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CONVERGENCE RESULTS FOR LUPAŞ-KANTOROVICH OPERATORS INVOLVING PÓLYA DISTRIBUTION

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Abstract. In this paper, we investigate the Lupaş-Kantorovich operators with the Pólya distribution introduced by Agrawal et al. in 2016. Our focus is on understanding certain approximation properties. Specifically, we examine a Voronovskaya-type result related to second moduli of continuity, the Chebyshev-Grüss inequality and two Grüss-Voronovskaya theorems for Lupaş-Kantorovich operators.

Keywords: Chebyshev-Grüss inequality, Grüss-Voronovskaya theorem, Pólya distribution, First-order modulus.

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1. Introduction

Kantorovich [19] presented the following significant modification of Bernstein polynomials for $\zeta \in L_1[0,1]$ (the class of Lebesgue integrable functions on $[0,1]$):

$$K_{\mathbb{m}}(\zeta;\varkappa) = (\mathbb{m}+1)\sum_{k=0}^{\mathbb{m}} p_{\mathbb{m},k}(\varkappa)\int_0^1 \phi_{\mathbb{m},k}(t)\zeta(t)dt, \quad \varkappa \in [0,1],$$

where $p_{\mathbb{m},k}(\varkappa) = \binom{\mathbb{m}}{k}\varkappa^k(1-\varkappa)^{\mathbb{m}-k}$ and another function $\phi_{\mathbb{m},k}(t)$ that represents a certain interval $[k/(\mathbb{m}+1),(k+1)/(\mathbb{m}+1)]$. Many researchers have studied this topic and written papers about it. Some of the related papers include references like [1, 14, 26, 30, 31]. One particular set of operators, called Lupaş operators [22], has been widely studied recently.

$$P_{\mathbb{m}}^{(1/\mathbb{m})}(\zeta;\varkappa) = \sum_{k=0}^{\mathbb{m}} p_{\mathbb{m},k}^{(1/\mathbb{m})}(\varkappa)\zeta\left(\frac{k}{\mathbb{m}}\right) = \frac{2(\mathbb{m}!)}{(2\mathbb{m})!}\sum_{k=0}^{\mathbb{m}}\binom{\mathbb{m}}{k}\prod_{\nu=0}^{k-1}(\mathbb{m}\varkappa+\nu)\prod_{\mu=0}^{\mathbb{m}-k-1}(\mathbb{m}-\mathbb{m}\varkappa+\mu)\zeta\left(\frac{k}{\mathbb{m}}\right). \quad (1)$$

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The Lupaş operators (1), rooted in the fundamental basis $p_{\mathbb{m},k}^{(1/\mathbb{m})}$, are derived from the Pólya distribution and can be seen as a specialized instance of the operator class initially introduced by Stancu [29]. Gupta and Rassias [17] examined a Durrmeyer type integral modification of the operators (1) and provided insights into their local, asymptotic behavior and global characteristics. Additionally, Agrawal et al. [4] investigated the Bézier variant of the Gupta and Rassias operators [17] and explored its direct approximation outcomes.

Agrawal et al. [3] introduced a Kantorovich type modification of the Pólya distribution-derived operators and previously studied by Lupaş and Lupaş [22], considering $\zeta \in C[0,1]$. The proposed modification is formulated as follows:

$$\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa) \;=\; (\mathbb{m}+1)\sum_{k=0}^{\mathbb{m}} p_{\mathbb{m},k}^{(1/\mathbb{m})}(\varkappa)\int_{\frac{k}{\mathbb{m}+1}}^{\frac{k+1}{\mathbb{m}+1}} \zeta(t)\,dt, \quad \varkappa \in [0,1], \tag{2}$$

where $p_{\mathbb{m},k}^{(1/\mathbb{m})}(\varkappa)$ is defined above.

In their study, Gonska and Tachev [16] looked into Grüss-type inequalities that apply to linear positive operators, using second-order moduli of smoothness. They also explored the applications of these inequalities to Bernstein operators. Acu et al. [2] derived results regarding the non-multiplicativity of positive linear operators that replicate constant functions. Additionally, Gal and Gonska [11] utilized the Grüss-type inequality for Bernstein operators to establish a theorem called the Grüss-Voronovskaja-type theorem. Rahman et al. [28] presented Kantorovich variant of Lupaş operators based on Pólya distribution with shifted knots and obtained rate of convergence of these operators. In [24], the authors introduced Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution in the polynomial weighted space and studied convergence properties of these operators by using Korovkin's theorem. Kilicman et al. [20] considered Stancu-Baskakov–Durrmeyer type operators and obtained direct results.

This paper aims to investigate the quantitative Voronovskaya-type outcome concerning second moduli of continuity, the Chebyshev-Grüss inequality, and two Grüss-Voronovskaya theorems applicable to the operators (2).

2. Basic Results

Lemma 2.1. *[3] For $e_i = t^i, i = 0,1,2,3,4$ we have*

(1) $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(e_0;\varkappa) = 1$;

(2) $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(e_1;\varkappa) = \dfrac{2\mathbb{m}\varkappa+1}{2(\mathbb{m}+1)}$;

(3) $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(e_2;\varkappa) = \dfrac{3\mathbb{m}^3\varkappa^2+9\mathbb{m}^2\varkappa-3\mathbb{m}^2\varkappa^2+3\mathbb{m}\varkappa+\mathbb{m}+1}{3(\mathbb{m}+1)^3}$;

(4) $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(e_3;\varkappa) = \dfrac{1}{4(\mathbb{m}+1)^4(\mathbb{m}+2)}\Big\{4\varkappa^3(\mathbb{m}^5+3\mathbb{m}^4+2\mathbb{m}^3)+6\varkappa^2(\mathbb{m}^4+\mathbb{m}^3-2\mathbb{m}^2)+4x(\mathbb{m}^3+9\mathbb{m}^2+2\mathbb{m})+(\mathbb{m}+1)(\mathbb{m}+2)\Big\}$;

(5) $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(e_4;\varkappa) = \dfrac{\mathbb{m}^4\varkappa^4}{(\mathbb{m}+1)^4}+\dfrac{\varkappa(1-\varkappa)(60\varkappa^2\mathbb{m}^7+60\mathbb{m}^5\varkappa^2+180\mathbb{m}^6\varkappa-60\mathbb{m}^5\varkappa+130\mathbb{m}^5-10\mathbb{m}^4)}{5\mathbb{m}(\mathbb{m}+1)^5(\mathbb{m}+2)(\mathbb{m}+3)}$
$+\dfrac{\mathbb{m}\varkappa}{(\mathbb{m}+1)^4}+\dfrac{2\varkappa^2\mathbb{m}^2(\mathbb{m}^3+10\mathbb{m}^2-3\mathbb{m}-10)+8\varkappa^2\mathbb{m}^2(1+2\mathbb{m})-12\mathbb{m}^4\varkappa^3}{(\mathbb{m}+1)^5(\mathbb{m}+2)}$.

Remark 2.1. *By simple applications of Lemma 2.1, we have*

$$\begin{aligned}
\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa);\varkappa) &= \frac{1-2\varkappa}{2(\mathrm{m}+1)};\\
\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^2;\varkappa) &= \frac{\varkappa(1-\varkappa)(2\mathrm{m}^2-\mathrm{m}-1)}{(\mathrm{m}+1)^3}+\frac{1}{3(\mathrm{m}+1)^2};\\
\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^3;\varkappa) &= \frac{(1+2\mathrm{m})(9\mathrm{m}^2+\mathrm{m}-2)\varkappa^3}{(\mathrm{m}+1)^4(\mathrm{m}+2)}\\
&+\frac{\left(8-2(1+2\mathrm{m})(-2+\mathrm{m}+9\mathrm{m}^2)-4\mathrm{m}(-3+\mathrm{m}(11+18\mathrm{m}))\right)\varkappa^2}{4(\mathrm{m}+1)^4(\mathrm{m}+2)}\\
&+\frac{(-8-2\mathrm{m}(\mathrm{m}+3)+2\mathrm{m}(-3+\mathrm{m}(11+18\mathrm{m})))\varkappa}{4(\mathrm{m}+1)^4(\mathrm{m}+2)}+\frac{(2+\mathrm{m}(\mathrm{m}+3))}{4(\mathrm{m}+1)^4(\mathrm{m}+2)};\\
\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^4;\varkappa) &= \frac{(1+2\mathrm{m})(6\mathrm{m}^4-75\mathrm{m}^3-64\mathrm{m}^2-\mathrm{m}+6)\varkappa^4}{(\mathrm{m}+1)^5(\mathrm{m}+2)(\mathrm{m}+3)}+\frac{2(1+2\mathrm{m})(6\mathrm{m}^4-75\mathrm{m}^3-64\mathrm{m}^2-\mathrm{m}+6)\varkappa^3}{(\mathrm{m}+1)^5(\mathrm{m}+2)(\mathrm{m}+3)}\\
&+\frac{2(1+2\mathrm{m})(3\mathrm{m}^4-48\mathrm{m}^3-40\mathrm{m}^2-\mathrm{m}+6)\varkappa^2}{(\mathrm{m}+1)^5(\mathrm{m}+2)(\mathrm{m}+3)}+\frac{1}{5(\mathrm{m}+1)^4}\\
&+\frac{(1+2\mathrm{m})(21\mathrm{m}^3+16\mathrm{m}^2+\mathrm{m}-6)\varkappa}{(\mathrm{m}+1)^5(\mathrm{m}+2)(\mathrm{m}+3)}.
\end{aligned}$$

Lemma 2.2. *For* $\mathrm{m}\in\mathbb{N}$*, we have*

$$\begin{aligned}
\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^2;\varkappa) &\le \frac{1}{2(\mathrm{m}+1)};\\
\frac{\left|\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^3;\varkappa)\right|}{\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^2;\varkappa)} &\le \frac{9}{2(\mathrm{m}+1)};\\
\frac{\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^4;\varkappa)}{\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}((t-\varkappa)^2;\varkappa)} &\le \frac{6(\mathrm{m}+12)}{(\mathrm{m}+1)^2}.
\end{aligned}$$

The theorem by Păltănea [27] presented below serves as a crucial foundation for providing a more clear outcome in relation to classical moduli for continuous functions and further elaboration, refer to [14].

Theorem 2.1. *[27] Given a positive linear operator* $L: C(I)\to\mathcal{F}(I)$*, where* I *represents an arbitrary interval and* $\mathcal{F}(I)$ *denotes the space of real functions defined on* I*, the following holds true for any* $\zeta\in C(I)$*,* $\varkappa\in I$ *and* $0<h<\frac{1}{2}\mathit{length}(I)$

$$\begin{aligned}
|L(\zeta;\varkappa)-\zeta(\varkappa)| &\le |L(e_0;\varkappa)-1|\cdot|\zeta(\varkappa)|+\frac{1}{h}|L(e_1-\varkappa;\varkappa)|\omega(\zeta;h)\\
&+\left[(Le_0)(\varkappa)+\frac{1}{2h^2}L((e_1-\varkappa)^2;\varkappa)\right]\omega_2(\zeta;h).
\end{aligned}$$

Theorem 2.2. *For all* $\zeta\in C[0,1]$ *and* $\mathrm{m}\ge 3$*, we have*

$$\|\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta-\zeta\|_\infty \le \frac{1}{\sqrt{2(1+\mathrm{m})}}\omega_1\left(\zeta;\frac{1}{\sqrt{(1+\mathrm{m})}}\right)+\frac{9}{8}\omega_2\left(\zeta;\frac{1}{\sqrt{(\mathrm{m}+1)}}\right).$$

3. A quantitative Voronovskaya result

An early and highly significant result is the Voronovskaja theorem [3] for the Kantorovich-type modification of operators is

$$\lim_{\mathrm{m}\to\infty}\mathrm{m}\left(\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)-\zeta(\varkappa)\right)=\frac{1}{2}(1-2\varkappa)\zeta'(\varkappa)+\varkappa(1-\varkappa)\zeta''(\varkappa). \tag{3}$$

This section origins can be traced back to a little-known booklet by Videnskij, where a quantitative version of the renowned Voronovskaya theorem for the classical Bernstein operators is presented (refer to [32]). Subsequently, this estimate was further generalized and enhanced in ([15]). In another study ([14]), an application of this result was demonstrated for Kantorovich operators. In this paper, we aim to further improve upon the previous findings.

Theorem 3.1. *When* $\mathbb{m} \geq 1$ *and* $\zeta \in C^2[0,1]$*, The following is true.*

$$\left\| \mathbb{m}\left(\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}\zeta - \zeta\right) - \left(\frac{(1-2\varkappa)\zeta'(\varkappa)}{2} + \varkappa(1-\varkappa)\zeta''(\varkappa)\right)\right\|_\infty \leq \frac{15}{8}\left\{\frac{1}{\sqrt{1+\mathbb{m}}}\omega_1\left(\zeta''; \frac{1}{\sqrt{1+\mathbb{m}}}\right) + \omega_2\left(\zeta''; \frac{1}{\sqrt{1+\mathbb{m}}}\right)\right\} + \frac{1}{2(\mathbb{m}+1)}\left(\|\zeta'\|_\infty + \|\zeta''\|_\infty\right). \quad (4)$$

Proof. From [15, Thm. 3], we get

$$\left|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa) - \zeta(\varkappa) - \mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(t-\varkappa;\varkappa)\zeta'(\varkappa) - \frac{1}{2}\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}\left((e_1-\varkappa)^2;\varkappa\right)\zeta''(\varkappa)\right|$$
$$\leq \mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}((e_1-\varkappa)^2;\varkappa)\left\{\frac{|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}((e_1-\varkappa)^3;\varkappa)|}{\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}((e_1-\varkappa)^2;\varkappa)}\frac{5}{6h}\omega_1(\zeta'';h) + \left(\frac{3}{4} + \frac{\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}((e_1-\varkappa)^4;\varkappa)}{\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}((e_1-\varkappa)^2;\varkappa)}\cdot\frac{1}{16h^2}\right)\omega_2(\zeta'';h)\right\}.$$

Applying Lemma 2.2 and Remark 2.1, we get

$$\left|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa) - \zeta(\varkappa) - \frac{1-2\varkappa}{2(\mathbb{m}+1)}\zeta'(\varkappa) - \frac{1}{2}\left[\frac{\varkappa(1-\varkappa)(2\mathbb{m}^2-\mathbb{m}-1)}{(\mathbb{m}+1)^3} + \frac{1}{3(\mathbb{m}+1)^2}\right]\zeta''(\varkappa)\right|$$
$$\leq \left[\frac{\varkappa(1-\varkappa)(2\mathbb{m}^2-\mathbb{m}-1)}{(\mathbb{m}+1)^3} + \frac{1}{3(\mathbb{m}+1)^2}\right]\left\{\frac{45}{12h(\mathbb{m}+1)}\omega_1(\zeta'';h) + \left(\frac{3}{4} + \frac{6(\mathbb{m}+12)}{16h^2(\mathbb{m}+1)^2}\right)\omega_2(\zeta'';h)\right\},$$

and for $h = \dfrac{1}{\sqrt{\mathbb{m}+1}}$. By multiplying both sides by $\mathbb{m}$, we obtain,

$$\left|\mathbb{m}\left[\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa) - \zeta(\varkappa)\right] - \frac{\mathbb{m}}{\mathbb{m}+1}\frac{(1-2\varkappa)}{2}\zeta'(\varkappa) - \frac{1}{2}\left[\varkappa(1-\varkappa)\frac{\mathbb{m}(2\mathbb{m}^2-\mathbb{m}-1)}{(\mathbb{m}+1)^3} + \frac{\mathbb{m}}{3(\mathbb{m}+1)^2}\right]\zeta''(\varkappa)\right|$$
$$\leq \frac{15}{8}\left\{\frac{1}{\sqrt{\mathbb{m}+1}}\omega_1\left(\zeta''; \frac{1}{\sqrt{1+\mathbb{m}}}\right) + \omega_2\left(\zeta''; \frac{1}{\sqrt{1+\mathbb{m}}}\right)\right\}.$$

We may write

$$\left|\mathbb{m}\left[\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa) - \zeta(\varkappa)\right] - \frac{(1-2\varkappa)}{2}\zeta'(\varkappa) - \varkappa(1-\varkappa)\zeta''(\varkappa)\right|$$
$$\leq \left|\mathbb{m}\left[\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa) - \zeta(\varkappa)\right] - \frac{\mathbb{m}}{\mathbb{m}+1}\frac{(1-2\varkappa)}{2}\zeta'(\varkappa) - \frac{1}{2}\left[\varkappa(1-\varkappa)\frac{\mathbb{m}(2\mathbb{m}^2-\mathbb{m}-1)}{(\mathbb{m}+1)^3} + \frac{\mathbb{m}}{3(\mathbb{m}+1)^2}\right]\zeta''(\varkappa)\right|$$
$$+ \left|\frac{(1-2\varkappa)}{2}\frac{1}{1+\mathbb{m}}\zeta'(\varkappa) + \varkappa(1-\varkappa)\frac{\mathbb{m}^2}{(1+\mathbb{m})^3}\zeta''(\varkappa) - \frac{\mathbb{m}}{6(\mathbb{m}+1)^2}\zeta''(\varkappa)\right|$$
$$\leq \frac{15}{8}\left\{\frac{1}{\sqrt{\mathbb{m}+1}}\omega_1\left(\zeta''; \frac{1}{\sqrt{\mathbb{m}+1}}\right) + \omega_2\left(\zeta''; \frac{1}{\sqrt{\mathbb{m}+1}}\right)\right\} + \frac{1}{2(\mathbb{m}+1)}\left(\|\zeta'\|_\infty + \|\zeta''\|_\infty\right).$$

□

4. Chebyshev-Grüss inequality for Lupaş Kantorovich operators

Theorem 4.1. *[2] Let* $H : C[a,b] \to C[a,b]$ *represent a positive, linear operator that fulfills the condition* $He_0 = e_0$*. Now consider the following:*

$$D(\zeta,\beta;\varkappa) := H(\beta\zeta;\varkappa) - H(\beta;\varkappa)\cdot H(\zeta;\varkappa).$$

Then for $\zeta, \beta \in C[a,b]$ *and* $\varkappa \in [a,b]$ *fixed one has*

$$|D(\zeta,\beta;\varkappa)| \leq \frac{1}{4}\tilde{\omega}\left(\zeta; 2\sqrt{H\left((e_1-\varkappa)^2;\varkappa\right)}\right)\cdot\tilde{\omega}\left(\beta; 2\sqrt{H\left((e_1-\varkappa)^2;\varkappa\right)}\right).$$

Here, $\tilde{\omega}$ represents the least concave majorant of the first-order modulus ω_1, which is defined as follows

$$\tilde{\omega}(\zeta;t) = \sup\left\{\frac{(t-\varkappa)\omega_1(\zeta;y) + (y-t)\omega_1(\zeta;\varkappa)}{y-\varkappa} : 0 \le x \le t \le y \le b-a, x \ne y\right\}.$$

Therefore, the lack of multiplicativity observed in Lupaş-Kantorovich operators can be understood in a similar manner as described in

Theorem 4.2. *Regarding the Lupaş-Kantorovich operators $\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})} : C[0,1] \to C[0,1]$, a uniform inequality is given by*

$$\|\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\beta\|_\infty \le \frac{1}{4}\tilde{\omega}\left(\zeta; 2\sqrt{\frac{1}{2(\mathrm{m}+1)}}\right)\cdot\tilde{\omega}\left(\beta; 2\sqrt{\frac{1}{2(\mathrm{m}+1)}}\right), \mathrm{m} \ge 1, \tag{5}$$

for all $\zeta, \beta \in C[0,1]$.

5. Grüss-Voronovskaya theorems

Given ζ and β as twice continuously differentiable functions on $[0,1]$, we seek to find their limit

$$\lim_{\mathrm{m}\to\infty} \mathrm{m}\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta\zeta)(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta)(\varkappa)\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta)(\varkappa)\right] = 2X\zeta'(\varkappa)\beta'(\varkappa).$$

Given that $X = \varkappa(1-\varkappa)$ by straightforward calculation we have obtained a result,

$$\begin{aligned}
&\mathrm{m}\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta)(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta)(\varkappa)\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta)(\varkappa)\right] \\
&= \mathrm{m}\left\{\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta)(\varkappa) - \zeta(\varkappa)\beta(\varkappa) - \left(\frac{X'}{2\mathrm{m}}(\zeta' g + \beta'\zeta) + \frac{X}{\mathrm{m}}(\beta\zeta'' + 2\zeta'\beta' + \beta''\zeta)\right)\right. \\
&\quad -\beta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta)(\varkappa) - \zeta(\varkappa) - \left(\frac{X'}{2\mathrm{m}}\zeta' + \frac{X}{\mathrm{m}}\zeta''\right)\right] \\
&\quad -\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta)(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta)(\varkappa) - \beta(\varkappa) - \left(\frac{X'}{2\mathrm{m}}\beta' + \frac{X}{\mathrm{m}}\beta''\right)\right] \\
&\quad \left. +\frac{2X}{\mathrm{m}}\zeta'(\varkappa)\beta'(\varkappa) + \left(\frac{X'}{2\mathrm{m}}\beta' + \frac{X}{\mathrm{m}}\beta''\right)\left[\zeta(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta)(\varkappa)\right]\right\},
\end{aligned}$$

passing to the limit it easily follows

$$\lim_{\mathrm{m}\to\infty} \mathrm{m}\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta)(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta)(\varkappa)\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta)(\varkappa)\right] = 2X\zeta'(\varkappa)\beta'(\varkappa).$$

Theorem 5.1. *Let $\zeta, \beta \in C^2[0,1]$. Then for each $\varkappa \in [0,1]$*

$$\|\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta\cdot\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\beta - \frac{2}{\mathrm{m}}X\zeta'\beta'\|_\infty = \begin{cases} o(1), & \zeta,\beta \in C^2[0,1], \\ O\left(\frac{1}{\sqrt{m}}\right), & \zeta,\beta \in C^3[0,1], \\ O\left(\frac{1}{m}\right), & \zeta,\beta \in C^4[0,1]. \end{cases}$$

Proof. The method described in [1] is followed, where we generate three Voronovskaya-type expressions using the given difference, combined with other remaining terms. The Voronovskaya limit for Kantorovich operators is, as a reminder,

$$\frac{1}{2}(X\zeta')' = \frac{1}{2}X\zeta''(\varkappa) + \frac{1}{2}X'\zeta'(\varkappa),$$

where $X := \varkappa(1-\varkappa)$, so $X' = 1 - 2\varkappa$.

For $\zeta, \beta \in C^2[0,1]$ one has

$$\begin{aligned}
&\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta;\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa) - \frac{2}{\mathrm{m}}X\zeta'(\varkappa)\beta'(\varkappa)\\
&= \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta\beta;\varkappa) - (\zeta\beta)(\varkappa) - \frac{1}{\mathrm{m}}\left(\frac{X'(\zeta\beta)'}{2} + X(\zeta\beta)''\right)\\
&- \zeta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa) - \beta(\varkappa) - \left(\frac{X'\beta'}{2} + X\beta''\right)\right] - \beta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa) - \zeta(\varkappa) - \left(\frac{X'\zeta'}{2} + X\zeta''\right)\right]\\
&+ \left[\beta(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa)\right]\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa) - \zeta(\varkappa)\right]\\
&- \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa) - \frac{2}{\mathrm{m}}X\zeta'\beta' + (\zeta\beta)(\varkappa) + \frac{1}{\mathrm{m}}\left(\frac{X'(\zeta\beta)'}{2} + X(\zeta\beta)''\right)\\
&+ \zeta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa) - \beta(\varkappa) - \left(\frac{X'\beta'}{2} + X\beta''\right)\right] + \beta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa) - \zeta(\varkappa) - \left(\frac{X'\zeta'}{2} + X\zeta''\right)\right]\\
&- \left[\beta(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa)\right]\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa) - \zeta(\varkappa)\right].
\end{aligned}$$

Below, we will estimate the first three lines. Our goal is to demonstrate that the sum of these three lines equates to 0.

At present, we will omit the argument $\varkappa$. The following expression holds:

$$\begin{aligned}
&- \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa) - \frac{2}{\mathrm{m}}X\zeta'\beta' + (\zeta\beta)(\varkappa) + \frac{1}{\mathrm{m}}\left(\frac{X'(\zeta\beta)'}{2} + X(\zeta\beta)''\right)\\
&+ \zeta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa) - \beta(\varkappa) - \left(\frac{X'\beta'}{2} + X\beta''\right)\right] + \beta(\varkappa)\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa) - \zeta(\varkappa) - \left(\frac{X'\zeta'}{2} + X\zeta''\right)\right]\\
&- \left[\beta(\varkappa) - \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\beta;\varkappa)\right]\left[\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa) - \zeta(\varkappa)\right]\\
&= -\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta \cdot \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\beta - \frac{2}{\mathrm{m}}X\zeta'\beta' + \zeta\beta + \frac{1}{2\mathrm{m}}\left(X'\zeta'\beta + X'\zeta\beta'\right) + \frac{1}{\mathrm{m}}X\left(\zeta''\beta + 2\zeta'\beta' + \zeta\beta''\right)\\
&+ f\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\beta - \zeta\beta - (\frac{1}{2\mathrm{m}}\zeta X'\beta' + \frac{1}{\mathrm{m}}\zeta X\beta'') + \beta\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta - \zeta\beta - (\frac{1}{2\mathrm{m}}\beta X'\zeta' + \frac{1}{\mathrm{m}}\beta X\zeta'')\\
&- \beta\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta + \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\beta \cdot \mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}\zeta + \zeta\beta - \zeta\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}g = 0.
\end{aligned}$$

We will use the Voronovskaya previously presented to estimate the first two lines previously specified. Specifically, for $h \in C^2[0,1]$, the estimate states:

$$\begin{aligned}
\left\|\mathrm{m}\left(\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}h - h\right) - \left(\frac{X'h'}{2} + Xh''\right)\right\|_\infty \quad \leq \quad & \frac{15}{8}\left\{\frac{1}{\sqrt{\mathrm{m}+1}}\omega_1\left(\zeta''; \frac{1}{\sqrt{\mathrm{m}+1}}\right) + \omega_2\left(\zeta''; \frac{1}{\sqrt{\mathrm{m}+1}}\right)\right\}\\
& + \frac{1}{2(\mathrm{m}+1)}\left(\|\zeta'\|_\infty + \|\zeta''\|_\infty\right) =: U(h,\mathrm{m}).
\end{aligned}$$

We will use Theorem 2.2, which establishes that for $h \in C^2[0,1]$, to estimate the third line., we obtain the following result:

$$\|\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}h - h\|_\infty \leq \frac{1}{2\mathrm{m}}\|h'\|_\infty + \frac{9}{8\mathrm{m}}\|h''\|_\infty = O\left(\frac{1}{\mathrm{m}}\right).$$

Combining these inequalities, we obtain the following result:

$$\|\mathcal{D}_{m}^{*(1/m)}(\zeta\beta) - \mathcal{D}_{m}^{*(1/m)}\zeta \cdot \mathcal{D}_{m}^{*(1/m)}\beta - \frac{1}{m}X\zeta'\beta'\|_\infty \leq U(\zeta\beta, m) + \|\zeta\|_\infty U(\beta, m) + \|\beta\|_\infty U(\zeta, m) + O\left(\frac{1}{m^2}\right)$$
$$= \begin{cases} o(1), & \zeta, \beta \in C^2[0,1], \\ O\left(\frac{1}{\sqrt{m}}\right), & \zeta, \beta \in C^3[0,1], \\ O\left(\frac{1}{m}\right), & \zeta, \beta \in C^4[0,1]. \end{cases}$$

□

Next, our goal is to provide a theorem like Grüss-Voronovskaya theorem, under the condition that both ζ and g are elements of $C^1[0,1]$.

Theorem 5.2. *Assuming ζ and β belong to $C^1[0,1]$ and $m \geq 1$, it can be established that a constant called C exists that is independent of m, ζ, β and $\varkappa$, satisfying the following*

$$\left\|\mathcal{D}_{m}^{*(1/m)}(\zeta\beta) - \mathcal{D}_{m}^{*(1/m)}\zeta \cdot \mathcal{D}_{m}^{*(1/m)}\beta - \frac{2X}{m}\zeta'\beta'\right\|_\infty \leq \frac{C}{m}\Big\{\omega_3\left(\zeta', m^{-\frac{1}{6}}\right)\omega_3\left(\beta', m^{-\frac{1}{6}}\right)$$
$$+ \|\zeta'\|_\infty\omega_3\left(\beta', m^{-\frac{1}{6}}\right) + \|\beta'\|_\infty\omega_3\left(\zeta', m^{-\frac{1}{6}}\right)$$
$$+ \max\left\{\frac{\|\zeta'\|_\infty}{m^{\frac{1}{2}}}, \omega_3\left(\zeta', m^{-\frac{1}{6}}\right)\right\}\max\left\{\frac{\|\beta'\|_\infty}{m^{\frac{1}{2}}}, \omega_3\left(\beta', m^{-\frac{1}{6}}\right)\right\}\Big\}.$$

Proof. Let

$$E_{m}(\zeta, \beta; \varkappa) = \mathcal{D}_{m}^{*(1/m)}(\zeta\beta; \varkappa) - \mathcal{D}_{m}^{*(1/m)}(\zeta; \varkappa)\mathcal{D}_{m}^{*(1/m)}(\beta; \varkappa) - \frac{2\varkappa(1-\varkappa)}{m}\zeta'(\varkappa)\beta'(\varkappa), \tag{6}$$

Let C be a constant that remains independent of m, ζ, β, $\varkappa$ and it is allowed to vary throughout the proof.

For $\zeta, \beta \in C^1[0,1]$ fixed and $u, v \in C^4[0,1]$ arbitrary, one has

$$|E_{m}(\zeta, \beta; \varkappa)| = |E_{m}(\zeta - u + u, \beta - v + v; \varkappa)| \tag{7}$$
$$\leq |E_{m}(\zeta - u, \beta - v; \varkappa)| + |E_{m}(u, \beta - v; \varkappa)| + |E_{m}(\zeta - u, v; \varkappa)| + |E_{m}(u, v; \varkappa)|.$$

Considering the function $h(\varkappa) = x$ for $\varkappa \in [0,1]$, and utilizing [2, Theorem 4.1], it can be deduced that there exist values η and θ within the interval $[0,1]$ such that

$$\mathcal{D}_{m}^{*(1/m)}(\zeta\beta; \varkappa) - \mathcal{D}_{m}^{*(1/m)}(\zeta; \varkappa)\mathcal{D}_{m}^{*(1/m)}(\beta; \varkappa) = \zeta'(\eta)\beta'(\theta)\left[\mathcal{D}_{m}^{*(1/m)}m(h^2; \varkappa) - (\mathcal{D}_{m}^{*(1/m)}m(h; \varkappa))^2\right]$$
$$= \zeta'(\eta)\beta'(\theta)\left\{\varkappa(1-\varkappa)\frac{2m^2}{(m+1)^3} + \frac{1}{12(m+1)^2}\right\}. \tag{8}$$

From (6) and (8) we get

$$|mE_{m}(\zeta, \beta; \varkappa)| \leq \left[\varkappa(1-\varkappa)\frac{2m^3}{(m+1)^3} + \frac{m}{12(m+1)^2} + 2\varkappa(1-\varkappa)\right]\|\zeta'\|_\infty\|\beta'\|_\infty$$
$$\leq 4\left[\varkappa(1-\varkappa) + \frac{1}{24(m+1)}\right]\|\zeta'\|_\infty\|\beta'\|_\infty. \tag{9}$$

Using Theorem 3.1 for $\zeta \in C^4[0,1]$ we obtain

$$\left|m\left[\mathcal{D}_{m}^{*(1/m)}(\zeta; \varkappa) - \zeta(\varkappa)\right] - \left(\frac{1}{2}X'\zeta'(\varkappa) + X\zeta''(\varkappa)\right)\right| \leq C\frac{1}{m}\left(\|\zeta'\|_\infty + \|\zeta''\|_\infty + \|\zeta'''\|_\infty + \|\zeta^{(4)}\|_\infty\right).$$

But, for $\zeta \in C^{m}[a,b]$, $m \in \mathbb{N}$ one has (see [12, Remark 2.15])

$$\max_{0 \leq k \leq m}\left\{\|\zeta^{(k)}\|\right\} \leq C\max\left\{\|\zeta\|_\infty, \|\zeta^{(m)}\|_\infty\right\}.$$

Therefore,

$$\left|\mathbb{m}\left[\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa)-\zeta(\varkappa)\right]-\left(\frac{1}{2}X'\zeta'(\varkappa)+X\zeta''(\varkappa)\right)\right|\leq\frac{C}{\mathbb{m}}\max\left\{\|\zeta'\|_{\infty},\|\zeta^{(4)}\|_{\infty}\right\}\tag{10}$$

For $u,v\in C^4[0,1]$, employing the identical decomposition as in the proof of Theorem 5.1, along with the relation (10) and Theorem 2.2, yields:

$$\begin{aligned}|E_{\mathbb{m}}(u,v;\varkappa)|\leq&\left|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(uv;\varkappa)-(uv)(\varkappa)-\frac{1}{\mathbb{m}}\left(\frac{X'(uv)'}{2}+X(uv)''\right)\right|\\&+|u(\varkappa)|\left|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}\mathbb{m}(v;\varkappa)-v(\varkappa)-\frac{1}{\mathbb{m}}\left(\frac{X'(v)'}{2}+X(v)''\right)\right|\\&+|v(\varkappa)|\left|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}\mathbb{m}(u;\varkappa)-u(\varkappa)-\frac{1}{\mathbb{m}}\left(\frac{X'(u)'}{2}+X(u)''\right)\right|\\&+\left|\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(u;\varkappa)-u(\varkappa)\right|\left|v(\varkappa)-\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(v;\varkappa)\right|\\&\leq\frac{C}{\mathbb{m}^2}\max\left\{\|v'\|_{\infty},\|v^{(4)}\|_{\infty}\right\}\max\left\{\|u'\|_{\infty},\|u^{(4)}\|_{\infty}\right\}.\end{aligned}\tag{11}$$

By utilizing the relationships (7), (9), and (11), we derive:

$$\begin{aligned}|E_{\mathbb{m}}(\zeta,\beta;\varkappa)|\leq&\frac{4}{\mathbb{m}}\left[\varkappa(1-\varkappa)+\frac{1}{24(\mathbb{m}+1)}\right]\{\|(\zeta-u)'\|_{\infty}\|(\beta-v)'\|_{\infty}+\|u'\|_{\infty}\|(\beta-v)'\|_{\infty}\\&+\|(\zeta-u)'\|_{\infty}\|v'\|_{\infty}\}+\frac{C}{\mathbb{m}^2}\max\left\{\|u'\|_{\infty},\|u^{(4)}\|_{\infty}\right\}\max\left\{\|v'\|_{\infty},\|v^{(4)}\|_{\infty}\right\}.\end{aligned}$$

Utilizing [13, Lemma 3.1] with the specific values $r=1$, $s=2$, $\zeta_{h,3}=u$, and $g_{h,3}=v$, one can conclude that for all $h\in(0,1]$ and any $\mathbb{m}\in\mathbb{N}$, the subsequent outcome ensues:

$$\begin{aligned}|E_{\mathbb{m}}(\zeta,\beta;\varkappa)|\leq&\frac{C}{\mathbb{m}}\left\{\omega_3(\zeta',h)\omega_3(\beta',h)+\frac{1}{h}\omega_1(\zeta,h)\omega_3(\beta',h)+\frac{1}{h}\omega_1(\beta,h)\omega_3(\zeta',h)\right\}\\&+\frac{C}{\mathbb{m}^2}\max\left\{\frac{1}{h}\omega_1(\zeta,h),\frac{1}{h^3}\omega_3(\zeta',h)\right\}\cdot\max\left\{\frac{1}{h}\omega_1(\beta,h),\frac{1}{h^3}\omega_3(\beta',h)\right\}\\&\leq\frac{C}{\mathbb{m}}\{\omega_3(\zeta',h)\omega_3(\beta',h)+\|\zeta'\|_{\infty}\omega_3(\beta',h)+\|\beta'\|_{\infty}\omega_3(\zeta',h)\}\\&+\frac{C}{\mathbb{m}^2}\max\left\{\|\zeta'\|_{\infty},\frac{1}{h^3}\omega_3(\zeta',h)\right\}\cdot\max\left\{\|\beta'\|_{\infty},\frac{1}{h^3}\omega_3(\beta',h)\right\}.\end{aligned}$$

Choosing $h=\mathbb{m}^{-\frac{1}{6}}$, we obtain

$$\begin{aligned}|E_{\mathbb{m}}(\zeta,\beta;\varkappa)|\leq&\frac{C}{\mathbb{m}}\left\{\omega_3\left(\zeta',\mathbb{m}^{-\frac{1}{6}}\right)\omega_3\left(\beta',\mathbb{m}^{-\frac{1}{6}}\right)\right.\\&+\|\zeta'\|_{\infty}\omega_3\left(\beta',\mathbb{m}^{-\frac{1}{6}}\right)+\|\beta'\|_{\infty}\omega_3\left(\zeta',\mathbb{m}^{-\frac{1}{6}}\right)\\&\left.+\max\left\{\frac{\|\zeta'\|_{\infty}}{\mathbb{m}^{\frac{1}{2}}},\omega_3\left(\zeta',\mathbb{m}^{-\frac{1}{6}}\right)\right\}\max\left\{\frac{\|\beta'\|_{\infty}}{\mathbb{m}^{\frac{1}{2}}},\omega_3\left(\beta',\mathbb{m}^{-\frac{1}{6}}\right)\right\}\right\}.\end{aligned}$$

□

6. Numerical Examples

Example 1 Let's consider the function $\zeta(\varkappa)=x\cos(2\pi\varkappa)$ (blue), and $\mathbb{m}$ takes on the values of 20 (green), 40 (black) and 60 (red). To calculate the convergence of the operator

$\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)$, we need to find the resulting values of $\mathcal{D}_{20}^{*(1/20)}(\zeta;\varkappa)$, $\mathcal{D}_{40}^{*(1/40)}(\zeta;\varkappa)$ and $\mathcal{D}_{60}^{*(1/60)}(\zeta;\varkappa)$ and plot them on a graph to observe the convergence (see Fig. 1).

In Fig. 2, we consider the function $\zeta(\varkappa) = e^{\varkappa/2} sin(2\pi\varkappa)$ (blue) over the interval [0,1]. We want to approximate $\zeta(\varkappa)$ using the operators $\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)$ with the same values of m as mentioned above.

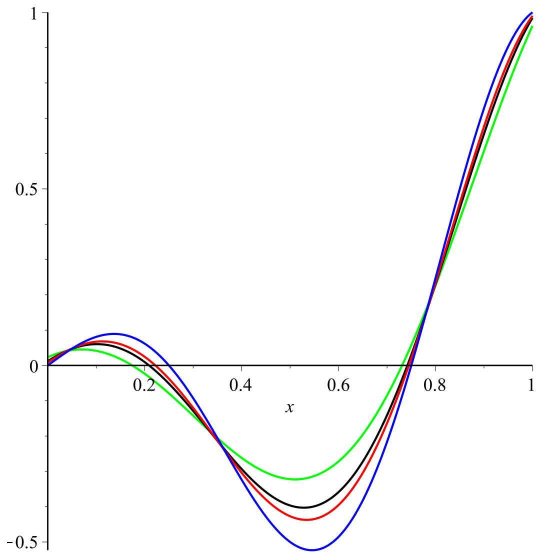


FIGURE 1. Convergence of $\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)$ to the function $\zeta = \varkappa\cos(2\pi\varkappa)$

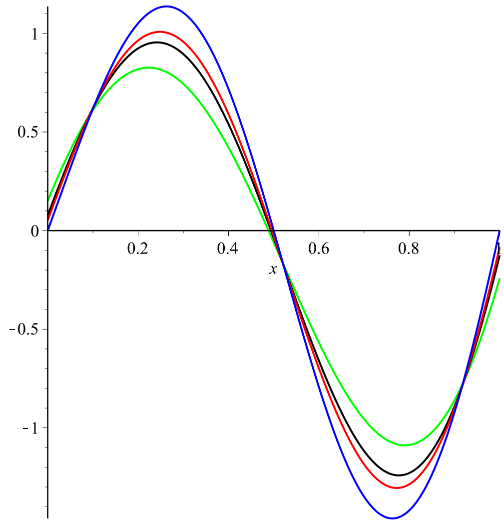
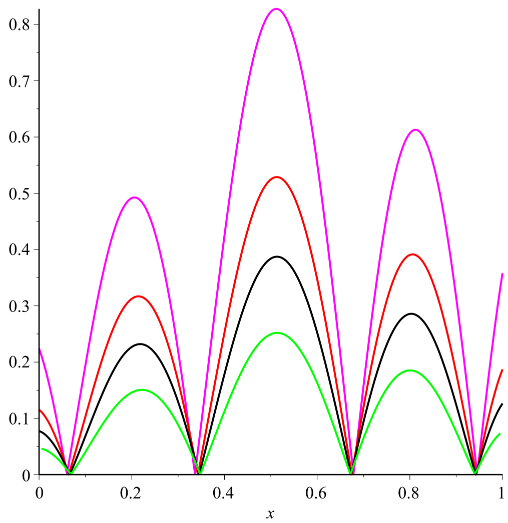


FIGURE 2. Convergence of $\mathcal{D}_{\mathrm{m}}^{*(1/\mathrm{m})}(\zeta;\varkappa)$ to the function $\zeta = e^{\varkappa/2} sin(2\pi\varkappa)$

Example 2 The error of approximation of the operators $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa)$ for the function $\zeta(\varkappa) = e^{\varkappa/2}sin(3\pi\varkappa)$ for the parameter values of $\mathbb{m} = 20$(magenta), $\mathbb{m} = 40$(red), $\mathbb{m} = 60$(black) and $\mathbb{m} = 100$(green) display in Fig 3.

In Fig 4. the error of approximation of the operators for these parametric values given above and $\zeta(\varkappa) = e^{\varkappa/2}\varkappa^3 cos(2\pi\varkappa)$



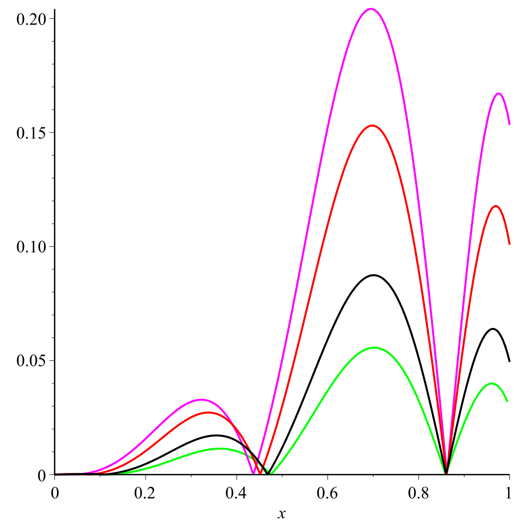


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FIGURE 3. Error Estimation for $\zeta(\varkappa) = e^{\varkappa/2}sin(3\pi\varkappa)$

FIGURE 4. Error Estimation for $\zeta(\varkappa) = e^{\varkappa/2}\varkappa^3 cos(2\pi\varkappa)$

In the following Table 1, we compute the error of approximation of our operators $\mathcal{D}_{\mathbb{m}}^{*(1/\mathbb{m})}(\zeta;\varkappa)$ for $\mathbb{m} = 10, 12, 15$ and choosing $\zeta(x) = e^{\varkappa/2}sin(3\pi\varkappa)$. It is clear that as the value of m increase, the error in the approximation decreases.
In Table 2 the error of approximation of the operators for these parametric values given above and $\zeta(\varkappa) = e^{\varkappa/2}\varkappa^3 cos(2\pi\varkappa)$

Table 1
Error of Approximation

x	$\mathbb{m} = 20$	$\mathbb{m} = 40$	$\mathbb{m} = 60$	$\mathbb{m} = 100$
0.1	0.1945025870	0.1033129686	0.0684706734	0.0402513756
0.2	0.4915419726	0.3125567724	0.2269116751	0.1459553959
0.3	0.2196810664	0.1662171768	0.1293796904	0.0884946036
0.4	0.4245874718	0.2522668128	0.1785494834	0.1123953306
0.5	0.8226269746	0.5246252843	0.3839043061	0.249519885
0.6	0.5534530758	0.3477578682	0.2526809541	0.1631100470
0.7	0.1572114733	0.1195320347	0.0935918499	0.0644665227
0.8	0.6067663002	0.390426022	0.285613593	0.185214316
0.9	0.3147213901	0.180371304	0.124308016	0.076008683

Table 2

Error of Approximation

x	$m = 20$	$m = 40$	$m = 60$	$m = 100$
0.1	0.002083614655	0.0001687009370	0.0001146988573	0.0001777434717
0.2	0.01574719572	0.007473866316	0.004659942057	0.002571991109
0.3	0.03193337888	0.02031423066	0.01448561258	0.009106804121
0.4	0.01857838831	0.01824155074	0.01478340294	0.01029813713
0.5	0.0503439904	0.0203904410	0.0117708627	0.0059396078
0.6	0.1522033031	0.0854503366	0.0593011557	0.0367544846
0.7	0.2040420394	0.1224020242	0.08743306548	0.05564851887
0.8	0.1190841417	0.0716987646	0.0511491561	0.0324780937
0.9	0.0782548446	0.0478337885	0.0347606458	0.0225255760

7. Conclusions

We investigated the Lupaş-Kantorovich operators with the Pólya distribution. We studied an interesting results the quantitative Voronovskaya-type outcome concerning second moduli of continuity, the Chebyshev-Grüss inequality, and two Grüss-Voronovskaya theorems applicable to the operators (2). In the end, we display graphical representations illustrating the convergence of operators. As we increase the value of m, we observed a reduction in error, bringing us closer to approximating our function.

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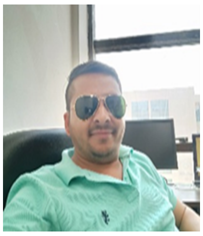

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