} - I$, we consider distance based association scheme here. The above PBIB-design exhibits 3-class association scheme with parameters of second kind as $n_1 = n-1$, $n_2 = n-1$ and $n_3 = 1$ with $P_1 = \begin{bmatrix} 0 & n-2 & 0 \\ n-2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_2 = \begin{bmatrix} n-2 & 0 & 1 \\ 0 & n-2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & n-1 & 0 \\ n-1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. □

From Remark 3.1, it follows that crown graph $K_{n,n} - I$ being unique eccentric vertex graph, the hypergraph $H_1(K_{n,n} - I)$ is distance regular with intersection array $(2n-2, 1 : 1, 2n-2)$ and is also strongly regular with parameters $(2n, 2n-2, 2n-4, 2n-2)$.

Next subsection deals with Johnson graph which is another huge class of distance regular graphs named after Selmer M. Johnson, which is defined from system of sets.

4.2. **Johnson Graph** $J(n,k)$**.** Johnson graphs are a special class of undirected graphs defined from system of sets. The vertices of Johnson graph $J(n,k)$ are the k-element subsets of an n-element set where two vertices are adjacent when intersection of two vertices (subsets) contains $k-1$ elements.

Since we are considering graphs of diameter 3, we take Johnson graphs having k value 3.

Example: In the following figure we depict the Johnson graph $J(6,3)$ along with its labeling. Vertices of the graph are 3-element subsets of the set $\{1,2,3,4,5,6\}$ and hence, are $\{123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456\}$. Any two vertices are adjacent if their labels differ by only one element. For example, the adjacencies of vertex 123 are 124, 125, 126, 134, 135, 136, 234, 235 and 236.

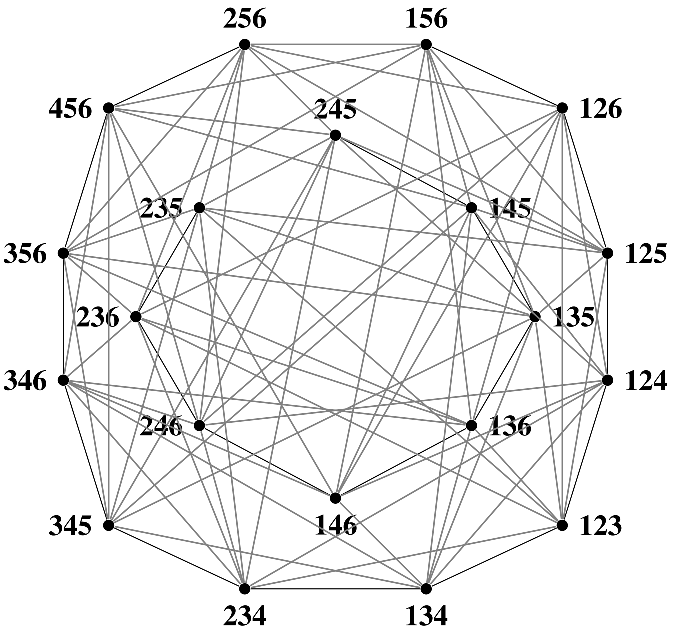

FIGURE 2. $J(6,3)$

Below we give a general result obtained for diametral design arising from hypergraph H_1 of Johnson graph $J(n,3)$.

Theorem 4.2. *The collection of all diametral paths in hypergraph H_1 of Johnson graph $J(n,3)$ where $n \geq 6$ forms a PBIB-design having parameters* $(v,b,r,k,\lambda_1,\lambda_2,\mu) = \Big(\binom{n}{3}, \dfrac{\binom{n}{3}\binom{n-3}{3}9(n-4)}{2}, \binom{n-3}{3}9(n-4)+\dfrac{3(n-3)(n-4)^2(n-5)}{4}, 3, (n-4)(n-5), 2(n-5)^2, 9(n-4)\Big)$.

Proof. As explained earlier, Johnson graph $J(n,3)$ has vertices as 3-element subsets of an n-element set. Therefore, there are $\binom{n}{3}$ such subsets or vertices in $J(n,3)$. There is an edge between any two vertices if their Hamming distance is 1. Clearly, $J(n,3)$ is a regular graph of regulariy $3(n-3)$, as the number of vertices at Hamming distance 1 from any 3-element subset is $3(n-3)$. Suppose N_1 denotes the number of neighbours of each vertex, we get $N_1 = 3(n-3)$. Let the number of second neighbours of a vertex be denoted by N_2 and is equal to $\dfrac{\binom{3}{1}(n-3)(n-4)}{2}$.

This can be realized by simple combinatorial choices allowed for labels of 3-element subsets as vertices of $J(n,3)$. Hence, the remaining $\binom{n-3}{3}$ vertices are third neighbours of a vertex, denoted as N_3, which are also its eccentric vertices.

Now consider the hypergraph model $H_1(J(n,3))$. Here all vertices at distance 1 and 2 from a vertex, say, x in $J(n,3)$ forms neighbours of x in $H_1(J(n,3))$. Therefore, number of first neighbours of x in $H_1(J(n,3))$ is $3(n-3)+\frac{\binom{3}{1}(n-3)(n-4)}{2}$ and remaining $\binom{n-3}{3}$ vertices are second neighbours or eccentric vertices. Consider a diametral path between any pair of eccentric vertices say x and z. Following cases arise depending on the label of intermediate vertex say y.

Case i) y has either two elements common with x (or z) and one element common with z (or x). Two common elements can be chosen in $\binom{3}{2}$ ways and the third element be any element of z giving 9 such vertices.

Case ii) one element common with x and one element common with z.
The element in the label of y common with x can be chosen in three ways. Similarly the element common with z can also be chosen in three ways and the third element in the label of y can be any of the remaining $n-6$ elements. Hence there are $9(n-6)$ possible ways of choosing such intermediate vertex y.

Combining both cases, we get $9(n-4)$ diametral paths between each pair of eccentric vertices. Since each vertex has $\binom{n-3}{3}$ eccentric vertices, there are $\binom{n-3}{3}9(n-4)$ diametral paths with a fixed initial vertex. Since there are $\binom{n}{3}$ vertices in $H_1(J(n,3))$, we get a total of $\frac{\binom{n}{3}(\binom{n-3}{3}9(n-4))}{2}$ distinct diametral paths in $H_1(J(n,3))$. Since diameter is 2, block size is 3.

Consider a diameteral path $x-y-z$.
Suppose x and y share two elements in common in their labels, then z will have one of the elements from y which is the non-common element between x and y and the remaining two elements of z can take any of the remaining $n-4$ and $n-5$ elements respectively. Hence there are $\frac{(n-4)(n-5)}{2}$ diametral paths with x and y together. Let this be denoted by λ_1^*. Similarly the number of diametral paths of the form $y-x-w$ can be counted. Hence we get the value of λ_1 as $(n-4)(n-5)$ which is the number of diametral paths containing both x and y where x and y share two common elements.
Suppose x and y share one element in common then z can be chosen in three ways.

(1) Both the elements in y which are not present in x can be retained in z and the third element of z can be chosen in $n-5$ ways.
(2) One of the elements in y which is not in x can be retained in z and the other two elements of z can be chosen in $n-5$ and $n-6$ ways respectively.
(3) Same as in (ii) wherein the other element of y which is not in x is retained in z.

Hence by counting we get $(n-5)^2$ diameteral paths of the form $x-y-z$ which is denoted as λ_2^*. Counting the number of diameteral paths of the form $y-x-w$ on similar lines, we get $\lambda_2 = 2(n-5)^2$.
Clearly as explained above $\mu = 9(n-4)$ which is the number of diametral paths containing any pair of eccentric vertices.

Now let us count the repetition number of the design. There are $\binom{n-3}{3}9(n-5)$ diametral paths with x as the end vertex and $\frac{N_1\lambda_1^*}{2}+\frac{N_2\lambda_2^*}{2}$ diametral paths with x as intermediate vertex. Thus the repetition number is $\binom{n-3}{3}9(n-4)+\frac{3}{4}(n-3)(n-4)^2(n-5)$. Thus the design parameters obtained are as given in statement of the theorem.

Now we give distance based association scheme for the design obtained. The above PBIB-design exhibits 3-class association scheme with parameters of second kind as $n_1 = 3(n-3)$, $n_2 = \dfrac{3(n-3)(n-4)}{2}$ and $n_3 = \binom{n-3}{3}$ with $P_1 = \begin{bmatrix} (n-2) & 2(n-4) & 0 \\ 2(n-4) & (n-4)^2 & \dfrac{(n-4)(n-5)}{2} \\ 0 & \dfrac{(n-4)(n-5)}{2} & \binom{n-4}{3} \end{bmatrix}$,

$P_2 = \begin{bmatrix} 4 & (2n-8) & (n-5) \\ (2n-8) & \dfrac{(n-5)(n+2)}{2} & (n-5)(n-6) \\ (n-5) & (n-5)(n-6) & \binom{n-5}{3} \end{bmatrix}$ and

$$P_3 = \begin{bmatrix} 0 & 9 & 3(n-6) \\ 9 & 9(n-6) & \dfrac{3(n-6)(n-7)}{2} \\ 3(n-6) & \dfrac{3(n-6)(n-7)}{2} & \binom{n-6}{3} \end{bmatrix}. \qquad \square$$

Note: Johnson graph $J(6,3)$ is a unique eccentric vertex graph. Hence from Remark 3.1, $H_1(J(6,3))$ is a strongly regular graph with parameters $(20, 18, 16, 18)$.

The next subsection deals with one of the large class of distance regular graphs named after the American mathematician Richard Hamming, which finds wide range of applications in several fields of mathematics and computer science.

4.3. **Hamming graph** $H(d,q)$**.** A Hamming graph $H(d,q)$, sometimes also denoted q^d is a cartesian product of d copies of complete graph K_q. Therefore, $H(d,q)$ has q^d vertices. These are distance regular graphs with diameter d. Let S be a set of q elements and d a positive integer. The Hamming graph $H(d,q)$ has vertex set S^d, the set of ordered d-tuples of elements of S, or sequences of length d from S. Any two vertices are adjacent if they differ in precisely 1 coordinate, that is, their Hamming distance is 1.

Since we are dealing with graphs of diameter 3, we consider Hamming graphs of the form $H(3,q)$.

Example: Consider Hamming graph $H(3,3)$. Let the set S be $\{0,1,2\}$. Now the vertex set of $H(3,3)$ is the ordered 3-tuples of the elements of set S and hence,
$S^3 = \{000, 001, 002, 010, 011, 012, 020, 021, 022, 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, 202, 210, 211, 212, 220, 221, 222\}$.
Any two vertices are adjacent if they differ in exactly one coordinate. For example, vertex 000 has vertices 001, 002, 010, 020, 100 and 200 as its adjacent vertices.

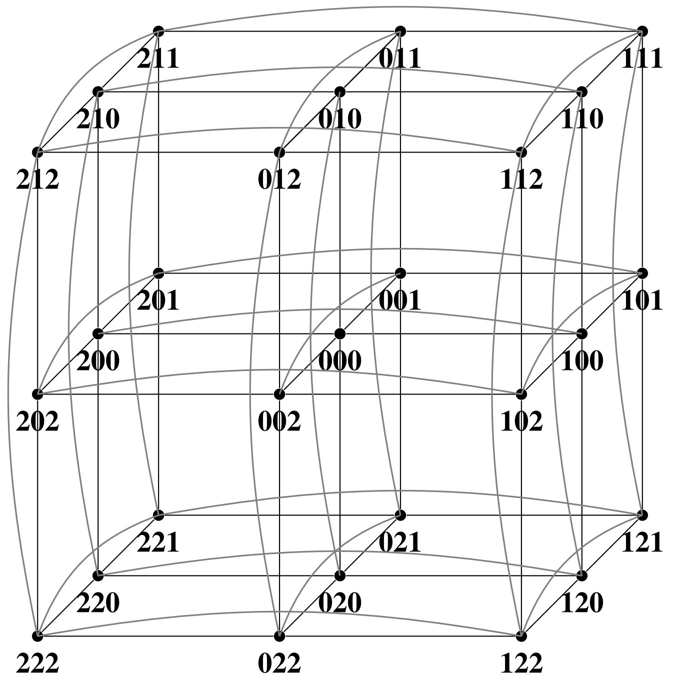

FIGURE 3. $H(3,3)$

The following result gives the design parameters arising from hypergraph H_1 of Hamming graph $H(3,q)$.

Theorem 4.3. *The collection of all diametral paths in hypergraph H_1 of Hamming graph $H(3,q)$ forms a PBIB-design having parameters* $(v,b,r,k,\lambda_1,\lambda_2,\mu) = (q^3, 3q^3(q-1)^4, 9(q-1)^4, 3, 2(q-1)^2, 2(q-1)(2q-3), 6(q-1))$.

Proof. Hamming graph $H(3,q)$ has q^3 number of vertices which are 3-tuples and each coordinate can take q values. Any two vertices are adjacent if their Hamming distance is 1. Consider a vertex say, x in $H(3,q)$. Its adjacent vertices are such that thier labels have any two coordinates same as that of x and the third coordinate can take any of the remaining $(q-1)$ values. The coordinates that remain unchanged can be chosen in $\binom{3}{2}$ ways. Hence there are $3(q-1)$ adjacencies for each vertex. Similar counting yields the number of vertices at distance 2 in $H(3,q)$ as $3(q-1)^2$ and the number of eccentric vertices or vertices at distance 3 as $(q-1)^3$.

Consider the hypergraph $H_1(H(3,q))$. As mentioned earlier, the vertices which were at distance 1 and 2 in $H(3,q)$ become adjacent in $H_1(H(3,q))$. Hence each vertex has $3q(q-1)$ vertices at distance 1 and $(q-1)^3$ vertices at distance 2 in $H_1(H(3,q))$. Consider a diametral path $x-y-z$ in $H_1(H(3,q))$. Now we take different cases for counting the number of diametral paths.

Case i) If y is at Hamming distance 1 (or 2) from x and at Hamming distance 2 (or 1) from z.
In this case the coordinates of y are such that they differ in just one coordinate from x and that particular coordinate takes the respective coordinate value of z. Hence we can get only 3 such diametral paths with y having this property.

Case ii) If y is at Hamming distance 2 from both x and z.
In this case, vertex y is such that two of its coordinates differ from x as well as from z and none of the coordinates of x and z are similar, since they are eccentric to each other. Taking one coordinate

each from both x and z in y, the third coordinate can take any of the remaining $q-2$ values. Totally, there are $6(q-2)$ such triple y possible.

Hence there are $3+3+6(q-2) = 6(q-1)$ diametral paths between any pair of ecentric vertices. There are $(q-1)^3$ number of eccentric vertices for each vertex and q^3 vertices in $H(3,q)$. Thus, there are $6q^3(q-1)^4$ diametral paths in $H_1(H(3,q))$. But each of them is counted twice. Hence there are $3q^3(q-1)^4$ distinct diametral paths in $H_1(H(3,q))$.

Consider a vertex x. Counting the number of diametral paths on similar lines as above, the number of diametral paths with x as one of the terminal vertices is $6(q-1)^4$ and with x as intermediate vertex is $3(q-1)^4$. Thus, repetition number of the design is $9(q-1)^4$.

Now let us count the number of diametral paths in $H_1(H(3,q))$ containing vertices x and y where x and y are at Hamming distance 1 in $H(3,q)$. The terminal vertex of such a diametral path be vertex say, z which is eccentric to x. Clearly the label of z would be such that it has one coordinate same as that of y which is not in x and other 2 coordinates can take any of the remaining $q-1$ values each. Hence there are $(q-1)^2$ diametral paths with x and y as terminal and intermediate vertex respectively. Similarly there are $(q-1)^2$ diametral paths with y and x as terminal and intermediate vertex respectively. Hence $\lambda_1 = 2(q-1)^2$.

To get the value of λ_2, we consider any two vertices say, x and y which are at Hamming distance 2 in $H(3,q)$. Now consider a diametral path with initial vertex x, intermediate vertex y and terminal vertex z. While counting the number of such diametral paths, we encounter two cases.

Case i) z is at Hamming distance 1 from y in $H(3,q)$.
Clearly, labels of z and y has two same coordinates and the third coordinate of z can take any one element from set S except the one that is common in both x and y. Hence, there are $(q-1)$ such diametral paths $x-y-z$.

Case ii) z is at Hamming distance 2 from y in $H(3,q)$.
In this case, labels of z and y share a common element. Of the remaining two coordinates of z, one coordinate can take any element from S except the one that is common in both x and y and the other coordinate can take $(q-2)$ values. Hence there are $2(q-1)(q-2)$ such diametral paths $x-y-z$ corresponding to each non-common element between labels of x and y.

Combining both cases, we get $(q-1)(2q-3)$ diametral paths $x-y-z$ in $H_1(H(3,q))$. Similar counting yields the number of diametral paths of the form $y-x-w$ as $(q-1)(2q-3)$. Hence there are $2(q-1)(2q-3)$ diametral paths in $H_1(H(3,q))$ conatining x and y which are at Hamming distance 2 in $H(3,q)$, thus giving the value of λ_2 as $2(q-1)(2q-3)$.

Since there are $6(q-1)$ diametral paths between every pair of eccentric vertices in $H_1(H(3,q))$, the value of μ follows.

The above PBIB-design exhibits 3-class Hamming distance based association scheme with parameters of second kind as $n_1 = 3(q-1)$, $n_2 = 3(q-1)^2$ and $n_3 = (q-1)^3$ with

$$P_1 = \begin{bmatrix} q-2 & 2(q-1) & 0 \\ 2(q-1) & 2(q-1)(q-2) & (q-1)^2 \\ 0 & (q-1)^2 & (q-1)^2(q-2) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 2 & 2(q-2) & (q-1) \\ 2(q-2) & (q-2)^2+2(q-1) & 2(q-1)(q-2) \\ (q-1) & 2(q-1)(q-2) & (q-1)(q-2)^2 \end{bmatrix} \text{ and}$$

$$P_3 = \begin{bmatrix} 0 & 3 & 3(q-2) \\ 3 & 6(q-2) & 3(q-2)^2 \\ 3(q-2) & 3(q-2)^2 & (q-2)^3 \end{bmatrix}.$$

□

5. Conclusion

Berge [2] introduced hypergraphs as a means to generalize the graph approach. Graphs only support pairwise relationships, whereas hypergraphs preserve multi-adic relationships and therefore become a natural modeling of collaboration networks and various other situations. Due to recent advancements in the field of hypergraph theory, it relatively finds a wide range of applications in mathematical modeling of real world problems, engineering, computer science, chemistry, to name a few. As combinatorial design theory basically deals with set systems, we have in this paper found the parameters of PBIB-designs arising from hypergraphs of some known families of graphs. Here we have given construction of three families of graphs namely, Crown graph $K(n,n) - I$, Johnson graph $J(n,3)$ and Hamming graph $H(3,q)$ which are distance-regular graphs of diameter 3. We have taken hypergraph H_1 model of these graphs and obtained generalized expressions for parameters of diametral designs obtained from these hypergraphs by taking vertices belonging to diametral paths as blocks. We have also given the distance based association scheme for each of the above designs.

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