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IDEAL CONVERGENCE IN FUZZY METRIC SPACES

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Abstract. In this paper, the concepts of $\mathcal{K}-$convergence, $\mathcal{K}$-Cauchy sequences, $\mathcal{K}^*$-convergence, and $\mathcal{K}^*$-Cauchy sequences in fuzzy metric spaces is proposed. Also, a few fundamental properties of these concepts are investigated. Then, the concepts of $\mathcal{K}$-limit and $\mathcal{K}$-cluster points of a sequence in these spaces is defined. Afterwards, some of their basic properties is examined. Finally, the need for further research is discussed.

Keywords: $\mathcal{K}$-convergence, $\mathcal{K}$-Cauchy sequences, $\mathcal{K}$-limit points, $\mathcal{K}$-cluster points, fuzzy metric spaces

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1. Introduction

Kostyrko et al. [11] have proposed the concept of ideal convergence ($\mathcal{K}-$convergence) in 2000, which is a generalization of the concepts of Cauchy and statistical convergence [6], considering the ideal $\mathcal{K}$ of some subset of positive integers. Also, they have studied the concept of convergence closely related to $\mathcal{K}-$convergence ($\mathcal{K}^*-$convergence).

Definition 1.1. [11] *A class of subsets $\mathcal{K} \subseteq P(\mathbb{E})$, where $\mathbb{E}$ is a non-empty set, is called an ideal on $\mathbb{E}$, if*

(1) $\emptyset \in \mathcal{K}$,
(2) $R, U \in \mathcal{K} \Rightarrow R \cup U \in \mathcal{K}$,
(3) $(R \in \mathcal{K} \wedge U \subseteq R) \Rightarrow U \in \mathcal{K}$.

Definition 1.2. [11] *$\mathcal{K}$, is an ideal on $\mathbb{E}$, is said to be a non-trivial ideal such that $\mathbb{E} \notin \mathcal{K}$ and $\mathcal{K} \neq \emptyset$. In addition, the ideal $\mathcal{K}$, which is a non-trivial ideal, is called an admissible ideal, if $\{\{a\} : a \in \mathbb{E}\} \subseteq \mathcal{K}$.*

Definition 1.3. [11] *A class of subsets $\emptyset \neq \mathcal{F} \subseteq P(\mathbb{E})$, where $\mathbb{E}$ is a non-empty set, is referred to as a filter on $\mathbb{E}$, if*

(1) $\emptyset \notin \mathcal{F}$,

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(2) $R, U \in \mathcal{F} \Rightarrow R \cap U \in \mathcal{F}$,
(3) $(R \in \mathcal{F} \wedge R \subseteq U) \Rightarrow U \in \mathcal{F}$.

Remark 1.1. [11] *The filter $\mathcal{F}(\mathcal{K}) = \{\mathbb{E} \setminus S : S \in \mathcal{K}\}$ on $\mathbb{E}$ is said to be the associated filter with $\mathcal{K}$.*

In the following part of the study, the ideal $\mathcal{K}$ is the ideal that satisfies the condition $\mathcal{K} \subseteq P(\mathbb{N})$.

Definition 1.4. [11] *Let $(\mathbb{E}, d)$ be a metric space, $\mathcal{K}$ be a non-trivial ideal, (a_k) be a sequence in $\mathbb{E}$, and $a_0 \in \mathbb{E}$. Then, a sequence (a_k) is said to be ideal convergent ($\mathcal{K}-$convergent) to a_0, if, for all $\eta > 0$,*

$$A_\eta = \{k \in \mathbb{N} : d(a_k, a_0) \geq \eta\} \in \mathcal{K},$$

and is denoted by $\mathcal{K} - \lim_{k\to\infty} a_k = a_0$ or $a_k \xrightarrow{\mathcal{K}} a_0$ as $k \to \infty$.

Remark 1.2. *In Definition 1.4, convergence, in the Cauchy sense, means $\mathcal{K}-$convergence if $\mathcal{K}$ is an admissible ideal.*

Definition 1.5. [11] *Let $(\mathbb{E}, d)$ be a metric space, $\mathcal{K}$ be a non-trivial ideal, (a_k) be a sequence in $\mathbb{E}$, and $a_0 \in \mathbb{E}$. Then, a sequence (a_k) is called $\mathcal{K}^*-$convergent to a_0, if there exists a set*

$$T = \{t_1 < t_2 < \ldots < t_n < \ldots\} \in \mathcal{F}(\mathcal{K})$$

such that

$$\lim_{n\to\infty} d(a_{t_n}, a_0) = 0.$$

Besides, Dems [5] has studied the concept of ideal Cauchy ($\mathcal{K}-$Cauchy) sequences by extending the concept of Cauchy sequences to ideals. Afterwards, Nabiyev et al. [16] have suggested $\mathcal{K}^*-$Cauchy sequences. Also, they have proposed these sequences relation to $\mathcal{K}-$Cauchy sequences.

Definition 1.6. [16] *Let $(\mathbb{E}, d)$ be a metric space, $\mathcal{K}$ be an admissible ideal, and (a_k) be a sequence in $\mathbb{E}$. Then, a sequence (a_k) is referred to as an ideal Cauchy ($\mathcal{K}-$Cauchy) sequence in $\mathbb{E}$, if, for all $\eta > 0$, there exists an $K = K(\eta)$ such that*

$$A_\eta = \{k \in \mathbb{N} : d(a_k, a_K) \geq \eta\} \in \mathcal{K}.$$

Definition 1.7. [16] *Let $(\mathbb{E}, d)$ be a metric space, $\mathcal{K}$ be an admissible ideal, and (a_k) be a sequence in $\mathbb{E}$. Then, a sequence (a_k) is said to be a $\mathcal{K}^*-$Cauchy sequence in $\mathbb{E}$, if there exists a set*

$$T = \{t_1 < t_2 < \ldots < t_n < \ldots\} \in \mathcal{F}(\mathcal{K})$$

such that

$$\lim_{n,p\to\infty} d(a_{t_n}, a_{t_p}) = 0.$$

Fuzzy set theory was defined by Zadeh [20] in 1965. Since then, it has been applied to a variety of domains, including artificial intelligence, decision making, and image analysis, where uncertainty is handled precisely through degrees of membership, providing a flexible approach to problem solving. In particular, the concept of fuzzy metric spaces (FMSs) was first explored by Kramosil and Michalek [12] as well as by Kaleva and Seikkala [10]. George and Veeramani [9] have redefined the original definition of fuzzy metrics to construct Hausdorff topology. Now, we recall some basic definitions.

Definition 1.8. [10] *A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is referred to as a triangular norm (t-norm), if, for all $a_1, a_2, a_3 \in [0,1]$,*

(1) $a_1\diamondsuit a_2 = a_2\diamondsuit a_1$,
(2) $a_1\diamondsuit(a_2\diamondsuit a_3) = (a_1\diamondsuit a_2)\diamondsuit a_3$,
(3) $a_1\diamondsuit a_2 \leq a_1\diamondsuit a_3$ *whenever* $a_2 \leq a_3$,
(4) $a_1\diamondsuit 1 = 1\diamondsuit a_1 = a_1$

are satisfied.

Example 1.1. [20] *According to the Definition 1.8, the following operators are a t-norm:*

(1) $a\diamondsuit b = ab$,
(2) $a\diamondsuit b = \min\{a, b\}$.

Definition 1.9. [9] *Let* $\mathbb{E}$ *be an arbitrary set,* $\diamondsuit$ *be a continuous t-norm, and* $\mho$ *be a fuzzy set on* $\mathbb{E}^2 \times (0,\infty)$*. Then, an FMS is the triplet* $(\mathbb{E},\mho,\diamondsuit)$*, if the following conditions are hold, for all an* $s, v > 0$ *and* $a_1, a_2, a_3 \in \mathbb{E}$,

(1) $\mho(a_1, a_2, 0) > 0$,
(2) $\mho(a_1, a_2, s) = 1 \Leftrightarrow a_1 = a_2$,
(3) $\mho(a_1, a_2, s) = \mho(a_2, a_1, s)$,
(4) $\mho(a_1, a_3, s + v) \geq \mho(a_1, a_2, s)\diamondsuit\mho(a_2, a_3, v)$,
(5) *The function* $(\mho)_{a_1a_2} : (0,\infty) \to [0, 1]$*, defined by* $(\mho)_{a_1a_2} = \mho(a_1, a_2, s)$ *for all* $s \in (0, 1)$*, is continuous.*

Example 1.2. [12] *Let* $x\diamondsuit y = xy$ *for* $x, y \in [0, 1]$*. Then, for all* $s > 0$ *and* $a, b \in \mathbb{R}$*, the triplet* $(\mathbb{R},\mho,\diamondsuit)$ *is an FMS where*

$$\mho(a, b, s) = \left(e^{\frac{|a-b|}{s}}\right)^{-1}.$$

Definition 1.10. [9] *Let* $(\mathbb{E},\mho,\diamondsuit)$ *be an FMS and* $a \in \mathbb{E}$*. Then, for all* $\eta \in (0, 1)$ *and* $s > 0$*, the set*

$$D(a, \eta, s) = \{b \in \mathbb{E} : \mho(a, b, s) > 1 - \eta\}$$

is said to be the $\eta-$*disc about* a.

Definition 1.11. [10] *Let* $(\mathbb{E},\mho,\diamondsuit)$ *be an FMS,* (a_k) *be a sequence in* $\mathbb{E}$*, and* $a_0 \in \mathbb{E}$*. Then, a sequence* (a_k) *is referred to as convergent to* a_0*, if, for all* $\eta \in (0, 1)$ *and* $s > 0$*, there exists* $k_\eta \in \mathbb{N}$ *such that* $k \geq k_\eta$ *implies*

$$\mho(a_k, a_0, s) > 1 - \eta$$

or equivalently

$$\lim_{k\to\infty} \mho(a_k, a_0, s) = 1,$$

and is denoted by $\mho - \lim\limits_{k\to\infty} a_k = a_0$ *or* $a_k \xrightarrow{\mho} a_0$ *as* $k \to \infty$.

Definition 1.12. [10] *Let* $(\mathbb{E},\mho,\diamondsuit)$ *be an FMS and* (a_k) *be a sequence in* $\mathbb{E}$*. Then, a sequence* (a_k) *is said to be Cauchy sequence in* $\mathbb{E}$*, if, for all* $\eta \in (0, 1)$ *and* $s > 0$*, there exists* $k_\eta \in \mathbb{N}$ *such that* $k, K \geq k_\eta$ *implies*

$$\mho(a_k, a_K, s) > 1 - \eta$$

or equivalently

$$\lim_{k,K\to\infty} \mho(a_k, a_K, s) = 1.$$

Recently, Mihet [14] examined the concept of pointwise convergence, which is less strong than Cauchy convergence. Furthermore, the concept of $s-$ convergence has been proposed by Gregori et al. [7]. Morillas and Sapena [15] have introduced the concept of standard convergence. The concept of strong convergence, which is stronger than Cauchy convergence, was introduced by Gregori and Miňana [8]. Li et al. [13] introduced the concepts of statistical convergence and statistical Cauchy sequences in FMSs and analyzed their fundamental properties.

The concept of convergence of sequences has always been a topic of interest in mathematics, and much of the work in this area in the last 20 years has focused on the study of fuzzy versions of fundamental concepts and properties of topology and functional analysis. As a result of this motivation, researchers have studied ideal convergence and similar types of convergence in various spaces [1, 2, 3, 4, 17, 18, 19]. With the same motivation, we define the notion of ideal convergence, which is more general than statistical convergence, in fuzzy metric spaces. The ideal convergence in FMSs is defined in this article, as a result of previous research on this topic.

2. $\mho(\mathcal{K})$-convergence and $\mho(\mathcal{K})$-Cauchy sequence

In this section, we propose definitions and theorems for $\mathcal{K}-$convergence and $\mathcal{K}-$Cauchy sequences in FMSs. Furthermore, we analyze some important characteristics. It must be noted that in the remainder of the paper, unless otherwise stated, $\mathbb{E}$ is an FMS and $\mathcal{K} \subseteq P(\mathbb{N})$. Also, the condition (AP) will be referred to in Definition 3.3 in [11].

Definition 2.1. *Let $\mathcal{K}$ be a non-trivial ideal. Then, a sequence (a_k) in $\mathbb{E}$ is referred to as $\mho(\mathcal{K})-$convergent to a_0, if, for all $\eta \in (0,1)$ and $s > 0$,*

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \eta\} \in \mathcal{K},$$

and is denoted by $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$ or $a_k \xrightarrow{\mho(\mathcal{K})} a_0$ as $k \to \infty$. The element a_0 is said to be the $\mathcal{K}-$limit of the sequence (a_k).

Example 2.1.

(1) *If choose $\mathcal{K} = \mathcal{K}_f = \{T \subseteq \mathbb{N} : T$ is a finite set$\}$, then $\mathcal{K}-$ convergence is the same as Cauchy convergence in FMSs.*

(2) *If choose $\mathcal{K} = \mathcal{K}_\delta = \{T \subseteq \mathbb{N} : \delta(T) = 0\}$, then $\mathcal{K}-$ convergence is the same as statistical convergence* [6].

Example 2.2. *Let the class of T_k, which is an infinite set for all $k \in \mathbb{N}$, be a decomposition in $\mathbb{N}$. Then, K is an admissible ideal in $\mathbb{N}$, where*

$$K = \{A \subset \mathbb{N} : k \in \mathbb{N} \ni A \subseteq T_1 \cup T_2 \cup \cdots \cup T_k\}.$$

Remark 2.1. *The Cauchy convergence in FMSs means $\mathcal{K}$-convergence, if $\mathcal{K}$ is an admissible ideal.*

The following theorem presents, well-known in Cauchy convergence, which gives whether the following expressions satisfy at ideal convergence:
I. Every constant sequence $\{a_0, a_0, ..., a_0, ...\}$ is convergent to a_0.
II. The limit of converge sequence's can be determined by uniquely.
III. Each subsequence of the converge sequence converges to the same limit.

Theorem 2.1. *Let (a_k) is a sequence in $\mathbb{E}$ and $a_0, a_1 \in \mathbb{E}$.*

(1) *Every constant sequence $\{a_0, a_0, ..., a_0, ...\}$ is $\mho(\mathcal{K})$-convergent to a_0.*

(2) *If $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$ and $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_1$ are provided, then $a_0 = a_1$.*

(3) *If $\mathcal{K}$ contains an infinite set and $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$ is satisfied, then sequence (a_k) has a subsequence (b_k) that does not $\mho(\mathcal{K})$-converge to the element a_0.*

Proof.

(1) It is obvious that every constant sequence $\{a_0, a_0, ..., a_0, ...\}$ is $\mho(\mathcal{K})$-convergent to a_0.

(2) Suppose that $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$, $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_1$, $a_0 \neq a_1$. Choose $s > 0$ and $\eta = \frac{1}{k}$, $(k = 2, 3, \ldots)$. Hence, from Remark 1.1 and assumption, the sets $H_1 = \mathbb{N} \setminus A = \{k \in \mathbb{N} : \mho(a_k, a_0, s) > 1 - \eta\}, H_2 = \mathbb{N} \setminus B = \{k \in \mathbb{N} : \mho(a_k, a_1, s) > 1 - \eta\} \in \mathcal{F}(\mathcal{K})$, and so $H_1 \cap H_2 \in \mathcal{F}(\mathcal{K})$, too. Therefore, there is an $m \in \mathbb{N}$ such that

$$\mho(a_m, a_0, s) > 1 - \eta \quad \text{and} \quad \mho(a_m, a_1, s) > 1 - \eta.$$

From this $\mho(a_0, a_1, s) = 1$ which is a contradiction to $a_0 \neq a_1$.

(3) Assume that $K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq \mathbb{N}$ is an infinite set and it is contained in $\mathcal{K}$. Put

$$\mathbb{N} \setminus K = \{t_1 < t_2 < \ldots < t_n < \ldots\}.$$

Since $\mathcal{K}$ is an admissible ideal, $\mathbb{N}$ can not belong to $\mathcal{K}$, and so $\mathbb{N} \setminus T$ is infinite. Define the sequence (a_k) as follows

$$a_k := \begin{cases} a_0, & \text{if } k \in K, \\ a_1, & \text{if } k \in \mathbb{N} \setminus K. \end{cases}$$

Obviously $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$. In addition, the subsequence (a_{t_n}) of (a_k) is stationary and therefore $\mho(\mathcal{K}) - \lim_{n\to\infty} a_{t_n} = a_1$ (see proposition (1)). Hence, sequence (a_k) has a subsequence that does not $\mho(\mathcal{K})$-converge to the element a_0.

□

Definition 2.2. *Let $\mathcal{K}$ be an admissible ideal. Then, a sequence (a_k) in $\mathbb{E}$ is referred to as $\mho(\mathcal{K})-$Cauchy sequence in $\mathbb{E}$, if, for all $\eta \in (0, 1)$ and $s > 0$, there exists a $K \in \mathbb{N}$ such that*

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(a_k, a_K, s) \leq 1 - \eta\} \in \mathcal{K}.$$

Theorem 2.2. *Let $\mathcal{K}$ is an admissible ideal. Then, every $\mho(\mathcal{K})-$convergent sequence in $\mathbb{E}$ is an $\mho(\mathcal{K})-$Cauchy sequence in $\mathbb{E}$.*

Proof. Let $a_0 \in \mathbb{E}$ and $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$. Then, for all $\eta \in (0, 1)$ and $s > 0$,

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \eta\} \in \mathcal{K}.$$

Given the definition of admissible ideal, there exists a $K \in \mathbb{N}$ such that $K \notin A_{s,\eta}$.

Let

$$B = \{k \in \mathbb{N} : \mho(a_k, a_K, s) \leq \delta(\eta)\}$$

such that $\delta(\eta) = (1 - \eta)\diamondsuit(1 - \eta)$ and $K \notin A_{s,\eta}$. Considering the following inequality

$$\mho(a_k, a_K, s) \geq \mho\left(a_k, a_0, \frac{s}{2}\right) \diamondsuit \mho\left(a_K, a_0, \frac{s}{2}\right).$$

Therefore, if $k \in B$, then

$$\delta(\eta) = (1 - \eta)\diamondsuit(1 - \eta) \geq \mho\left(a_k, a_0, \frac{s}{2}\right) \diamondsuit \mho\left(a_K, a_0, \frac{s}{2}\right).$$

Moreover, $\mho(a_K, a_0, s) > 1 - \eta$ satisfies because $K \notin A_{s,\eta}$. Therefore, $\mho(a_k, a_0, s) \leq 1 - \eta$, then $k \in A_{s,\eta}$. In this case, $B \subseteq A_{s,\eta} \in \mathcal{K}$ for all $\eta \in (0, 1)$ and $s > 0$. Hence, (a_k) is a $\mho(\mathcal{K})-$Cauchy sequence. □

Definition 2.3. *Let $\mathcal{K}$ be an admissible ideal. Then, a sequence (a_k) in $\mathbb{E}$ is called $\mho(\mathcal{K}^*)$-convergent to $a_0 \in \mathbb{E}$, if there exists a set*

$$T = \{t_1 < t_2 < \ldots < t_n < \ldots\} \in \mathcal{F}(\mathcal{K})$$

such that

$$\lim_{n\to\infty} \mho(a_{t_n}, a_0, s) = 1, \tag{1}$$

and is denoted by $\mho(\mathcal{K}^*) - \lim_{k\to\infty} a_k = a_0$ or $a_k \xrightarrow{\mho(\mathcal{K}^*)} a_0$ as $k \to \infty$.

Theorem 2.3. *Let $\mathcal{K}$ be an admissible ideal. Then, $\mho(\mathcal{K}^*) - \lim_{k\to\infty} a_k = a_0$ implies that $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$.*

Proof. Let $\mho(\mathcal{K}^*) - \lim_{k\to\infty} a_k = a_0$. Hence, there is a set $T = \mathbb{N} \setminus K = \{t_1 < t_2 < \ldots < t_n < \ldots\} \in \mathcal{F}(\mathcal{K})$ such that (1) holds. Then, $K \in \mathcal{K}$.

Let $\eta \in (0,1)$ and $s > 0$. By (1), there exists a $k_\eta \in \mathbb{N}$ such that $\mho(a_k, a_0, s) > 1 - \eta$ for $k > k_\eta$. Put $A_{s,\eta} = \{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \eta\}$. Then,

$$A_{s,\eta} \subseteq K \cup \{t_1, t_2, \ldots, t_{k_\eta}\}. \tag{2}$$

Since $K \in \mathcal{K}$ and $\mathcal{K}$ is an admissible ideal,

$$K \cup \{t_1, t_2, \ldots, t_{k_\eta}\} \in \mathcal{K}$$

and hence $A_{s,\eta} \in \mathcal{K}$. □

The following Example 2.3 states that the converse of Theorem 2.3 does not always hold.

Example 2.3. *Let $(\mathbb{R}, |.|)$ be a metric space and $x \diamondsuit y = xy$ for all $x, y \in [0,1]$. Then, for all $s > 0$ and $a, b \in \mathbb{R}$, the triplet $(\mathbb{R}, \mho, \diamondsuit)$ is an FMS where*

$$\mho(a,b,s) = \frac{s}{s + |a-b|}.$$

Put the FMS $(\mathbb{R}, \mho, \diamondsuit)$ and $\mathcal{K} = K$ (see Example 2.2). Assume that a_0 is accumulation point of $\mathbb{R}$. Then, there exists a sequence (a_k) in $\mathbb{R}$ such that $\lim_{k\to\infty} \mho(a_k, a_0, s) = 1$. Define

$$b_k := \begin{cases} a_j, & \text{if } k \in T_j, \\ a_0, & \text{if } k \notin T_j. \end{cases} \quad \text{for } j = 1, 2, \ldots$$

Choose $s > 0$ and $n \in \mathbb{N}$ such that $\frac{1}{n} < \eta$ for $\eta \in (0,1)$. Thus,

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(b_k, a_0, s) \leq 1 - \eta\} \subseteq \left\{k \in \mathbb{N} : \mho(b_k, a_0, s) \leq 1 - \frac{1}{n}\right\} \subseteq \bigcup_{l=1}^{n} T_l.$$

Hence, according to the concept of ideal $A_{s,\eta} \in \mathcal{K}$ and so $\mho(\mathcal{K}) - \lim_{k\to\infty} b_k = a_0$. Now, assume that $\mho(\mathcal{K}^*) - \lim_{k\to\infty} b_k = a_0$. Then, there exists a set $T = \{t_1 < t_2 < \ldots t_k\} \in \mathcal{K}$ such that

$$\lim_{\substack{k\to\infty \\ t_k \in T}} \mho(b_{t_k}, a_0, s) = 1.$$

Using of the concept of ideal $\mathcal{K}$, there exists a $l \in \mathbb{N}$ such that

$$T \subseteq T_1 \cup T_2 \cup \ldots \cup T_l.$$

But by notation used in proof of Theorem 2.3 and $T_{l+1} \subseteq \mathbb{N} \setminus T$, $b_{t_k} = \frac{1}{l+1}$ for infinitely many of k's. As result, $\lim_{k\to\infty} \mho(b_{t_k}, a_0, s) = 1$ can not be true.

Theorem 2.4. *Let $\mathcal{K}$ be an admissible ideal, (a_k) is a sequence in $\mathbb{E}$, and $a_0 \in \mathbb{E}$.*

(1) *If the $\mathcal{K}$ ideal has the condition (AP), then $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$ implies $\mho(\mathcal{K}^*) - \lim_{k\to\infty} a_k = a_0$.*
(2) *If $\mathbb{E}$ has at least one accumulation point and $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$ implies $\mho(\mathcal{K}^*) - \lim_{k\to\infty} a_k = a_0$, then $\mathcal{K}$ has the property (AP).*

Proof.

(1) Let $a_k \xrightarrow{\mho(\mathcal{K})} a_0$ and $\mathcal{K}$ has the condition (AP). Hence, for all $\eta \in (0,1)$ and $s > 0$, the set

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \eta\} \in \mathcal{K}.$$

Put

$$T_1 = \left\{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq \frac{1}{2}\right\},$$

$$T_l = \left\{k \in \mathbb{N} : \frac{l-1}{l} < \mho(a_k, a_0, s) \leq \frac{l}{l+1}\right\} \text{ for } l \geq 2.$$

Obviously $T_l \cap T_n = \emptyset$ for $l \neq n$ and $T_l \in \mathcal{K}$ $(l = 1, 2, \ldots)$. Since $\mathcal{K}$ satisfies (AP), there are sets $S_n \subseteq \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $T_n \Delta S_n$ is a finite set and $S = \bigcup_{n=1}^{\infty} S_n \in \mathcal{K}$. It suffices to prove that

$$\lim_{\substack{k\to\infty \\ k\in T}} \mho(a_k, a_0, s) = 1, \tag{3}$$

where $T = \mathbb{N} \setminus S$. Let $\mu \in (0,1)$ and $s > 0$. Choose a $m \in \mathbb{N}$ such that $\frac{1}{m} < \mu$. Then,

$$\{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \mu\} \subseteq \bigcup_{n=1}^{m+1} T_n.$$

From the additivity of $\mathcal{K}$, $\bigcup_{n=1}^{m+1} T_n \in \mathcal{K}$. Since, for all $n \in \mathbb{N}$, $T_n \Delta S_n$ exists a finite set, there is a $k_\mu \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{m+1} S_n \cap (k_\mu, \infty) = \bigcup_{n=1}^{m+1} T_n \cap (k_\mu, \infty).$$

If $k \notin S$ for $k > k_\mu$, then $k \notin \bigcup_{n=1}^{m+1} S_n$. Hence, $k \notin \bigcup_{n=1}^{m+1} T_n$. Since

$$\{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \mu\} \subseteq \bigcup_{n=1}^{m+1} T_n,$$

$k \in \{k \in \mathbb{N} : \mho(a_k, a_0, s) > 1 - \mu\}$. Consequently, (3) hold.

(2) Suppose $a_0 \in \mathbb{E}$ is an accumulation point of $\mathbb{E}$ and $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$ implies $\mho(\mathcal{K}^*) - \lim_{k\to\infty} a_k = a_0$. Hence, there exists a sequence (a_k) of distinct elements of $\mathbb{E}$ such that $a_k \neq a_0$ for any k and $\lim_{k\to\infty} \mho(a_k, a_0, s) = 1$. Let $\{T_1, T_2, \ldots\}$ be a disjoint class of nonempty sets in $\mathcal{K}$. Define a sequence (b_k) in the following way:

$b_k = a_j$ if $k \in T_j$ and $b_k = a_0$ if $k \notin T_j$, for all j. Let $\eta \in (0,1)$ and $s > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \eta$. Then,

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(b_k, a_0, s) \leq 1 - \eta\} \subset T_1 \cup T_2 \cup \cdots \cup T_n.$$

Hence, $A_{s,\eta} \in \mathcal{K}$ and so $\mho(\mathcal{K}) - \lim_{k\to\infty} b_k = a_0$. By virtue of our assumption, $\mho(\mathcal{K}^*) - \lim_{k\to\infty} b_k = a_0$. Therefore, there exists a set $S \in \mathcal{K}$ such that $\mathbb{N} \setminus S \in \mathcal{F}(\mathcal{K})$ and

$$\lim_{\substack{n\to\infty \\ k_n \in \mathbb{N}\setminus S}} \mho(b_{k_n}, a_0, s) = 1. \tag{4}$$

Put $S_j = T_j \cap S$, for $j \in \mathbb{N}$. Then, $S_j \in \mathcal{K}$, for all $j \in \mathbb{N}$. Moreover, $\bigcup_{j=1}^{\infty} S_j = S \cap \bigcup_{j=1}^{\infty} T_j \subset S$ and thus $\bigcup_{j=1}^{\infty} S_j \in \mathcal{K}$. From (4), for all $\eta \in (0,1)$ and $s > 0$,

$$\{k_n \in \mathbb{N} : \mho(b_{k_n}, a_0, s) \leq 1 - \eta\} \subseteq \mathbb{N} \setminus S$$

has a finite number of elements. By

$$A_{s,\eta} \subseteq T_1 \cup T_2 \cup \cdots \cup T_n$$

the set $T_j \cap \mathbb{N} \setminus S$ is a finite set. Then, there exists a $k_\eta \in \mathbb{N}$ such that

$$T_j \subseteq (T_j \cap S) \cup \{n_1, n_2, \cdots, n_{k_\eta}\}.$$

Hence,

$$T_j \Delta S_j \subseteq T_j \cap (\mathbb{N} \setminus S)$$

and we have $T_j \Delta S_j$ is finite.

□

Theorem 2.5. *If $\mathbb{E}$ has no accumulation point and $\mathcal{K}$ is an admissible ideal, then $\mathcal{K}$- and $\mathcal{K}^*$-convergence are the same.*

Proof. Let $\mathbb{E}$ has no accumulation points, $\mathcal{K}$ is an admissible ideal, and $a_k \xrightarrow{\mho(\mathcal{K})} a_0$. Hence, there exists $s > 0$ and $\eta \in (0,1)$ such that

$$B(a_0, \eta, s) = \{b \in \mathbb{E} : \mho(b, a_0, s) > 1 - \eta\} = \{a_0\}.$$

Also, thanks to Theorem 2.3, it suffices to prove that $a_k \xrightarrow{\mho(\mathcal{K}^*)} a_0$ as $k \to \infty$. Now, from the assumption $\{k \in \mathbb{N} : \mho(a_k, a_0, s) \leq 1 - \eta\} \in \mathcal{K}$. Hence,

$$\{k \in \mathbb{N} : \mho(a_k, a_0, s) > 1 - \eta\} = \{k \in \mathbb{N} : a_k = a_0\} \in \mathcal{F}(\mathcal{K}),$$

and obviously $a_k \xrightarrow{\mho(\mathcal{K}^*)} a_0$. □

Theorem 2.6. *Let $\mathcal{K}$ be an ideal which satisfies the condition (AP). Then, the following situations are equivalent:*

(1) $\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0$,
(2) *There exist $b = (b_k), c = (c_k) \in \mathbb{E}$ such that $a = b + c$, $\lim_{k\to\infty} \mho(b_k, a_0, s) = 1$ and $suppc \in \mathcal{K}$, where $suppc = \{k \in \mathbb{N} : c_k \neq \theta_{\mathbb{E}}\}$.*

Proof. Assume that $\mho(\mathcal{K}) - \lim\limits_{k\to\infty} a_k = a_0$. In that case, by Theorem 2.4, there exists a set $T \in \mathcal{F}(\mathcal{K})$, $T = \{t_1 < t_2 < \ldots < t_k < \ldots\}$ such that $\lim\limits_{k\to\infty} \mho(a_{t_k}, a_0, s) = 1$. Put the sequence $b = (b_k) \in \mathbb{E}$ as

$$b_k := \begin{cases} a_k, & k \in T, \\ a_0, & k \in \mathbb{N} \setminus T. \end{cases} \tag{5}$$

It is clear that $\lim\limits_{k\to\infty} \mho(b_k, a_0, s) = 1$. Further, let $c_k = a_k - b_k$, $k \in \mathbb{N}$. $\{k \in \mathbb{N} : c_k \neq 0\} \in \mathcal{K}$ is satisfied, because

$$suppc = \{k \in \mathbb{N} : x_k \neq y_k\} \subset \mathbb{N} \setminus T \in \mathcal{K}.$$

In addition, $suppc \in \mathcal{K}$ and by (5), $a = b + c$ is wrote.

Now, let $b = (b_k)$ and $c = (c_k)$ be two sequence in $\mathbb{E}$. This sequences satisfy $a = b + c$, $\lim\limits_{k\to\infty} \mho(b_k, a_0, s) = 1$ and $suppc \in \mathcal{K}$. It must be proved that

$$\mho(\mathcal{K}) - \lim_{k\to\infty} a_k = a_0. \tag{6}$$

Assume that $T = \{t_k \in \mathbb{N} : c_{t_k} = 0\} \subset \mathbb{N}$. $T \in \mathcal{F}(\mathcal{K})$ is satisfied, because $suppc = \{t_k \in \mathbb{N} : c_{t_k} \neq 0\} \in \mathcal{K}$. Hence, $a_k = b_k$ if $k \in T$. Therefore, it is obtained that there exists a set $T = \{t_1 < t_2 < \ldots\} \in \mathcal{F}(\mathcal{K})$ such that

$$\lim_{\substack{k\to\infty \\ t_k \in T}} \mho(a_{t_k}, a_0, s) = 1.$$

By Theorem 2.4, (6) is hold. □

Definition 2.4. *Let $\mathcal{K}$ be an admissible ideal. Then, a sequence (a_k) in $\mathbb{E}$ is said to be $\mho(\mathcal{K}^*)$-Cauchy sequence in $\mathbb{E}$, if there exists a set*

$$T = \{t_1 < t_2 < \ldots < t_k < \ldots\} \in \mathcal{F}(\mathcal{K})$$

such that

$$\lim_{\substack{k,p\to\infty \\ t_k, t_p \in T}} \mho(a_{t_k}, a_{t_p}, s) = 1. \tag{7}$$

Theorem 2.7. *Let $\mathcal{K}$ be an admissible ideal. If a sequence (a_k) in $\mathbb{E}$ is a $\mho(\mathcal{K}^*)$-Cauchy sequence in $\mathbb{E}$, then it is a $\mho(\mathcal{K})$-Cauchy sequence.*

Proof. Assume that (a_k) be a $\mho(\mathcal{K}^*)$-Cauchy sequence. Hence, there exists a set

$$T = \mathbb{N} \setminus K = \{t_1 < t_2 < \ldots < t_k < \ldots\} \in \mathcal{F}(\mathcal{K})$$

such that, for all $\eta \in (0, 1)$ and $s > 0$, $t_k, t_p > k_\eta$ implies $\mho(a_{t_k}, a_{t_p}, s) > 1 - \eta$.

Choose $N = t_{k_\eta + 1}$. Then, for all $\eta \in (0, 1)$ and $s > 0$,

$$t_k > k_\eta \Rightarrow \mho(a_{t_k}, a_N, s) > 1 - \eta.$$

It must been note that $K \in \mathcal{K}$ and

$$A_{s,\eta} = \{k \in \mathbb{N} : \mho(a_k, a_N, s) \leq 1 - \eta\} \subset K \cup \{t_1 < t_2 < \ldots < t_{k_\eta}\}. \tag{8}$$

From here

$$K \cup \{t_1 < t_2 < \ldots < t_{k_\eta}\} \in \mathcal{K}.$$

Hence, the sequence (a_k) is a $\mathcal{K}-$Cauchy sequence. □

Theorem 2.8. *Let $\mathcal{K}$ be an admissible ideal. Then, $\mho(\mathcal{K})-$Cauchy sequence in $\mathbb{E}$ implies that $\mho(\mathcal{K}^*)-$Cauchy sequence in $\mathbb{E}$ if and only if the ideal $\mathcal{K}$ has the condition (AP).*

Proof. Let a sequence (a_k) be a $\mho(\mathcal{K})-$Cauchy sequence in $\mathbb{E}$ and the ideal $\mathcal{K}$ has the condition (AP). Hence, there exists a K such that, for all $\eta \in (0,1)$ and $s > 0$,

$$\{k \in \mathbb{N} : \mho(a_k, a_K, s) \leq 1 - \eta\} \in \mathcal{K}.$$

Choose

$$T_j = \left\{k \in \mathbb{N} : \mho(a_k, a_{m_j}, s) > \frac{j-1}{j}\right\} \text{ for } j = 1, 2, \ldots,$$

where $m_j = K(\frac{j-1}{j})$. For $j = 1, 2, \ldots$, $T_j \in \mathcal{F}(\mathcal{K})$ is obvious. Since $\mathcal{K}$ has the condition (AP), then by Lemma 4 in [16] there exists a set T such that $T \in \mathcal{F}(\mathcal{K})$ and $T \setminus T_j$ is finite for all j.

Suppose that $\eta \in (0,1)$ and $n \in \mathbb{N}$ such that $n > \frac{1}{\eta}$. Hence, there exists $i(n)$ such that $k \in T_n$ and $m \in T_n$ for all $k, m > i(n)$, where $k, m \in T$. Therefore, $\mho(a_k, a_{m_n}, s) > \frac{n-1}{n}$ and $\mho(a_m, a_{m_n}, s) > \frac{n-1}{n}$ for all $k, m > i(n)$. In that case,

$$\mho(a_k, a_m, s) \geq \mho\left(a_k, a_{m_n}, \frac{s}{2}\right) \diamondsuit \mho\left(a_m, a_{m_n}, \frac{s}{2}\right) > (1-\eta) \diamondsuit (1-\eta) = \delta(\eta) \text{ for } m, k > i(n).$$

Consequently,

$$\lim_{\substack{k,m\to\infty \\ k,m\in T}} \mho(a_k, a_m, s) = 1.$$

□

Theorem 2.9. *Let $\mathcal{K}$ be an admissible ideal. If a sequence (a_k) in $\mathbb{E}$ is a $\mho(\mathcal{K}^*)-$convergent sequence, then it is a $\mho(\mathcal{K}^*)-$ Cauchy sequence.*

Proof. Let $a_k \xrightarrow{\mho(\mathcal{K}^*)} a_0$. Then, there exist a set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\} \in \mathcal{F}(\mathcal{K})$ such that

$$\lim_{\substack{k\to\infty \\ t_k\in T}} \mho(a_{t_k}, a_0, s) = 1.$$

Considering the following inequality

$$\mho(a_{t_k}, a_{t_p}, s) \geq \mho\left(a_{t_k}, a_0, \frac{s}{2}\right) \diamondsuit \mho\left(a_{t_p}, a_0, \frac{s}{2}\right) > (1-\eta) \diamondsuit (1-\eta) = \delta(\eta)$$

it is clear that

$$\lim_{k,p\to\infty} \mho(a_{t_k}, a_{t_p}, s) = 1.$$

Consequently, there exists a set $T \in \mathcal{F}(\mathcal{K})$ such that $\lim_{k,p\to\infty} \mho(a_{t_k}, a_{t_p}, s) = 1$, i.e., the sequence (a_k) is a $\mho(\mathcal{K}^*)-$ Cauchy sequence. □

3. $\mho(\mathcal{K})-$Limit Points and $\mho(\mathcal{K})$-Cluster Points

In this section, some introductory definitions, theorem, and proposition of the $\mho(\mathcal{K})-$limit point and the $\mho(\mathcal{K})-$cluster point in the FMS are presented.

Definition 3.1. *Let $\mathcal{K}$ be an admissible ideal, $a = (a_k)$ be a sequence in $\mathbb{E}$, and $a_0 \in \mathbb{E}$.*

(1) *An element a_0 is said to be a $\mho(\mathcal{K})$-limit point of a, if there is a set*

$$T = \{t_1 < t_2 < \ldots\} \notin \mathcal{K}$$

and

$$\lim_{\substack{k\to\infty \\ t_k\in T}} \mho(a_{t_k}, a_0, s) = 1$$

and the set of all $\mho(\mathcal{K})$-limit points of a sequence a is denoted by $\mho(\mathcal{K})(\Lambda_a)$.

(2) *An element a_0 is called a $\mho(\mathcal{K})$-cluster point of a, if, for all $\eta \in (0,1)$ and $s > 0$,*
$$\{k \in \mathbb{N} : \mho(a_k, a_0, s) > 1 - \eta\} \notin \mathcal{K}$$
and the set of all $\mho(\mathcal{K})$-cluster points of a sequence a is denoted by $\mho(\mathcal{K})(\Gamma_a)$.

Proposition 3.1. *Let $\mathcal{K}$ be an admissible ideal and $a = (a_k)$ be a sequence in $\mathbb{E}$. Then, $\mho(\mathcal{K})(\Lambda_a) \subset \mho(\mathcal{K})(\Gamma_a)$.*

Proof. Let $a_0 \in \mho(\mathcal{K})(\Lambda_a)$, then there exists a set $T = \{t_1 < t_2 < \ldots\} \notin \mathcal{K}$ such that
$$\lim_{k\to\infty} \mho(a_{t_k}, a_0, s) = 1. \tag{9}$$
Take $s > 0$, $\eta \in (0,1)$. According to (9), there exists $k_\eta \in \mathbb{N}$ such that for $t_k > k_\eta$, $\mho(a_{t_k}, a_0, s) > 1 - \eta$. Hence
$$T \setminus \{t_1, t_2, \ldots, t_{k_\eta}\} \subset \{k \in \mathbb{N} : \mho(a_k, a_0, s) > 1 - \eta\}$$
and thus $\{k \in \mathbb{N} : \mho(a_k, a_0, s) > 1 - \eta\} \notin \mathcal{K}$ which means that $a_0 \in \mho(\mathcal{K})(\Gamma_a)$. □

Theorem 3.1. *Let $\mathcal{K}$ be an admissible ideal. Then, the set $\mho(\mathcal{K})(\Gamma_a)$ is closed in $\mathbb{E}$.*

Proof. Let $b \in \overline{\mho(\mathcal{K})(\Gamma_a)}$, $\eta \in (0,1)$, and $s > 0$. Then, $a_0 \in B(b, \eta, s) \cap \mho(\mathcal{K})(\Gamma_a)$. Suppose that $\delta \in (0,1)$, $s > 0$ such that
$$B(a_0, \delta, s) \subset B(b, \eta, s).$$
Hence,
$$\{k \in \mathbb{N} : \mho(a_0, a_k, s) > 1 - \delta\} \subset \{k \in \mathbb{N} : \mho(b, a_k, s) > 1 - \eta\}.$$
Consequently, $\{k \in \mathbb{N} : \mho(b, a_k, s) > 1 - \eta\} \notin \mathcal{K}$ and $b \in \mho(\mathcal{K})(\Gamma_a)$. □

4. Conclusions

This study examines the notion of ideal convergence, which extends both statistical convergence and Cauchy convergence in FMSs. It employs George and Veeramani's definition of an FMS [9]. Hence, this definition makes it possible to analyze the convergence of statistically non-convergent sequences. Additionally, this paper examines $\mho(\mathcal{K})^*-$convergent and $\mho(\mathcal{K})-$Cauchy sequences, along with $\mho(\mathcal{K})^*-$Cauchy sequences, and analyses their fundamental properties. Furthermore, it defines $\mho(\mathcal{K})$-limit points and $\mho(\mathcal{K})-$cluster points in FMSs while exploring the correlation between these concepts.

In further work, we also believe that rough convergence in an FMS can be defined using the concepts and results presented here, and its basic properties can be studied.

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