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STRESS-STRENGTH MODEL OF m-COMPONENTS SYSTEM IN CENSORED DATA

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ABSTRACT. In this paper, we consider the statistical estimation using Bayesian inference for a multi-component stress-strength parameter. This involves non-identical component strengths under a progressive censoring scheme, within the context of the exponentiated Weibull distribution. The paper addresses the problem in three scenarios. Firstly, it assumes that stress and strength share two common and unknown parameters, along with one unknown different parameter. To obtain the Bayes estimate, the paper utilizes the Markov Chain Monte Carlo (MCMC) method. Secondly, in the scenario where stress and strength share two common and known parameters, along with one unknown different parameter, the paper employs the MCMC method and Lindley's approximation. Thirdly, in the general case, the paper utilizes the MCMC method to obtain the Bayes estimate. A Monte Carlo simulation study is used to compare various estimates. An application concerning monthly water capacities of the Shasta reservoir in California is finally provided.

Keywords: Progressive censoring scheme, Multi-component stress-strength parameter, MCMC method, Lindley's approximation

AMS Subject Classification: 62F15, 62N05

1. Introduction

A simple modification of the two-parameter Weibull distribution, called the exponentiated Weibull (EW) distribution, was proposed by [18] and further studied by [19] and [20] with applications to bus-motor failure data and flood data. They added a new shape parameter λ to the Weibull distribution. The probability density function (pdf) and cumulative distribution function (cdf) of the EW distribution are respectively, as follows:

$$f(x) = \lambda\alpha\theta x^{\theta-1}e^{-\alpha x^{\theta}}(1-e^{-\alpha x^{\theta}})^{\lambda-1}, \quad x>0, \tag{1}$$

$$F(x) = (1-e^{-\alpha x^{\theta}})^{\lambda}, \qquad x>0, \tag{2}$$

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where $\alpha,\ \theta,\ \lambda > 0$, α is a scale parameter and θ and λ are shape parameters. In following, a EW with the pdf in (1) is denoted by $EW(\alpha, \theta, \lambda)$. The particular case for $\lambda = 1$ is the Weibull distribution. The particular case for $\theta = 2$ is the generalized Rayleigh (GR) distribution. The particular case for $\theta = 1$ is generalized exponential (GE) distribution due to [16]. The particular cases for $\lambda = 1$ and $\theta = 1,\ 2$ are the exponential and Rayleigh distributions. The hazard function (hf) of the EW distribution is $h(x) = \frac{\lambda\alpha\theta x^{\theta-1} e^{-\alpha x^{\theta}} (1-e^{-\alpha x^{\theta}})^{\lambda-1}}{1-(1-e^{-\alpha x^{\theta}})^{\lambda}}$, $x > 0$, which is monotonically increasing when $\theta > 1$ and $\theta\lambda > 1$, monotonically decreasing when $\theta < 1$ and $\theta\lambda < 1$, bathtub shaped when $\theta > 1$ and $\theta\lambda < 1$ and unimodal shaped when $\theta < 1$ and $\theta\lambda > 1$. Figure 1 shows possible shapes of the pdf and the hf of the EW distribution. The exponentiated Weibull distribution is a

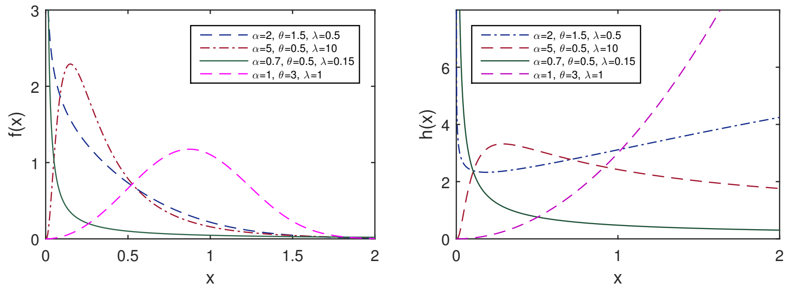

FIGURE 1. The shape of hf (right) and pdf (left) of EW distribution.

good distribution in context of reliability and censoring data. [21] considered the Bayesian estimation for this distribution. Bayesian estimation for the exponentiated Weibull model under Type-II progressive censoring is studied by [6]. [1] discussed the estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes. [22] obtained the estimation of reliability in multicomponent stress-strength based on two parameter exponentiated Weibull Distribution. Finally, reliability investigation of exponentiated Weibull distribution using IPL through numerical and artificial neural network modeling is considered by [25].

In reliability theory, the statistical inference on the reliability parameter is so important for the researchers. The stress-strength parameter is defined by:

$$R = P(X > Y) = \int_{-\infty}^{\infty} \big(1 - F_X(y)\big) dF_Y(y),$$

where Y is the stress variable with cdf $F_Y(\cdot)$ and X is the strength variable with cdf $F_X(\cdot)$. While the applied strength is greater than its stress, the system is reliable. This model, for the first time, is introduced by [4].

A system with more that component is called the multi-component system. This system has one common stress component and k strength components, which are independent and identical. While at least s from k strength components exceed its stress, the system is reliable. The multi-component stress-strength parameter, firstly, is defined by [3] as follows:

$$R_{s,k} = \sum_{p=s}^{k} \binom{k}{p} \int_{-\infty}^{\infty} (1 - F_X(y))^p \big(F_X(y)\big)^{k-p} dF_Y(y),$$

where, Y is the stress variable with cdf $F_Y(\cdot)$ and $(X_1, \ldots, X_k)$ are independent and identically distributed (i.i.d) random variables as the strength variables with cdf $F_X(\cdot)$.

Recently, this model has been studied by some researchers such as [7] (bivariate Kumaraswamy distribution), [13] (progressive censored of Kumaraswamy distribution), [14] (adaptive progressive censored of Weibull distribution), [26] (progressive censoring of Burr III distribution), [24] (progressive censoring of Topp-Leone distribution), [15] (progressive censoring of power Lindley distribution). In such system, [23] developed the multi-component stress-strength parameter with $\boldsymbol{k} = (k_1, k_2, \dots, k_m)$ components as

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{i=1}^{m} \binom{k_i}{p_i} \right) \int_{-\infty}^{\infty} \prod_{i=1}^{m} \left(\big(1 - F_i(y)\big)^{p_i} \big(F_i(y)\big)^{k_i - p_i} \right) dF_Y(y), \tag{3}$$

where, k_i components are of i, $i = 1, \dots, m$ type and $F_i(\cdot)$ is cdf of the strengths for the i-th type components. In this condition, it is assumed that all strength components are affected by one common stress Y with cdf $F_Y(\cdot)$, so, the system is available as long as at least $\boldsymbol{s} = (s_1, \dots, s_m)$ of $\boldsymbol{k}$ strength components exceed its stress. Recently, inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ have been studied by [12], for two components of strength variables. Also, this model is studied by [11] for the modified Weibull extension distribution, in progressive censored data. Moreover, [10] have studied $R_{\boldsymbol{s},\boldsymbol{k}}$ for progressive first failure censoring samples, when the strengths and stress variables follow the modified Kumaraswamy distribution. Very recently, [8] considered family of Kumaraswamy generalized distribution in estimation of multi-component reliability under the adaptive hybrid progressive censoring schemes. In this paper, this model is considered for the EW distribution.

Type-I and Type-II censoring schemes are two most important censoring schemes and hybrid scheme is derived by mixing of theses two schemes. As the active units cannot be removed during the experiment by these schemes, so, the progressive censoring scheme is introduced. The monograph by [2] is a complete and comprehensive reference on this scheme. The progressive censoring scheme is described as follows. During the test, assuming that N units are placed on the experiment, R_1 units randomly are removed from the test, at the first failure time, R_2 units randomly are removed from the test, at the second failure time and so on. Consequently, R_n units randomly are removed from the test along, at the n-th failure time. So, in this scheme, progressive censoring scheme is $\{R_1, \cdots, R_n\}$ and the progressive sample is $\{X_{1:n:N}, \cdots, X_{n:n:N}\}$, such that $R_1 + \cdots + R_n + n = N$. Here after, the progressive censoring sample is presented by $\{X_1, \cdots, X_n\}$. The joint pdf for the censoring sample $X_1 < \cdots < X_n$ with the pdf $f(\cdot)$ and cdf $F(\cdot)$ is given by

$$f(x_1, \cdots, x_n) \propto \prod_{i=1}^{n} f(x_i)\big(1 - F(x_i)\big)^{R_i}, \quad 0 < x_1 < \cdots < x_n < \infty.$$

In this paper, we study the Bayesian inference for $R_{\boldsymbol{s},\boldsymbol{k}}$ parameter when the progressive censoring samples follows the EW distribution. This problem is quite general, and it encompasses various scenarios, including:

- The $R_{\boldsymbol{s},\boldsymbol{k}}$ parameter can be transformed into $R_{\boldsymbol{s},\boldsymbol{k}}$ with two non-identical-component case when $\boldsymbol{k} = (k_1, k_2, 0, \cdots, 0)$.
- The $R_{\boldsymbol{s},\boldsymbol{k}}$ parameter can be transformed into $R_{s,k}$ parameter when $\boldsymbol{k} = (k, 0, \cdots, 0)$.
- The $R_{\boldsymbol{s},\boldsymbol{k}}$ parameter can be transformed into $R = P(X < Y)$ parameter when $\boldsymbol{k} = (1, 0, \cdots, 0)$.
- The progressive censoring scheme can be adapted into Type-II censoring when $R_1 = \cdots = R_{n-1} = 0, R_n = N - n, w = 1$.
- The progressive censoring scheme can be adapted into complete sample case when $R_1 = \cdots = R_n = 0, w = 1$.

- The exponentiated Weibull distribution can be reduced to the Weibull distribution when $\lambda = 1$.
- The exponentiated Weibull distribution can be reduced to the generalized Rayleigh distribution when $\theta = 2$.
- The exponentiated Weibull distribution can be reduced to the generalized exponential distribution when $\theta = 1$.
- The exponentiated Weibull distribution can be reduced to the exponential distribution when $\lambda = 1,\ \theta = 1$
- The exponentiated Weibull distribution can be reduced to the Rayleigh distribution when $\lambda = 1,\ \theta = 2$.

Along with all the advantages of this model, the most important benefit of it is the generality, so that, by addressing this problem, we can automatically tackle several related problems and scenarios (about 30 cases). About the limitation of this model, we should say that MCMC method which is widely used in this paper has limited resources depending on personal computer. So, though accuracy is important, but the speed comparisons show that the MCMC method has a very slow speed and this point is the most important limitation of this model.

In fact, similar works published by some authors. But each work has spacial unique properties. For example, [11] considered multi-component reliability inference in modified Weibull extension distribution and progressive censoring scheme. Also, [9] studied m-component reliability model in Bayesian inference on modified Weibull distribution. We should note that combination of progressive censoring, multi-component stress-strength parameter and exponentiated Weibull distribution is not considered in any works and this is the most important difference between this works and other papers.

The layout of this paper is as follows. In Section 2, we consider the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$, when the common parameters α and θ are unknown. For this aim, the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ have been developed by using the Markov Chain Monte Carlo (MCMC) method due to the lack of explicit forms and the highest posterior density (HPD) credible intervals is constructed. In Section 3, we consider the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$, when the common parameters α and θ are known. For this aim, the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ have been developed by using MCMC and Lindley's approximation methods and HPD credible intervals is constructed. In Section 4, we considered the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ in general case. In Section 5, we compare different methods using the simulation study and analyze one real data, for illustrative aims.

2. Inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ with unknown common α and θ

Let $X_1 \sim EW(\alpha, \theta, \lambda_1), \ldots, X_m \sim EW(\alpha, \theta, \lambda_m)$ and $Y \sim EW(\alpha, \theta, \lambda)$ be the independent random variables. So, we obtain the multi-component stress-strength parameter,

$R_{\boldsymbol{s},\boldsymbol{k}}$, from (1) and (2), as follows:

$$
\begin{aligned}
R_{\mathbf{s},\mathbf{k}} &= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\int_0^\infty \lambda\alpha\theta y^{\theta-1}e^{-\alpha y^\theta}(1-e^{-\alpha y^\theta})^{\sum_{l=1}^{m}\lambda_l(k_l-p_l)+\lambda-1} \\
&\times\prod_{l=1}^{m}\left(1-(1-e^{-\alpha y^\theta})^{\lambda_l}\right)^{p_l}dy\ Put: t=1-e^{-\alpha y^\theta} \\
&= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\lambda\int_0^1 t^{\sum_{l=1}^{m}\lambda_l(k_l-p_l)+\lambda-1}\prod_{l=1}^{m}(1-t^{\lambda_l})^{p_l}dt \\
&= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{p_1}\cdots\sum_{q_m=0}^{p_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\binom{p_1}{q_1}\cdots\binom{p_m}{q_m}
\end{aligned}
$$

$$
\times(-1)^{\sum_{l=1}^{m}q_l}\lambda\int_0^1 t^{\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda-1}dt=\sum_{\boldsymbol{k}}^{\boldsymbol{s}}\sum_{\boldsymbol{p}}^{\boldsymbol{q}}\frac{(-1)^{\sum_{l=1}^{m}q_l}\lambda}{\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda}. \tag{4}
$$

where $\sum_{\boldsymbol{k}}^{\boldsymbol{s}}=\sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}$, $\sum_{\boldsymbol{p}}^{\boldsymbol{q}}=\sum_{q_1=0}^{p_1}\cdots\sum_{q_m=0}^{p_m}\binom{p_1}{q_1}\cdots\binom{p_m}{q_m}$. As we know, in multi-component system, the likelihood function can be derived by the following samples:

$$
X_l=\overset{\text{Observed strength variables}}{\begin{pmatrix} X_{11}^{(l)} & \dots & X_{1k_l}^{(l)} \\ \vdots & \ddots & \vdots \\ X_{n1}^{(l)} & \dots & X_{nk_l}^{(l)}\end{pmatrix}},\ l=1,\dots,m,\ \text{and}\qquad Y=\overset{\text{Observed stress variables}}{\begin{pmatrix} Y_1 \\ \vdots \\ Y_n\end{pmatrix}}\qquad ,
$$

where $\{X_{i1}^{(l)},\dots,X_{ik_1}^{(l)}\}$, $i=1,\dots,n,\ l=1,\dots,m$ are l progressive censoring samples from $EW(\alpha,\theta,\lambda_l)$ with schemes $\{k_l,R_1^{(l)},\dots,R_{k_l}^{(l)}\}$. Also, $\{Y_1,\dots,Y_n\}$ is a progressive censoring sample from $EW(\alpha,\theta,\lambda)$ with scheme $\{n,S_1,\dots,S_n\}$. The above matrix representation shows that the observed strength variables is presented in a $n\times k_l$ matrix, so that the sample size in strength X_l is nk_l, $l=1,\cdots,m$. Also, the observed stress variable is presented in a n-dimensional vector, so that the sample size in stress Y is n. Now, we can obtained the likelihood function of the parameters as

$$
L(\alpha,\theta,\lambda_1,\dots,\lambda_m,\lambda|\text{data})\propto\prod_{i=1}^{n}\left(\prod_{l=1}^{m}\Big(\prod_{j_l=1}^{k_l}f_l(x_{ij_l}^{(l)})\big(1-F_l(x_{ij_l}^{(l)})\big)^{R_j^{(l)}}\Big)\right)f(y_i)\big(1-F(y_i)\big)^{S_i}.
$$

In this section, under squared error loss functions, we study the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ where α, θ and $\lambda_1,\dots,\lambda_m$, as the random variables, follow the independent gamma distributions. As we know, the joint posterior density function, based on the observed censoring samples, can be derived as follows:

$$
\pi(\alpha,\theta,\lambda_1,\dots,\lambda_m,\lambda|\text{data})\propto L(\text{data}|\alpha,\theta,\lambda_1,\dots,\lambda_m,\lambda)\times\pi(\alpha)\pi(\theta)\big(\prod_{l=1}^{m}\pi(\lambda_l)\big)\pi(\lambda) \tag{5}
$$

where

$$\pi(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \alpha, a_1, b_1 > 0, \pi(\theta) \propto \theta^{c_1-1}e^{-d_1\theta}, \theta, c_1, d_1 > 0,$$
$$\pi(\lambda_l) \propto \lambda_l^{e_l-1}e^{-f_l\lambda_l}, \lambda_l, e_l, f_l > 0, l = 1, \ldots, m, \pi(\lambda) \propto \lambda^{e_{m+1}-1}e^{-f_{m+1}\lambda}, \lambda, e_{m+1}, f_{m+1} > 0.$$

The reasons, for choosing the gamma priors, can be explained as follows. The range of the unknown model parameters is positive. Also, because given α, θ and progressive censoring data, the full conditional distributions of λ_l, $l = 1, \cdots, m$ and λ are gamma distributions, so they are conjugated priors. Finally, by selecting these gamma priors, the calculations become a little bit easier. The equation (5) shows that we cannot derived the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ in the closed form. So, in following, it is approximated via the MCMC method. From the equation (5), the posterior pdfs of α, θ, $\lambda_1, \ldots, \lambda_m$ and λ can be derived as follows:

$$\pi(\alpha|\theta, \lambda_1, \ldots, \lambda_m, \lambda, \text{data}) \propto \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l-1}\prod_{i=1}^{n}\left(1 - e^{-\alpha y_i^\theta}\right)^{\lambda-1}$$
$$\times \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}\Big(1 - \big(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\big)^{\lambda_l}\Big)^{R_j^{(l)}}\prod_{i=1}^{n}\Big(1 - \big(1 - e^{-\alpha y_i^\theta}\big)^{\lambda}\Big)^{S_i}$$
$$\times \alpha^{n(\sum_{l=1}^{m}k_l+1)+a_1-1} e^{-\alpha\left(b_1+\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{k_l}(x_{ij}^{(l)})^\theta+\sum_{i=1}^{n}y_i^\theta\right)},$$
$$\pi(\theta|\alpha, \lambda_1, \ldots, \lambda_m, \lambda, \text{data}) \propto \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l-1}\prod_{i=1}^{n}\left(1 - e^{-\alpha y_i^\theta}\right)^{\lambda-1}$$
$$\times \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}\Big(1 - \big(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\big)^{\lambda_l}\Big)^{R_j^{(l)}}\prod_{i=1}^{n}\Big(1 - \big(1 - e^{-\alpha y_i^\theta}\big)^{\lambda}\Big)^{S_i}\Big(\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}(x_{ij}^{(l)})^{\theta-1}\Big)$$
$$\times \Big(\prod_{i=1}^{n}y_i^{\theta-1}\Big) \times \theta^{n(\sum_{l=1}^{m}k_l+1)+c_1-1} e^{-d_1\theta-\alpha\left(\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{k_l}(x_{ij}^{(l)})^\theta+\sum_{i=1}^{n}y_i^\theta\right)},$$
$$\pi(\lambda_l|\alpha, \theta, \text{data}) \propto \prod_{i=1}^{n}\prod_{j=1}^{k_l}\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l-1}\prod_{i=1}^{n}\prod_{j=1}^{k_l}\Big(1 - \big(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\big)^{\lambda_l}\Big)^{R_j^{(l)}},$$
$$\times \lambda_l^{nk_l+e_l-1}e^{-f_l\lambda_l},\ l = 1, \ldots, m,$$
$$\pi(\lambda|\alpha, \theta, \text{data}) \propto \prod_{i=1}^{n}\left(1 - e^{-\alpha y_i^\theta}\right)^{\lambda-1}\prod_{i=1}^{n}\Big(1 - \big(1 - e^{-\alpha y_i^\theta}\big)^{\lambda}\Big)^{S_i}\lambda^{n+e_{m+1}-1}e^{-f_{m+1}\lambda}.$$

It is notable that as the posterior pdfs of all parameters are not the well known distributions, so generating samples from them should be done by using the Metropolis-Hastings method. So, the following algorithm of Gibbs sampling cab be proposed:

1. Begin with $(\alpha_{(0)}, \theta_{(0)}, \lambda_{1(0)}, \ldots, \lambda_{m(0)}, \lambda_{(0)})$.
2. Set $t = 1$.
3. Generate $\alpha_{(t)}$ from $\pi(\alpha|\theta_{(t-1)}, \lambda_{1(t-1)}, \ldots, \lambda_{m(t-1)}, \lambda_{(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\alpha_{(t-1)}, 1)$ as proposal distribution.
4. Generate $\theta_{(t)}$ from $\pi(\theta|\alpha_{(t-1)}, \lambda_{1(t-1)}, \ldots, \lambda_{m(t-1)}, \lambda_{(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\theta_{(t-1)}, 1)$ as proposal distribution.

5 - m+4. Generate $\lambda_{l(t)}$ from $\pi(\lambda_l|\alpha_{(t-1)}, \theta_{(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\lambda_{l(t-1)}, 1)$ as proposal distribution.
m+5. Generate $\lambda_{(t)}$ from $\pi(\lambda|\alpha_{(t-1)}, \theta_{(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\lambda_{(t-1)}, 1)$ as proposal distribution.
m+6. Evaluate $R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{\boldsymbol{k}}^{\boldsymbol{s}}\sum_{\boldsymbol{p}}^{\boldsymbol{q}} \frac{(-1)^{\sum_{l=1}^{m} q_l} \lambda_{(t)}}{\sum_{l=1}^{m} \lambda_{l(t)}(k_l - p_l + q_l) + \lambda_{(t)}}$.
m+7. Set $t = t + 1$.
m+8. Repeat T times, steps 3 - m+7.

Consequently, under the squared error loss functions, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained by

$$\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T-M}\sum_{t=M+1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}, \tag{6}$$

where M is the burn-in period. Also, using the method of [5], we construct the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ as follows. First, order $R_{(1)\boldsymbol{s},\boldsymbol{k}}, \dots, R_{(T)\boldsymbol{s},\boldsymbol{k}}$ as $R_{\left((1)\boldsymbol{s},\boldsymbol{k}\right)} < \cdots < R_{\left((T)\boldsymbol{s},\boldsymbol{k}\right)}$ and then construct all the $100(1-\eta)\%$ confidence intervals of R, as:

$$\left(R_{\left((1)\boldsymbol{s},\boldsymbol{k}\right)}, R_{\left(([T(1-\eta)])\boldsymbol{s},\boldsymbol{k}\right)}\right), \dots, \left(R_{\left(([T\eta])\boldsymbol{s},\boldsymbol{k}\right)}, R_{\left(([T])\boldsymbol{s},\boldsymbol{k}\right)}\right),$$

where $[T]$ symbolizes the largest integer less than or equal to T. The HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ is the shortest length interval.

3. Inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ with known common α and θ

When the common parameters α and θ are known, the $R_{\boldsymbol{s},\boldsymbol{k}}$ can be derived as presented in equation (4). In this section, under squared error loss functions, we study the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ where $\lambda_1, \dots, \lambda_m$, as the random variables, follow the independent gamma distributions. Similar to Section 2, the posterior pdfs of $\lambda_1, \dots, \lambda_m$ and λ can be derived as follows:

$$\pi(\lambda_l|\alpha, \theta, \text{data}) \propto \prod_{i=1}^{n}\prod_{j=1}^{k_l}\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l - 1}\prod_{i=1}^{n}\prod_{j=1}^{k_l}\left(1 - \left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l}\right)^{R_j^{(l)}}$$
$$\times \lambda_l^{nk_l + e_l - 1} e^{-f_l\lambda_l},\ l = 1, \dots, m,$$
$$\pi(\lambda|\alpha, \theta, \text{data}) \propto \prod_{i=1}^{n}\left(1 - e^{-\alpha y_i^\theta}\right)^{\lambda - 1}\prod_{i=1}^{n}\left(1 - \left(1 - e^{-\alpha y_i^\theta}\right)^{\lambda}\right)^{S_i}\lambda^{n + e_{m+1} - 1}e^{-f_{m+1}\lambda}.$$

Similar to previous section, we use the Gibbs sampling and obtain

$$\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T}\sum_{t=1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{7}$$

Also, using the method of [5], we construct the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$.

3.1. Lindley's approximation.

One of the most important numerical methods to obtain the approximation Bayes estimation is proposed by [17]. This method can be explained as follows. Under the squared error loss, the Bayes estimation of $U(\psi)$, is

$$\mathbb{E}(u(\psi)|\text{data}) = \frac{\int u(\psi)e^{Q(\psi)}d\psi}{\int e^{Q(\psi)}d\psi}, \tag{8}$$

where $Q(\psi) = \rho(\psi) + \ell(\psi)$, $\rho(\psi)$ and $\ell(\psi)$ are logarithm of the prior density of ψ and log-likelihood function, respectively. The $\mathbb{E}(u(\psi)|\text{data})$ in equation (8), is approximated by [17] as follows:

$$\mathbb{E}(u(\psi)|\text{data}) = u + \frac{1}{2}\sum_i\sum_j(u_{i,j} + 2u_i\rho_j)\sigma_{i,j} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_p \ell_{i,j,k}\sigma_{i,j}\sigma_{k,p}u_p\Big|_{\psi=\widehat{\psi}}, \quad (9)$$

where $\psi = (\psi_1, \dots, \psi_m)$, $i, j, k, p = 1, \dots, m$, $\widehat{\psi}$ is the MLE of ψ, $u = u(\psi)$, $u_i = \partial u/\partial\psi_i$, $u_{i,j} = \partial^2 u/(\partial\psi_i\partial\psi_j)$, $\ell_{i,j,k} = \partial^3\ell/(\partial\psi_i\partial\psi_j\partial\psi_k)$, $\rho_j = \partial\rho/\partial\psi_j$, and $\sigma_{i,j} = (i,j)$-th element in the inverse of matrix $[-\ell_{i,j}]$ all evaluated at the MLE of parameters. The equation (9), for $m+1$ parameters, can be re-write as follows:

$$\widehat{u}^{Lin} = u + (\sum_{i=1}^{m+1} u_i d_i + d_{m+2} + d_{m+3}) + \frac{1}{2}\sum_{i=1}^{m+1} A_i(\sum_{j=1}^{m+1} u_j\sigma_{i,j}), \quad (10)$$

where

$$d_i = \sum_{j=1}^{m+1}\rho_j\sigma_{i,j},\ i = 1, \cdots, m+1,\ d_{m+2} = \sum_{\substack{i=1\\i<j}}^{m+1}\sum_{j=1}^{m+1} u_{i,j}\sigma_{i,j},\ d_{m+3} = \frac{1}{2}\sum_{i=1}^{m+1} u_{i,i}\sigma_{i,i},$$

$$A_i = \sum_{\substack{j=1\\j\le k}}^{m+1}\sum_{k=1}^{m+1}\ell_{j,k,i} \times \begin{cases} \sigma_{j,k} & j = k, \\ 2\sigma_{j,k} & j < k, \end{cases} \quad i = 1, \cdots, m+1.$$

For $(\psi_1, \cdots, \psi_m, \psi_{m+1}) \equiv (\lambda_1, \cdots, \lambda_m, \lambda)$ and $u \equiv R_{\boldsymbol{s},\boldsymbol{k}}$, we have

$$\rho_l = \frac{a_l - 1}{\lambda_l} - b_l,\ l = 1, \cdots, m,\ \rho_{m+1} = \frac{a_{m+1} - 1}{\lambda} - b_{m+1},$$

$$\ell_{l,l} = -\frac{nk_l}{\lambda_l^2} - \sum_{i=1}^{n}\sum_{j=1}^{k_l} R_j^{(l)}\frac{\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l}\log^2\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)}{\left(1 - \left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l}\right)^2},\ l = 1, \cdots, m,$$

$$\ell_{m+1,m+1} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n} S_i\frac{\left(1 - e^{-\alpha y_i^\theta}\right)^\lambda\log^2\left(1 - e^{-\alpha y_i^\theta}\right)}{\left(1 - \left(1 - e^{-\alpha y_i^\theta}\right)^\lambda\right)^2},\ \ell_{l,k} = 0,\ l = 1, \cdots, m+1, l \neq k.$$

The values of $\sigma_{i,j}, i, j = 1, \cdots, m+1$ can be obtained from $\ell_{i,j}, i, j = 1, \cdots, m+1,$. Also,

$$\ell_{l,l,l} = \frac{2nk_l}{\lambda_l^3} - \sum_{i=1}^{n}\sum_{j=1}^{k_l} R_j^{(l)}\frac{\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l}\log^3\left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)}{\left(1 - \left(1 - e^{-\alpha(x_{ij}^{(l)})^\theta}\right)^{\lambda_l}\right)^2},\ l = 1, \cdots, m,$$

$$\ell_{m+1,m+1,m+1} = \frac{2n}{\lambda^3} - \sum_{i=1}^{n} S_i\frac{\left(1 - e^{-\alpha y_i^\theta}\right)^\lambda\log^3\left(1 - e^{-\alpha y_i^\theta}\right)}{\left(1 - \left(1 - e^{-\alpha y_i^\theta}\right)^\lambda\right)^2},$$

and other $\ell_{i,j,k} = 0$. Also, $u_l = \sum_{\boldsymbol{k}}^{\boldsymbol{s}}\sum_{\boldsymbol{p}}^{\boldsymbol{q}}(-1)^{\sum_{l=1}^{m} q_l} \times A$ and $u_{l,k} = \sum_{\boldsymbol{k}}^{\boldsymbol{s}}\sum_{\boldsymbol{p}}^{\boldsymbol{q}}(-1)^{\sum_{l=1}^{m} q_l} \times B$, where

$$A = \begin{cases} \frac{\lambda(k_l-p_l+q_l)}{\left(\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda\right)^2} & l = 1,\cdots,m, \\ \frac{\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)}{\left(\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda\right)^2} & l = m+1, \end{cases}$$

and

$$B = \begin{cases} \frac{2\lambda(k_l-p_l+q_l)(k_k-p_k+q_k)}{\left(\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda\right)^3} & l,k = 1,\cdots,m, \\ -\frac{2\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)}{\left(\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda\right)^3} & l = m+1, \\ -\frac{(k_l-p_l+q_l)\left(\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)-\lambda\right)}{\left(\sum_{l=1}^{m}\lambda_l(k_l-p_l+q_l)+\lambda\right)^3} & l = 1,\cdots,m,\ k = m+1. \end{cases}$$

After obtaining the above values, Lindley's estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$, $\widehat{R}^{Lin}_{\boldsymbol{s},\boldsymbol{k}}$, can be obtained, from equation (10). It is notable that all parameters should be computed at $(\widehat{\lambda}_1,\cdots,\widehat{\lambda}_m,\widehat{\lambda})$, MLEs of $(\lambda_1,\cdots,\lambda_m,\lambda)$.

4. Inference on $R_{\boldsymbol{s},\boldsymbol{k}}$ in general case

Let $X_1 \sim EW(\alpha_1,\theta_1,\lambda_1),\ldots,X_m \sim EW(\alpha_m,\theta_m,\lambda_m)$ and $Y \sim EW(\alpha,\theta,\lambda)$ be the independent random variables. So, we obtain the multi-component stress-strength parameter, $R_{\boldsymbol{s},\boldsymbol{k}}$, from (1) and (2), as follows:

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{\boldsymbol{k}}^{\boldsymbol{s}}\int_0^\infty \lambda\alpha\theta y^{\theta-1}e^{-\alpha y^\theta}(1-e^{-\alpha y^\theta})^{\lambda-1}\prod_{l=1}^{m}\left(1-(1-e^{-\alpha_l y^{\theta_l}})^{\lambda_l}\right)^{p_l}$$
$$\times (1-e^{-\alpha_l y^{\theta_l}})^{\lambda_l(k_l-p_l)}dy.$$

In this section, under squared error loss functions, we study the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ where $\alpha_1,\ldots,\alpha_m$, α, $\theta_1,\ldots,\theta_m$, θ, $\lambda_1,\ldots,\lambda_m$ and λ, as the random variables, follow the independent gamma distributions as

$$\pi(\alpha_l) \propto \alpha^{a_l-1}e^{-b_l\alpha_l}, \alpha_l, a_l, b_l > 0, l = 1,\ldots,m, \pi(\alpha) \propto \alpha^{a_{m+1}-1}e^{-b_{m+1}\alpha}, \alpha, a_{m+1}, b_{m+1} > 0,$$
$$\pi(\theta_l) \propto \theta_l^{c_l-1}e^{-d_l\theta_l}, \theta_l, c_l, d_l > 0, l = 1,\ldots,m, \pi(\theta) \propto \theta^{c_{m+1}-1}e^{-d_{m+1}\theta}, \theta, c_{m+1}, d_{m+1} > 0,$$
$$\pi(\lambda_l) \propto \lambda_l^{e_l-1}e^{-f_l\lambda_l}, \lambda_l, e_l, f_l > 0, l = 1,\ldots,m, \pi(\lambda) \propto \lambda^{e_{m+1}-1}e^{-f_{m+1}\lambda}, \lambda, e_{m+1}, f_{m+1} > 0.$$

Similar to Section 2, we cannot derived the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ in the closed form. So, in following, it is approximated via the MCMC method. The posterior pdfs of the parameters can be derived as follows:

$$\pi(\alpha_l|\theta_l,\lambda_l,\text{data}) \propto \prod_{i=1}^{n}\prod_{j=1}^{k_l}\big(1-e^{-\alpha_l (x_{ij}^{(l)})^{\theta_l}}\big)^{\lambda_l-1}\prod_{i=1}^{n}\prod_{j=1}^{k_l}\Big(1-\big(1-e^{-\alpha(x_{ij}^{(l)})^{\theta_l}}\big)^{\lambda_l}\Big)^{R_j^{(l)}}$$

$$\times\, \alpha_l^{nk_l+a_l-1} e^{-\alpha_l\left(b_l+\sum_{i=1}^{n}\sum_{j=1}^{k_l}(x_{ij}^{(l)})^{\theta_l}\right)},\ l=1,\ldots,m,$$

$$\pi(\alpha|\theta\lambda,\text{data}) \propto \prod_{i=1}^{n}\big(1-e^{-\alpha y_i^{\theta}}\big)^{\lambda-1}\prod_{i=1}^{n}\Big(1-\big(1-e^{-\alpha y_i^{\theta}}\big)^{\lambda}\Big)^{S_i}\alpha^{n+a_{m+1}-1}e^{-\alpha\left(b_{m+1}+\sum_{i=1}^{n}y_i^{\theta}\right)},$$

$$\pi(\theta_l|\alpha_l,\lambda_l,\text{data}) \propto \prod_{i=1}^{n}\prod_{j=1}^{k_l}\big(1-e^{-\alpha(x_{ij}^{(l)})^{\theta}}\big)^{\lambda_l-1}\prod_{i=1}^{n}\prod_{j=1}^{k_l}\Big(1-\big(1-e^{-\alpha(x_{ij}^{(l)})^{\theta_l}}\big)^{\lambda}\Big)^{R_j^{(l)}}$$

$$\times\, \Big(\prod_{i=1}^{n}\prod_{j=1}^{k_l}(x_{ij}^{(l)})^{\theta_l-1}\Big)\theta_l^{nk_l+c_l-1}e^{-d_l\theta_l-\alpha_l\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{k_l}(x_{ij}^{(l)})^{\theta_l}},\ l=1,\ldots,m,$$

$$\pi(\theta|\alpha,\lambda,\text{data}) \propto \prod_{i=1}^{n}\big(1-e^{-\alpha y_i^{\theta}}\big)^{\lambda-1}\prod_{i=1}^{n}\Big(1-\big(1-e^{-\alpha y_i^{\theta}}\big)^{\lambda}\Big)^{S_i}\Big(\prod_{i=1}^{n}y_i^{\theta-1}\Big)\theta^{n+c_{m+1}-1}$$

$$\times\, e^{-d_{m+1}\theta-\alpha\sum_{i=1}^{n}y_i^{\theta}},$$

$$\pi(\lambda_l|\alpha,\theta,\text{data}) \propto \prod_{i=1}^{n}\prod_{j=1}^{k_l}\big(1-e^{-\alpha(x_{ij}^{(l)})^{\theta}}\big)^{\lambda_l-1}\prod_{i=1}^{n}\prod_{j=1}^{k_l}\Big(1-\big(1-e^{-\alpha(x_{ij}^{(l)})^{\theta}}\big)^{\lambda_l}\Big)^{R_j^{(l)}},$$

$$\times\, \lambda_l^{nk_l+e_l-1}e^{-f_l\lambda_l},\ l=1,\ldots,m,$$

$$\pi(\lambda|\alpha,\theta,\text{data}) \propto \prod_{i=1}^{n}\big(1-e^{-\alpha y_i^{\theta}}\big)^{\lambda-1}\prod_{i=1}^{n}\Big(1-\big(1-e^{-\alpha y_i^{\theta}}\big)^{\lambda}\Big)^{S_i}\lambda^{n+e_{m+1}-1}e^{-f_{m+1}\lambda}.$$

It is notable that as the posterior pdfs of all parameters are not the well known distributions, so generating samples from them should be done by using the Metropolis-Hastings method. So, the following algorithm of Gibbs sampling cab be proposed:

1. Begin with $(\alpha_{1(0)},\ldots,\alpha_{m(0)},\alpha_{(0)},\ \theta_{1(0)},\ldots,\theta_{m(0)},\theta_{(0)},\ \lambda_{1(0)},\ldots,\lambda_{m(0)},\lambda_{(0)})$.
2. Set $t=1$.
3 - m+2. Generate $\alpha_{l(t)}$ from $\pi(\alpha_l|\theta_{l(t-1)},\lambda_{l(t-1)},\text{data})$, using Metropolis-Hastings method, with $N(\alpha_{l(t-1)},1)$ as proposal distribution.
m+3. Generate $\alpha_{(t)}$ from $\pi(\alpha|\theta,\lambda,\text{data})$, using Metropolis-Hastings method, with $N(\alpha_{(t-1)},1)$ as proposal distribution.
m+4 - 2m+3. Generate $\theta_{l(t)}$ from $\pi(\theta_l|\alpha_{l(t-1)},\lambda_{l(t-1)},\text{data})$, using Metropolis-Hastings method, with $N(\theta_{l(t-1)},1)$ as proposal distribution.
2m+4. Generate $\theta_{(t)}$ from $\pi(\theta|\alpha_{(t-1)},\lambda_{(t-1)},\text{data})$, using Metropolis-Hastings method, with $N(\theta_{(t-1)},1)$ as proposal distribution.
2m+5 - 3m+4. Generate $\lambda_{l(t)}$ from $\pi(\lambda_l|\alpha_{l(t-1)},\theta_{l(t-1)},\text{data})$, using Metropolis-Hastings method, with $N(\lambda_{l(t-1)},1)$ as proposal distribution.
3m+5. Generate $\lambda_{(t)}$ from $\pi(\lambda|\alpha_{(t-1)},\theta_{(t-1)},\text{data})$, using Metropolis-Hastings method, with $N(\lambda_{(t-1)},1)$ as proposal distribution.

3m+6. Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{\boldsymbol{k}}^{\boldsymbol{s}} \int_0^\infty \lambda_{(t)}\alpha_{(t)}\theta_{(t)} y^{\theta_{(t)}-1} e^{-\alpha_{(t)} y^{\theta_{(t)}}} (1 - e^{-\alpha_{(t)} y^{\theta_{(t)}}})^{\lambda_{(t)}-1} \times \prod_{l=1}^{m} \big(1 - (1 - e^{-\alpha_{l(t)} y^{\theta_{l(t)}}})^{\lambda_{l(t)}}\big)^{p_l} (1 - e^{-\alpha_{l(t)} y^{\theta_{l(t)}}})^{\lambda_{l(t)}(k_l - p_l)} dy.$$

3m+7. Set $t = t + 1$.

3m+8. Repeat T times, steps 3 - 3m+7.

Consequently, under the squared error loss functions, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained by

$$\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T - M} \sum_{t=M+1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}, \tag{11}$$

where M is the burn-in period. Also, using the method of [5], we construct the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$.

5. Simulation study and data analysis

5.1. **Numerical experiment and discussion.** In this section, we compare different estimations, using the Monte Carlo simulation. For this aim, mean square errors (MSEs) is applied to compare the point estimates and average confidence lengths (AL) and coverage percentages (CP) are employed to compare the interval estimates. Some different censoring schemes, different parameter values and hyper-parameters are utilizing in implementation the simulation studies. The number of repetitions is 2000 and the number of repetitions in Gibbs sampling algorithm is $T = 3000$. Also, we note that the threshold of burn-in is 1000. Moreover, we consider the value 0.95 for the significance level and assume the simulated system has two strength components. To obtain the simulation results we use some censoring scheme which are provided in Table 1. As expected,

Table 1. Different censoring schemes.

(k_l, K_l)		C.S.	(n, N)		C.S.
(5,10)	R_1	(0,0,0,0,5)	(5,10)	S_1	(0,0,0,0,5)
	R_2	(5,0,0,0,0)		S_2	(5,0,0,0,0)
	R_3	(1,1,1,1,1)		S_3	(1,1,1,1,1)
(10,20)	R_4	$(0^{*9},10)$	(10,20)	S_4	$(0^{*9},10)$
	R_5	$(10,0^{*9})$		S_5	$(10,0^{*9})$
	R_6	(1^{*10})		S_6	(1^{*10})

we study three cases. In the first case, when the common parameters α and θ are unknown, we employ the equation (6) to obtain the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$. Also, we use $(\alpha, \theta, \lambda_1, \lambda_2, \lambda_3, \lambda) = (2, 1, 2, 1.5, 2.5, 3)$, Prior 1: $a_1, c_1, e_l = 0,\ b_1, d_1, f_l = 0,\ l = 1, \ldots, 3$, Prior 2: $a_1 = c_1 = e_l = 0.2,\ b_1, d_1, f_l = 0.4,\ l = 1, \ldots, 3$, to derive the simulation results which provided in Table 2. In the second case, when the common parameters α and θ are known, we employ the equations (7) and (10) to obtain the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$. Also, we use $(\alpha, \theta, \lambda_1, \lambda_2, \lambda_3, \lambda) = (1, 1.5, 2.5, 3, 2, 3.5)$, Prior 3: $a_1, c_1, e_l = 0,\ b_1, d_1, f_l = 0,\ l = 1, \ldots, 3$, Prior 4: $a_1, c_1, e_l = 0.3,\ b_1, d_1, f_l = 0.5,\ l = 1, \ldots, 3$, to derive the simulation results which provided in Table 3. In the general case, we employ the equation (11) to obtain the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$. Also, we use $(\alpha_1, \alpha_2, \alpha_3, \alpha, \theta_1, \theta_2, \theta_3, \theta, \lambda_1, \lambda_2, \lambda_3, \lambda) = (1, 2, 3, 2.5, 1.5, 1, 2.5, 3.5, 3.5, 3, 2.5, 2)$, Prior 5: $a_l, c_l, e_l = 0,\ b_l, d_l, f_l = 0,\ l = 1, \ldots, 3$, Prior 6: $a_l, c_l, e_l = 0.6,\ b_l, d_l, f_l = 0.8,\ l = 1, \ldots, 3$, to derive the simulation results which provided in Table 4.

Tables 2-4 show that the informative priors (Priors 2, 4 and 6) have the best performance based on the MSE, AL and CP values, in point and interval estimations. Also, it is observed that the MCMC method performs better than the Lindley's approximation. Now, MCMC diagnostics is considered, so that we provided the trace plot for different parameters and different censoring schemes. In all cases the trace plots show the convergence of the MCMC method. Some of trace plots are given in Figures 2. Moreover, as expected, by increasing n, for fixed $\boldsymbol{s}$ and $\boldsymbol{k}$, MSEs and ALs decrease and CPs increase. Also, by increasing k, for fixed $\boldsymbol{s}$ and n, MSEs and ALs decrease and CPs increase, in all cases. Two above results may occur due to the fact that, by increasing n, the number of failures increases and therefore, more information is gathered. So, the performance of estimates improved.

TABLE 2. Simulation results when common parameters are unknown.

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$	C.S	MCMC					
		Prior 1			Prior 2		
		MSE	AL	CP	MSE	AL	CP
(5,5,5,5,2,2,2)	(R_1, R_1, R_1, S_1)	0.0574	0.6532	0.941	0.0523	0.6222	0.945
	(R_2, R_2, R_2, S_2)	0.0584	0.6518	0.942	0.0520	0.6274	0.946
	(R_3, R_3, R_3, S_3)	0.0577	0.6537	0.940	0.0529	0.6291	0.946
(5,5,5,10,2,2,2)	(R_1, R_1, R_1, S_4)	0.0501	0.5674	0.944	0.0446	0.5119	0.948
	(R_2, R_2, R_2, S_5)	0.0503	0.5691	0.945	0.0450	0.5170	0.949
	(R_3, R_3, R_3, S_6)	0.0509	0.5624	0.944	0.0443	0.5134	0.949
(10,10,10,5,2,2,2)	(R_4, R_4, R_4, S_1)	0.0430	0.5034	0.948	0.0379	0.4733	0.950
	(R_5, R_5, R_5, S_2)	0.0432	0.5040	0.947	0.0380	0.4756	0.951
	(R_6, R_6, R_6, S_3)	0.0435	0.5081	0.948	0.0381	0.4755	0.950
(10,10,10,10,2,2,2)	(R_4, R_4, R_4, S_4)	0.0381	0.4428	0.950	0.0349	0.4037	0.953
	(R_5, R_5, R_5, S_5)	0.0385	0.4475	0.951	0.0356	0.4093	0.952
	(R_6, R_6, R_6, S_6)	0.0389	0.4430	0.950	0.0352	0.4011	0.953
(5,5,5,5,4,4,4)	(R_1, R_1, R_1, S_1)	0.0589	0.6329	0.942	0.0530	0.6013	0.946
	(R_2, R_2, R_2, S_2)	0.0570	0.6320	0.941	0.0537	0.6030	0.947
	(R_3, R_3, R_3, S_3)	0.0576	0.6348	0.940	0.0539	0.6040	0.946
(5,5,5,10,4,4,4)	(R_1, R_1, R_1, S_4)	0.0512	0.5517	0.945	0.0463	0.5290	0.948
	(R_2, R_2, R_2, S_5)	0.0515	0.5591	0.946	0.0468	0.5296	0.948
	(R_3, R_3, R_3, S_6)	0.0510	0.5544	0.944	0.0460	0.5281	0.948
(10,10,10,5,4,4,4)	(R_4, R_4, R_4, S_1)	0.0465	0.5231	0.949	0.0410	0.4879	0.950
	(R_5, R_5, R_5, S_2)	0.0460	0.5211	0.948	0.0407	0.4863	0.950
	(R_6, R_6, R_6, S_3)	0.0475	0.5234	0.949	0.0413	0.4806	0.950
(10,10,10,10,4,4,4)	(R_4, R_4, R_4, S_4)	0.0400	0.4519	0.951	0.0367	0.4110	0.953
	(R_5, R_5, R_5, S_5)	0.0395	0.4537	0.951	0.0360	0.4179	0.953
	(R_6, R_6, R_6, S_6)	0.0407	0.4529	0.950	0.0361	0.4138	0.954

5.2. **Real data analysis.** We analyze the monthly water capacity of the Shasta reservoir in California, available in https://cdec.water.ca.gov/dynamicapp/QueryMonthly?s=SHA, for the illustrative aims. Recently, [13] and [12] have considered this data. As we know understanding the probability of drought occurrence is so important in agriculture, so one scenario as follows can be considered. If the water capacity of a reservoir in a region on August, July and September at least two years out of next five years is more than the amount of water achieved on December of the previous year, there is not any drought. So, the probability of non-occurrence drought can be considered by $R_{\boldsymbol{s},\boldsymbol{k}}$.

By the above scenario, $X^{(1)}_{11}, ..., X^{(1)}_{15}$, $X^{(2)}_{11}, ..., X^{(2)}_{15}$ and $X^{(3)}_{11}, ..., X^{(3)}_{15}$ can be studied as the capacities of July, August and September from 1976 to 1980, $X^{(1)}_{21}, ..., X^{(1)}_{25}$, $X^{(2)}_{21}, ..., X^{(2)}_{25}$ and $X^{(3)}_{21}, ..., X^{(3)}_{25}$ as the capacities of July, August and September from 1982 to 1986, and so on $X^{(1)}_{81}, ..., X^{(1)}_{85}$, $X^{(2)}_{81}, ..., X^{(2)}_{85}$ and $X^{(3)}_{81}, ..., X^{(3)}_{85}$ as the capacities of July, August and September from 2018 to 2022. Moreover, $Y_1, Y_2, \ldots, Y_8$ are the capacity of December 1975, 1981 up to 2017. Only for simplifying the calculations, all data have been

TABLE 3. Simulation results when common parameters are known.

$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	MCMC						Lindley	
		Prior 3			Prior 4			Prior 3	Prior 4
		MSE	AL	CP	MSE	AL	CP	MSE	MSE
(5,5,5,5,2,2,2)	(R_1,R_1,R_1,S_1)	0.0503	0.5432	0.941	0.0467	0.5019	0.945	0.0543	0.0510
	(R_2,R_2,R_2,S_2)	0.0507	0.5481	0.940	0.0463	0.5042	0.946	0.0549	0.0512
	(R_3,R_3,R_3,S_3)	0.0500	0.5469	0.941	0.0468	0.5033	0.945	0.0540	0.0508
(5,5,5,10,2,2,2)	(R_1,R_1,R_1,S_4)	0.0432	0.4796	0.942	0.0382	0.4167	0.948	0.0516	0.0476
	(R_2,R_2,R_2,S_5)	0.0438	0.4785	0.943	0.380	0.4157	0.949	0.0510	0.0470
	(R_3,R_3,R_3,S_6)	0.0436	0.4734	0.942	0.389	0.4137	0.948	0.0517	0.0477
(10,10,10,5,2,2,2)	(R_4,R_4,R_4,S_1)	0.0376	0.4122	0.947	0.0312	03661	0.950	0.0478	0.0431
	(R_5,R_5,R_5,S_2)	0.0370	0.4180	0.946	0.318	0.3699	0.951	0.0479	0.0439
	(R_6,R_6,R_6,S_3)	0.0368	0.4136	0.947	0.316	0.3621	0.951	0.0473	0.0432
(10,10,10,10,2,2,2)	(R_4,R_4,R_4,S_4)	0.0318	0.3749	0.950	0.0263	0.3027	0.954	0.0428	0.0384
	(R_5,R_5,R_5,S_5)	0.0314	0.3769	0.951	0.0267	0.3051	0.953	0.0420	0.0381
	(R_6,R_6,R_6,S_6)	0.0316	0.3755	0.950	0.0260	0.3074	0.953	0.0429	0.0387
(5,5,5,5,4,4,4)	(R_1,R_1,R_1,S_1)	0.0509	0.5567	0.940	0.0452	0.5090	0.946	0.0546	0.0509
	(R_2,R_2,R_2,S_2)	0.0508	0.5537	0.942	0.458	0.5060	0.947	0.0547	0.0507
	(R_3,R_3,R_3,S_3)	0.0509	0.5591	0.941	0.0453	0.5017	0.946	0.0549	0.0502
(5,5,5,10,4,4,4)	(R_1,R_1,R_1,S_4)	0.0429	0.4662	0.944	0.0376	0.4215	0.949	0.0510	0.0482
	(R_2,R_2,R_2,S_5)	0.0436	0.4682	0.945	0.0370	0.4281	0.948	0.0517	0.0486
	(R_3,R_3,R_3,S_6)	0.0423	0.4631	0.945	0.0376	0.4237	0.949	0.0518	0.0480
(10,10,10,5,4,4,4)	(R_4,R_4,R_4,S_1)	0.0370	0.4037	0.948	0.0308	0.3751	0.951	0.0486	0.0430
	(R_5,R_5,R_5,S_2)	0.0379	0.4085	0.948	0.0300	0.3744	0.950	0.0489	0.0438
	(R_6,R_6,R_6,S_3)	0.0368	0.4069	0.949	0.0309	0.3792	0.950	0.0488	0.0439
(10,10,10,10,4,4,4)	(R_4,R_4,R_4,S_4)	0.0310	0.3794	0.951	0.0260	0.3146	0.952	0.0437	0.0380
	(R_5,R_5,R_5,S_5)	0.0312	0.3755	0.951	0.0267	0.3156	0.953	0.0439	0.382
	(R_6,R_6,R_6,S_6)	0.0319	0.3791	0.950	0.0269	0.3174	0.952	0.0440	0.0389

TABLE 4. Simulation results in general case.

$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	MCMC					
		Prior 5			Prior 6		
		MSE	AL	CP	MSE	AL	CP
(5,5,5,5,2,2,2)	(R_1,R_1,R_1,S_1)	0.0661	0.6050	0.942	0.0615	0.5193	0.946
	(R_2,R_2,R_2,S_2)	0.0654	0.6027	0.942	0.0618	0.5160	0.947
	(R_3,R_3,R_3,S_3)	0.0669	0.6033	0.943	0.0613	0.5177	0.946
(5,5,5,10,2,2,2)	(R_1,R_1,R_1,S_4)	0.0631	0.5299	0.945	0.0583	0.4377	0.949
	(R_2,R_2,R_2,S_5)	0.0637	0.5291	0.946	0.0579	0.4339	0.948
	(R_3,R_3,R_3,S_6)	0.0641	0.5278	0.945	0.0573	0.4380	0.949
(10,10,10,5,2,2,2)	(R_4,R_4,R_4,S_1)	0.0542	0.4830	0.949	0.0502	0.3920	0.950
	(R_5,R_5,R_5,S_2)	0.0538	0.4891	0.948	0.0509	0.3966	0.950
	(R_6,R_6,R_6,S_3)	0.0540	0.4866	0.948	0.0500	0.3940	0.951
(10,10,10,10,2,2,2)	(R_4,R_4,R_4,S_4)	0.0499	0.4311	0.950	0.0456	0.3246	0.953
	(R_5,R_5,R_5,S_5)	0.0498	0.4390	0.951	0.0450	0.3292	0.952
	(R_6,R_6,R_6,S_6)	0.0503	0.4374	0.951	0.0451	0.3277	0.953
(5,5,5,5,4,4,4)	(R_1,R_1,R_1,S_1)	0.0675	0.6113	0.943	0.0638	0.5266	0.946
	(R_2,R_2,R_2,S_2)	0.0670	0.6190	0.942	0.0630	0.5299	0.947
	(R_3,R_3,R_3,S_3)	0.0681	0.6187	0.942	0.0633	0.5266	0.945
(5,5,5,10,4,4,4)	(R_1,R_1,R_1,S_4)	0.0649	0.5388	0.945	0.0606	0.4449	0.948
	(R_2,R_2,R_2,S_5)	0.0650	0.5391	0.945	0.0609	0.4490	0.949
	(R_3,R_3,R_3,S_6)	0.0643	0.5374	0.946	0.0600	0.4467	0.949
(10,10,10,5,4,4,4)	(R_4,R_4,R_4,S_1)	0.0540	0.4733	0.947	0.0497	0.3953	0.951
	(R_5,R_5,R_5,S_2)	0.0549	0.4719	0.948	0.0483	0.3991	0.950
	(R_6,R_6,R_6,S_3)	0.0547	0.4766	0.948	0.0487	0.3978	0.951
(10,10,10,10,4,4,4)	(R_4,R_4,R_4,S_4)	0.0483	0.4394	0.950	0.0452	0.3277	0.953
	(R_5,R_5,R_5,S_5)	0.0487	0.4363	0.951	0.0450	0.3270	0.953
	(R_6,R_6,R_6,S_6)	0.0480	0.4385	0.950	0.0459	0.3266	0.954

divided by 4552000 acre-foot, total capacity of reservoir. We note that by this work the statistical inference has not been changed yet.

Now, we fit EW distribution on the these data sets, separately. The results are as follows. For X_1, $(\widehat{\alpha},\widehat{\theta},\widehat{\lambda}) = (3.3011, 11.1081, 0.2110)$ and the p-value= 0.9774. For X_2, $(\widehat{\alpha},\widehat{\theta},\widehat{\lambda}) = (4.8625, 9.1577, 0.2206)$ and the p-value= 0.6597. For X_3, $(\widehat{\alpha},\widehat{\theta},\widehat{\lambda}) = (6.2784, 9.7498, 0.1915)$ and the p-value= 0.3190. For Y, $(\widehat{\alpha},\widehat{\theta},\widehat{\lambda}) = (11.7338, 7.1837, 1.7953)$ and the p-value= 0.3340. The p-values show that EW provides adequate fits on data sets. Also, from the estimated parameters, we can consider the general case, only. The empirical distribution functions and PP-plots, for these data sets, are given in Figure 3. For

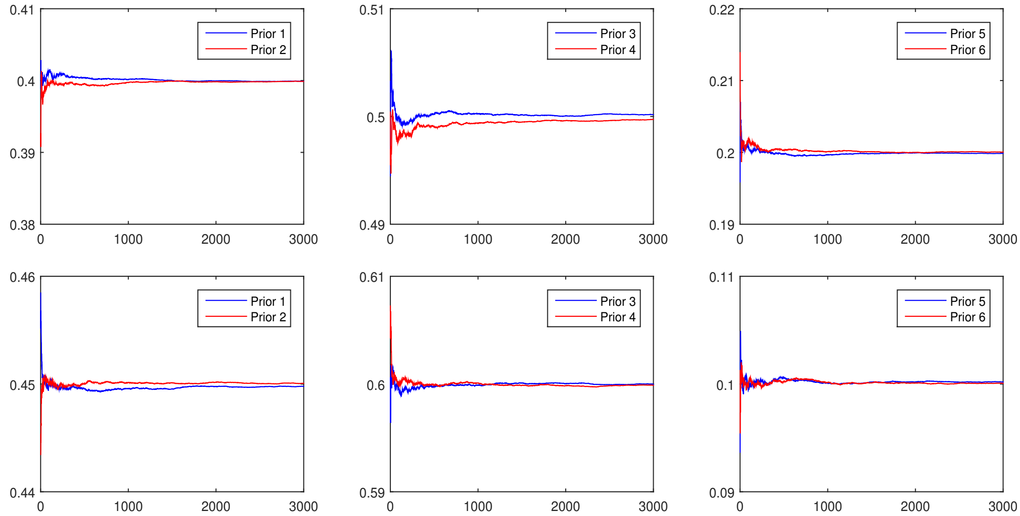

FIGURE 2. Trace plot for $R_{\boldsymbol{s},\boldsymbol{k}}$ in first line: $(\boldsymbol{k},\boldsymbol{s}) = (5,2)$, left $((R_1,R_1,R_1,S_1)$, case 1), center $((R_2,R_2,R_2,S_2)$, case 2) and right $((R_3,R_3,R_3,S_3)$, case 3). Second line: $(\boldsymbol{k},\boldsymbol{s}) = (10,4)$, left $((R_4,R_4,R_4,S_4)$, case 1), center $((R_5,R_5,R_5,S_5)$, case 2) and right $((R_6,R_6,R_6,S_6)$, case3).

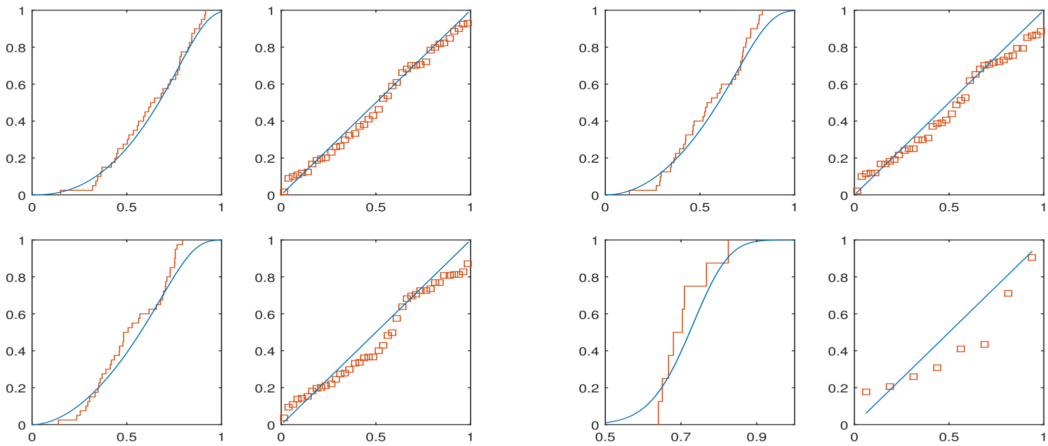

FIGURE 3. Empirical distribution function (left) and the PP-plot (right) for X_1 (left, top), X_2 (right, top), X_3 (left, bottom) and Y (right, bottom).

complete data set, by putting $\boldsymbol{s} = (2,2)$ and $\boldsymbol{k} = (5,5)$ with non-informative priors, $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding 95% HPD interval are 0.2387 and $(0.0811, 0.4377)$, respectively. MCMC diagnostics in the case of real data is provided. For this aim, the trace plot, auto-correlation plot of MCMC chains and the density plot of the posterior distribution of $R_{\boldsymbol{s},\boldsymbol{k}}$, for complete data in case $\boldsymbol{s} = (2,2)$ and non-informative priors, is presented in Figure 4. From this figure, the trace plot shows the convergence of the MCMC algorithm. Also, the auto-correlation plot shows the chain achieves stationarity and the density plots are symmetric and unimodal. Hence, we conclude that the MCMC chain is converged. Moreover, we monitor the overall convergence of MCMC chains in all cases of the real data, and these are omitted for the sake of brevity. Now, two different censoring progressive schemes are generated as follows:

Scheme 1: $R^{(1)} = R^{(2)} = R^{(3)} = [0, 0, 1, 0],\ S = [0, 0, 1, 0, 0, 0, 0],\ (\boldsymbol{k} = (4, 4), \boldsymbol{s} = (2, 2))$.

Scheme 2: $R^{(1)} = R^{(2)} = R^{(3)} = [1, 0, 1],\ S = [0, 1, 1, 0, 0, 0],\ (\boldsymbol{k} = (3, 3), \boldsymbol{s} = (2, 2))$.

For Scheme 1, with non-informative priors, $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding 95% HPD interval are obtained as 0.0940 and $(0.0724, 0.1812)$, respectively. For Scheme 2, with non-informative priors, $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding 95% HPD interval are obtained as 0.0332 and $(0.0100, 0.0653)$, respectively. With comparing point and interval estimates, we can see that Scheme 1 performs better than Scheme 2, as we expected.

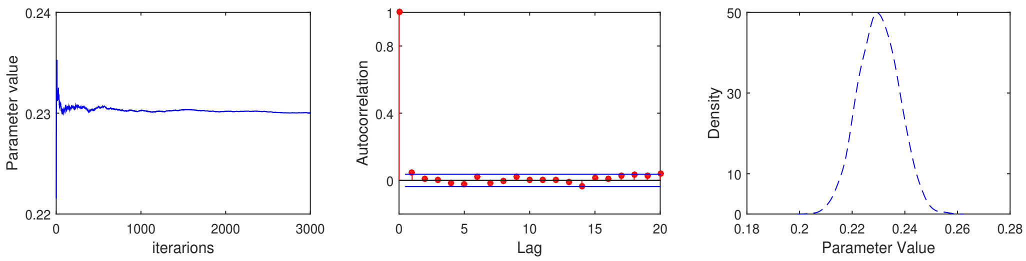

FIGURE 4. MCMC diagnostics for $R_{\boldsymbol{s},\boldsymbol{k}}$ in real data case.

6. Conclusion

We considered the Bayesian inference for multi-component stress-strength parameter with the non-identical-component strengths under the progressive censoring scheme, in exponentiated Weibull distribution. When the common parameters are unknown, known and in the general case, point and interval Bayesian estimations are obtained.

The Monte Carlo simulation study is employed to compare the different estimations. Using this method, we observed that the non-informative priors perform worse than informative ones, in point and interval estimates. Moreover, by increasing n, for fixed $\boldsymbol{s}$ and $\boldsymbol{k}$, and by increasing k, for fixed $\boldsymbol{s}$ and n, the performance of point and interval estimates improved. Two above results may occur due to the fact that, by increasing n, the number of failures increases and therefore, more information is gathered.

This work has the potential to be applied in the context of reliability theory and censored data analysis. So, further scope of this work can be conducted by extending multi-component stress-strength to a multi-component multi-stress-strength model. Also, the Bayesian inference can also arise in the multi-component reliability model, based on record values, in the presence of progressive censoring samples.

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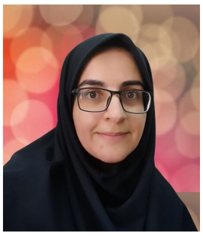

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