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ON CAPUTO-FABRIZIO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we established some new results concerning the existence, uniqueness and stability results for a nonlinear Caputo-Fabrizio fractional periodic Volterra-Fredholm integro-differential equation with the initial condition and boundary conditions via successive approximations method and Banach fixed point theorem. Examples are included for the illustration of the obtained results.

Keywords: Volterra-Fredholm integro-differential equation, Caputo-Fabrizio approach, successive approximations approach, Banach contraction, fixed-point theorem.

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1. Introduction

The subject of fractional order boundary value problems has been addressed by many researchers in recent years. The interest in the subject owes to its extensive applications in natural and social sciences. A few examples include chaos and fractional dynamics [1], financial economics [2], ecology [3], bioengineering [4], etc. In [5, 6], boundary value problems involving fractional derivatives of the Caputo, Riemann-Liouville, and Hadamard types, supplied with a range of boundary conditions, provide a number of intriguing findings. Riemann-Liouville, Erdelyi-Kober, Grunwald-Letnikov, Weyl, Hadamard, Riesz, and Caputo are a few of the definitions that we discuss here. A noteworthy characteristic of a fractional order differential operator emerged in its hereditary quality as compared to an integer order. Put differently, we forecast the future state of a process by its previous and present states when we describe it using a fractional operator [7]. However, the new definition suggested by Caputo and Fabrizio [8], which has all the characteristics of the old definitions, assumes two different representations for the temporal and spatial variables. They claimed that the classical definition given by Caputo appears to be particularly convenient for mechanical phenomena, related to plasticity, fatigue, damage, and with electromagnetic hysteresis. The main advantage of the Caputo-Fabrizio derivative

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(C-FD) is that the boundary conditions of the fractional differential equations with C-FD admit the same form as for the integer-order differential equations. On the other hand, the fractional C-FD has many significant properties, such as its ability in describing matter heterogeneities and configurations with different scales [9, 10].

The last ten years have seen the recognition of fractional differential equations as crucial instruments for describing mathematical modelling of processes in a variety of disciplines, including aerodynamics, physics, chemistry, engineering, statistics, control theory, signal and image processing, etc. [11, 12]. However, we see periodic movements everywhere in reality and in every branch of research [13]. Readers are referred to [14, 15, 16] for the latest work on the theory and applications of fractional differential equations for a range of issues by these scholars.

Due to their prevalence in several applied domains, including fractional power law [17] and heat transfer phenomena [18], integral-differential equations represent a significant research topic. Numerous scholars have also examined fractional integro-differential equations in conjunction with various boundary conditions, as demonstrated by [19, 20, 21, 22, 23, 24, 25, 26].

The analytic solutions of a viscous fluid with the fractional derivatives of Caputo and Caputo-Fabrizio are found in [27]. The authors of [28] modelled a Maxwell fluid using the fractional derivative with a nonsingular kernel and discovered semi-analytical solutions. Atangana-Baleanu and Caputo-Fabrizio, two of the most recent fractional derivatives models for a generalised Casson fluid, were compared in [29], and precise solutions were discovered. The existence of solutions for nonlinear differential equations has been investigated using various nonlinear analytic techniques because of the aforementioned applications [26, 30, 31].

In this paper, we investigate the existence, uniqueness and stability results of periodic solution of the following nonlinear fractional Volterra-Fredholm integro-differential equation

$$ {}_0^{CF}D_\gamma^\alpha(x(\gamma)) = h\left(\gamma, x(\gamma), \int_0^{a(\gamma)} \vartheta_1(\nu, x(\nu))d\nu, \int_0^T \vartheta_2(\nu, x(\nu))d\nu\right), \tag{1} $$

$$ x(0) = x_0,\ \gamma \in J := [0, T], \tag{2} $$

$$ x(0) = x(T) + \int_0^T H(x(\nu))d\nu, \tag{3} $$

where ${}_0^{\mathrm{CF}}D_\gamma^\alpha$ is the Caputo-Fabrizio fractional derivative ($\alpha \in (0,1]$), $\vartheta_1, \vartheta_2 \in C\left(J \times D_1, \mathbb{R}\right)$, D_1 be compact subset of $\mathbb{R}$, $a(\gamma)$ is continuous functions on $[0, T]$, the function $H(x(\nu)\ )$ is compact subset of R and periodic on γ of periodic T. We extend Picard's theorem to this problem, and by the successive approximation method, an iterative process is provided to obtain the periodic solution.

For considering the main Eq. (1), as

$$ {}_0^{CF}D_0^\alpha(x(0)) = 0. $$

Here, extra conditions have to be imposed to guarantee the existence of a solution, so we refer to Lemmas 3.1, 3.2 and 4.1 in [32], also, see Lemma 3.4 in [33]. This paper will discuss the existence, uniqueness, and stability results of non-trivial solution.

Studying this type of problems has attracted a special interest of many mathematicians due to the various applications of fractional differential equations in several bioengineering [4] fields. One of the main advantages of the Caputo-Fabrizio fractional derivative is the ability to combine all traditional fractional derivatives, and it satisfies the semigroup property, hence, generalized Caputo fractional derivative derivative is considered a generalized form of fractional derivatives.

2. Preliminaries

In this section, we recall some notations and definitions which are needed throughout this paper. Further, some lemmas and theorems are stated as preparations for the main results [8, 33, 34, 35, 36, 37, 38].

First, in the following, we provide some basic concepts and definitions in connection with the new Caputo-Fabrizio derivative.

Let $H^1(a,b) = \{\vartheta \mid \vartheta \in L^2(a,b), \vartheta' \in L^2(a,b)\}$, where $L^2(a,b)$ is the space of square integrable functions on the interval (a,b).

Definition 2.1. [11] *For a function $\vartheta : (0,\infty) \to \mathbb{R}$, the Caputo derivative of order $\alpha > 0$ of g is defined by*

$$ {}_0^{\gamma}D^{\alpha}\vartheta(\gamma) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\gamma} (\gamma-\nu)^{n-\alpha-1} \vartheta^{(n)}(\nu) d\nu \tag{4} $$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α, and $\Gamma(.)$ denotes the Gamma function, i.e., $\Gamma(z) = \int_0^{\infty} e^{-\gamma}\gamma^{z-1} d\gamma$.

Definition 2.2. [11] *Let ϑ be a function which is defined almost everywhere a.e on $[a,b]$, for $\alpha > 0$, we define*

$$ {}_a^{b}D^{-\alpha}\vartheta = \frac{1}{\Gamma(\alpha)} \int_a^{b} (b-\gamma)^{a-1} \vartheta(\gamma) d\gamma \tag{5} $$

provided that the integral (Lebesgue) exists.

Definition 2.3. [8] *Let ϑ be a given function in $H^1(a,b)$. The Caputo-Fabrizio derivative of fractional order $\alpha \in (0,1)$ is defined as*

$$ {}_a^{CF}D_{\gamma}^{\alpha}(\vartheta(\gamma)) = \left(\frac{N(\alpha)}{1-\alpha}\right) \int_a^{\gamma} \vartheta'(x) \exp\left[-\alpha\frac{\gamma-x}{1-\alpha}\right] dx, \tag{6} $$

where $N(\alpha)$ is a normalization function. Also, if a certain function ϑ does not satisfy in the restriction $\vartheta \in H^1(a,b)$, then its fractional derivative is redefined as

$$ {}_a^{CF}D_{\gamma}^{\alpha}(\vartheta(\gamma)) = \frac{\alpha N(\alpha)}{1-\alpha} \int_a^{\gamma} (\vartheta(\gamma)-\vartheta(x)) \exp\left[-\alpha\frac{\gamma-x}{1-\alpha}\right] dx. \tag{7} $$

Clearly, if one sets $\sigma = (1-\alpha)/\alpha \in (0,\infty)$ and $\alpha = 1/(1+\sigma) \in (0,1)$, then the CaputoFabrizio definition becomes

$$ {}_a^{CF}D_{\gamma}^{e}(\vartheta(\gamma)) = \frac{N(\sigma)}{\sigma} \int_a^{\gamma} \vartheta'(x) \exp\left[-\frac{\gamma-x}{\sigma}\right] dx, \tag{8} $$

where $N(0) = N(\infty) = 1$, and

$$ \lim_{\sigma\to 0} \exp\left[-\frac{\gamma-x}{\sigma}\right] = \delta(x-\gamma). \tag{9} $$

Also, the fractional derivative of order $(n+\alpha)$ when $n \geq 1$ and $\alpha \in [0,1]$ is defined by the following

$$ {}_a^{CF}D_{\gamma}^{(a+n)}(\vartheta(\gamma)) = \alpha^{CF}D_{\gamma}^{(a)}\left(D_{\gamma}^{(n)}\vartheta(\gamma)\right) \tag{10} $$

Definition 2.4. [8] *Let $\vartheta \in H^1(a,b)$, then its fractional integral of an arbitrary order is defined as follows:*

$$_{\alpha}C^{x}_{\gamma}(\vartheta(\gamma)) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}\vartheta(\gamma) + \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_a^{\gamma} \vartheta(\nu)d\nu, \gamma \geq 0. \tag{11}$$

It is clear, in view of the above definition, that the α th Caputo-Fabrizio derivative of function g is average between g and its first-order integral. Therefore,

$$\frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(\alpha)} = 1. \tag{12}$$

So, we arrive at the following

$$N(\alpha) = \frac{2}{2-\alpha}, 0 \leq \alpha \leq 1. \tag{13}$$

Lemma 2.1. *The periodic solution of the fractional integro-differential equation (1), with initial condition $x(0) = x_0$ and periodic boundary condition $x(0) = x(T)$ are defining the following integral equation*

$$\begin{aligned} & x(\gamma, x_0) \\ &= x_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h\left(\gamma, x(\gamma), \int_0^{a(\gamma)} \vartheta_1(\nu, x(\nu))d\nu, \int_0^T \vartheta_2(\nu, x(\nu))d\nu\right) \\ &- \left(\frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \frac{1}{\mathrm{T}} \int_0^T h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right) d\nu\right) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^{\gamma} \Bigg(h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right) \\ &- \frac{1}{\mathrm{T}} \int_0^T h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right) d\nu\Bigg)d\nu, \ \forall \ \gamma \in J. \end{aligned} \tag{14}$$

Proof. The proof is simple and can be derived as it is obtained from the original equation by applying the integral operator. .

Lemma 2.2. [38] *Let $\vartheta(\gamma)$ be a vector function which is defined in the interval $0 \leq \gamma \leq T$, then:*

$$\left|\int_0^{\gamma} \left(\vartheta(\nu) - \frac{1}{T}\int_0^T \vartheta(\nu)d\nu\right) d\nu\right| \leq \beta(\gamma)M$$

where $M = \max_{\gamma\in[0,T]} |\vartheta(\gamma)|$ and $\beta(\gamma) = 2\gamma\left(1 - \frac{\gamma}{T}\right), \max_{\gamma\in[0,T]} |\beta(\gamma)| \leq \frac{\mathrm{T}}{2}$.

Proof. *The proof follows directly from the estimate:*

$$\begin{aligned} \left|\int_0^{\gamma} \left(\vartheta(\nu) - \frac{1}{T}\int_0^T \vartheta(\nu)d\nu\right) d\nu\right| &\leq \left(1 - \frac{\gamma}{T}\right)\int_0^{\gamma} |\vartheta(\nu)|d\nu + \frac{\gamma}{T}\int_{\gamma}^T |\vartheta(\nu)|d\nu \\ &\leq \beta(\gamma)M. \end{aligned}$$

Theorem 2.1. [11] *(Banach fixed point theorem). Let $(B, \|.\|)$ be a Banach space and $P : B \to B$ be a contraction mapping i.e. Lipschitz continuous with Lipschitz constant $L \in [0,1)$. Then $\varphi \in B$ has a unique fixed point.*

3. Main Results

Some conditions are needed for investigate of the successive approximation for periodic solution of the system (1)-(2), suppose that the functions $h \in C([0,T] \times D_1 \times D_2 \times D_2, \mathbb{R})$, $\vartheta_1, \vartheta_2 \in C([0,T] \times D_1, \mathbb{R})$, D_1 and D_2 are compact subset of $\mathbb{R}, a(\gamma)$ is continuous functions on $[0,T]$, moreover define $|\cdot| = \max_{\gamma\in[0,T]} |\cdot|$, and satisfies the following hypothesis.

H_1: There exist positive constants $M, L_1, L_2, k_1, k_2, k_3$, and $L_{\vartheta_1}, L_{\vartheta_2}$, such that

$$|h(\gamma, x, y, z)| \leq M, \tag{15}$$

$$|\vartheta_1(\gamma, x)| \leq L_1, \tag{16}$$

$$|\vartheta_2(\gamma, x)| \leq L_2, \tag{17}$$

$$|h(\gamma, x_1, y_1, z_1) - h(\gamma, x_2, y_2, z_2)| \leq k_1 |x_1 - x_2| + k_2 |y_1 - y_2| + k_3 |z_1 - z_2|, \tag{18}$$

$$|\vartheta_1(\gamma, x_1) - \vartheta_1(\gamma, x_2)| \leq L_{\vartheta_1} |x_1 - x_2|, \tag{19}$$

$$|\vartheta_2(\gamma, x_1) - \vartheta_2(\gamma, x_2)| \leq L_{\vartheta_2} |x_1 - x_2|, \tag{20}$$

where $z_i^1 = \int_0^{a(\gamma)} \vartheta_1(\nu, x_i(\nu))\, d\nu$, $z_i^2 = \int_0^T \vartheta_2(\nu, x_i(\nu))\, d\nu$, for all $\gamma \in [0,T]$, $\quad x, x_1, x_2, \in D_1$ and $y_i, z_i \in D_2, \quad i = 1,2.$

H_2 : There exist positive constants a_T, such that for $\gamma \in [0,T]$,

$$|a(\gamma)| \leq a_T. \tag{21}$$

Define the non-empty set

$$\mathrm{D_h} = \mathrm{D_1} - M_1 \tag{22}$$

where

$$M_1 = \left(2(1-\alpha) + \frac{\alpha T}{2}\right) M.$$

Furthermore, we suppose that the following condition is valid:

$$\Lambda = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3) < 1. \tag{23}$$

Our main results separate to the following parts:

3.1. **Approximation of Periodic Solution.**

In this section, we study the periodic approximation solutions of the system (1)-(2). In the beginning, we define the following sequence of functions $\{\mathrm{x_{m+1}}\}_{m=0}^{\infty}$ given by the iterative formulas

$$\begin{aligned} & x_{m+1}(\gamma, x_0) \\ & = x_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h\left(\gamma, x_m(\gamma), \int_0^{a(\gamma)} \vartheta_1(\nu, x_m(\nu))\, d\nu, \int_0^T \vartheta_2(\nu, x_m(\nu))\, d\nu\right) \\ & - \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \frac{1}{T} \int_0^T h\left(\nu, x_m(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x_m(\tau))\, d\tau, \int_0^T \vartheta_2(\tau, x_m(\tau))\, d\tau\right) d\nu \\ & + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^\gamma \Bigg(h\left(\nu, x_m(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x_m(\tau))\, d\tau, \int_0^T \vartheta_2(\tau, x_m(\tau))\, d\tau\right) \\ & - \frac{1}{T} \int_0^T h\left(\nu, x_m(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x_m(\tau))\, d\tau, \int_0^T \vartheta_2(\tau, x_m(\tau))\, d\tau\right) d\nu\Bigg) d\nu\Bigg) d\nu \end{aligned} \tag{24}$$

For all $\gamma \in J, x_0(\gamma) = x_0, m = 0, 1, 2, \ldots$, then will be introduced by the following theorems.

Theorem 3.1. *If the nonlinear fractional integro-differential equations* (1)-(2) *satisfy the conditions* H_1*, and* H_2*, then the sequence of functions* (24)*, which are periodic in* γ *of period* T*, converges uniformly as* $m \to \infty$ *on the domain:-*

$$(\gamma, \mathrm{x}_0) \in [0, T] \times D_1 \tag{25}$$

to the limit functions x_θ *defined on the domain* (25) *which is periodic in* γ *of period* T *and satisfies the following integral equations:*

$$\begin{aligned}
&x(\gamma, x_0) \\
&= \mathrm{x}_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h\left(\gamma, x(\gamma), \int_0^{a(\gamma)} \vartheta_1(\nu, x(\nu))d\nu, \int_0^T \vartheta_2(\nu, x(\nu))d\nu\right) \\
&- \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \frac{1}{\mathrm{T}} \int_0^T h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right) d\nu \\
&+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^\gamma \Big(h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right) \\
&- \frac{1}{\mathrm{T}} \int_0^T h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right) d\nu\Big) d\nu,
\end{aligned} \tag{26}$$

on the domain (25)*, provided that*

$$|x(\gamma, x_0) - x_{m+1}(\gamma, x_0)| \leq \Lambda^{\mathrm{m}} (\mathrm{E} - \Lambda)^{-1} \mathrm{M}_1, \tag{27}$$

for all $\mathrm{m} \geq 0, \mathrm{x}_0 \in D$*, and* $\gamma \in \mathrm{J}$.

Proof. Setting $m = 0$ in the sequence of functions (24) and by using Lemma 2.2, we have

$$\begin{aligned}
|x_1(\gamma, x_0) - x_0| &\leq \left(\frac{4(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(\alpha)}\beta(\gamma)\right) M \\
&\leq \left(2(1-\alpha) + \frac{\alpha T}{2}\right) M = M_1,
\end{aligned}$$

for all $\gamma \in [0, T], x_0 \in D_h$ we get $x_1(\gamma, x_0) \in D_1$. Thus by mathematical induction, we find that

$$|x_m(\gamma, x_0) - x_0| \leq M_1 \tag{28}$$

mean that for all $\gamma \in [0, T], x_0 \in D_h$ we get $x_m(\gamma, x_0) \in D_1, m = 0, 1, 2, \ldots$

Now, we claim that the sequences of functions (24) are uniformly convergent on the domain (25). By the inequalities (18)-(21), we obtain

$$\begin{aligned}
|x_{m+1}(\gamma, x_0) - x_m(\gamma, x_0)| &\leq \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3) \\
&\quad |x_m(\gamma,, x_0) - x_{m-1}(\gamma,, x_0)| \\
&= \Lambda |x_m(\gamma,, x_0) - x_{m-1}(\gamma_r, x_0)|.
\end{aligned} \tag{29}$$

By mathematical induction, we obtain that

$$|x_{m+1}(\gamma, \mathrm{x}_0) - x_m(\gamma, \mathrm{x}_0)| \leq \Lambda^{\mathrm{m}} |x_1(\gamma, \mathrm{x}_0) - \mathrm{x}_0|. \tag{30}$$

Now from $m = 1, 2, \ldots$ and $p \geq 1$, we find that

$$|x_{m+p}(\gamma, \mathrm{x}_0) - x_m(\gamma, \mathrm{x}_0)| \leq \Lambda^{\mathrm{m}}(1-\Lambda)^{-1}\left(\left(2(1-\alpha) + \frac{\alpha T}{2}\right) M\right) \\ \leq \Lambda^{\mathrm{m}}(1-\Lambda)^{-1} M_1, \tag{31}$$

for all $\gamma \in [0,\ \mathrm{T}], x_0 \in \mathrm{D_h}$.

Since $\Lambda = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3) < 1$ and $\lim_{m\to\infty} \Lambda^{\mathrm{m}} = 0$, so that the right side of (31) tends to zero. Therefore the sequence of functions $x_m(\gamma, x_0), m = 1, 2, 3, \ldots$ is converges uniformly on the domain (25) to the limit function $\mathrm{x}(\gamma, \mathrm{x}_0)$ which is defined on the same domain. Let

$$\lim_{m\to\infty} x_m(\gamma, x_0) = \mathrm{x}_\theta(\gamma, x_0). \tag{32}$$

Since the sequence of functions (24) are periodic in γ of period T, then the limiting function $\mathrm{x}_\theta(\gamma, \mathrm{x}_0)$ is also periodic in γ of period T. By using the relation (32) and proceeding in (24) to limit, when $m \to \infty$, it is converging that the limiting function $x(\gamma, x_0)$ is the periodic solution of the integral equation (26).

Theorem 3.2. *If all assumptions of the Theorem 3.1 are satisfy, then* $\mathrm{x}(\gamma, \mathrm{x}_0)$ *is a unique solution of the system* (1)-(2).

Proof. Assume that $\hat{x}(\gamma, x_0)$ is another solution of the system (1)-(2), as follows

$$\begin{aligned} &\hat{x}(\gamma, x_0) \\ &= \mathrm{x}_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h\left(\gamma, \hat{x}(\gamma), \int_0^{a(\gamma)} \vartheta_1(\nu, \hat{x}(\nu))d\nu, \int_0^T \vartheta_2(\nu, \hat{x}(\nu))d\nu\right) \\ &- \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \frac{1}{\mathrm{T}} \int_0^T h\left(\nu, \hat{x}(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, \hat{x}(\tau))d\tau, \int_0^T \vartheta_2(\tau, \hat{x}(\tau))d\tau\right) d\nu \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^\gamma \left(h\left(\nu, \hat{x}(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, \hat{x}(\tau))d\tau, \int_0^T \vartheta_2(\tau, \hat{x}(\tau))d\tau\right)\right. \\ &\left. - \frac{1}{T} \int_0^T h\left(\nu, \hat{x}(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, \hat{x}(\tau))d\tau, \int_0^T \vartheta_2(\tau, \hat{x}(\tau))d\tau\right) d\nu\right) d\nu. \end{aligned} \tag{33}$$

Now, the difference between the two solutions $x(\gamma, \mathrm{x}_0)$ and $\hat{\mathrm{x}}(\gamma, \mathrm{x}_0)$, for all $\gamma \in [0,\ \mathrm{T}]$ and $\mathrm{x}_0 \in \mathrm{D_h}$, hence, by the inequalities (18)-(21), we get

$$\begin{aligned} &|\mathrm{x}(\gamma, \mathrm{x}_0) - \hat{\mathrm{x}}(\gamma, \mathrm{x}_0)| \\ &\leq \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_Y L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3) |\mathrm{x}(\gamma, \mathrm{x}_0) - \hat{\mathrm{x}}(\gamma, \mathrm{x}_0)| \\ &\leq \Lambda |\mathrm{x}(\gamma_r \mathrm{x}_0) - \hat{\mathrm{x}}(\gamma, \mathrm{x}_0)| \quad (4.11) \end{aligned} \tag{34}$$

By mathematical induction, we find that

$$|\mathrm{x}(\gamma, \mathrm{x}_0) - \hat{\mathrm{x}}(\gamma, \mathrm{x}_0)| \leq \Lambda^{\mathrm{m}} |\mathrm{x}(\gamma, \mathrm{x}_0) - \hat{\mathrm{x}}(\gamma, \mathrm{x}_0)| \tag{35}$$

From the condition (23), shows that the solution $\mathrm{x}(\gamma, \mathrm{x}_0) = \hat{\mathrm{x}}(\gamma, \mathrm{x}_0)$, thus $\mathrm{x}(\gamma, \mathrm{x}_0)$ is a unique periodic solution on the domain (25).

3.2. **Existence of Periodic Solutions of the system** (1)-(2)**.** The problem of the existence of the periodic solution for the system (1)-(2) is uniquely connected with the existence of the zeros of the functions:-

$$\mu(0,\mathrm{x}_0)=\frac{1}{\mathrm{T}}\int_0^T h\left(\nu,x(\nu),\int_0^{a(\nu)}\vartheta_1(\tau,x(\tau))d\tau,\int_0^T\vartheta_2(\tau,x(\tau))d\tau\right)d\nu. \quad (36)$$

Also, we define the sequences of functions $\mu_m(0,x_0)$ are approximately determined by the following:

$$\mu_{\mathrm{m}}(0,\mathrm{x}_0)=\frac{1}{\mathrm{T}}\int_0^T h\left(\nu,x_m(\nu),\int_0^{a(\nu)}\vartheta_1(\tau,x_m(\tau))\,d\tau,\int_0^T\vartheta_2(\tau,x_m(\tau))\,d\tau\right)d\nu. \quad (37)$$

Theorem 3.3. *If the hypotheses and all the conditions of the theorem 3.1 are given, the following inequalities are satisfied:-*

$$|\mu(0,\mathrm{x}_0)-\mu_{\mathrm{m}}(0,\mathrm{x}_0)|\le(k_1+a_TL_{\vartheta_1}k_2+TL_{\vartheta_2}k_3)\,\Lambda^{\mathrm{m}}(1-\Lambda)^{-1}M_1 \quad (38)$$

holds for all $\mathrm{m}\ge0$.

Proof. From equations (36) to (37), we obtain that

$$\begin{aligned}|\mu(0,\mathrm{x}_0)-\mu_m(0,\mathrm{x}_0)|&\le(k_1+a_TL_{\vartheta_1}k_2+TL_{\vartheta_2}k_3)\,|x(\gamma,\mathrm{x}_0)-x_m(\gamma,\mathrm{x}_0)|\\&\le(k_1+a_rL_{\vartheta_1}k_2+TL_{\vartheta_2}k_3)\,\Lambda^{\mathrm{m}}(1-\Lambda)^{-1}M_1.\end{aligned} \quad (39)$$

The inequality (38) is hold for all $\mathrm{m}\ge0$.

Theorem 3.4. *Let the function $h(\nu,x(\nu),z(\gamma))$ be defined on the intervals $[c,d]$ on R and periodic in γ of period T, suppose that for all $\mathrm{m}\ge0$, then the sequences of the functions $\mu_{\mathrm{m}}(0,x_0)$ which are defined in* (37) *satisfy the inequalities:-*

$$\left.\begin{aligned}&\min_{x_0\in[c,d]}\mu_{\mathrm{m}}(0,\mathrm{x}_0)\le-(k_1+a_\tau L_{\vartheta_1}k_2+TL_{\vartheta_2}k_3)\,\Lambda^{\mathrm{m}}(1-\Lambda)^{-1}M_1\\&\max_{x_0\in[c,d]}\mu_{\mathrm{m}}(0,\mathrm{x}_0)\ge(k_1+a_rL_{\vartheta_1}k_2+TL_{\vartheta_2}k_3)\,\Lambda^{\mathrm{m}}(1-\Lambda)^{-1}M_1.\end{aligned}\right\} \quad (40)$$

Then the system (1) *has a periodic solution* $\mathrm{x}(\gamma,\mathrm{x}_0)$ *such that* $\mathrm{x}_0\in[\mathrm{c},\mathrm{d}]=[\mathrm{c}+M_1,\ \mathrm{d}-M_1]$.

Proof. Let x_1 and x_2 be any points belonging to the intervals $[c,d]$, such that

$$\begin{aligned}\mu_{\mathrm{m}}(0,\mathrm{x}_1)&=\min_{x_0\in[c,d]}\mu_{\mathrm{m}}(0,\mathrm{x}_0)\\\mu_{\mathrm{m}}(0,\mathrm{x}_2)&=\max_{x_0\in[c,d]}\mu_{\mathrm{m}}(0,\mathrm{x}_0).\end{aligned} \quad (41)$$

By using inequalities (38) to (41), the following are obtained:-

$$\left.\begin{aligned}\mu(0,\mathrm{x}_1)=\mu_{\mathrm{m}}(0,\mathrm{x}_1)+(\mu(0,\mathrm{x}_1)-\mu_{\mathrm{m}}(0,\mathrm{x}_1))<0\\\mu(0,\mathrm{x}_2)=\mu_{\mathrm{m}}(0,\mathrm{x}_2)+(\mu(0,\mathrm{x}_2)-\mu_m(0,\mathrm{x}_2))>0\end{aligned}\right\} \quad (42)$$

and from the continuity of the functions $\mu(0,x_1),\mu(0,x_2)$ and the inequalities (42), then the isolated singular points $x^0\in[c,d]$ exist such that $\mu\left(0,x^0\right)=0$. This means that the system (1) has a periodic solution $\mathrm{x}(\gamma,\mathrm{x}_0)$.

3.3. **Stability of Periodic Solution of** (1)**.** In this section, we investigate the stability or periodic solution of (1).

Theorem 3.5. *Let the function $\mu(0,\mathrm{x}_0)$ be defined by the equation* (36) *where $\mathrm{x}(\gamma,\mathrm{x}_0)$ is a limit of the sequence of the function* (24)*, then the following inequalities yield:*

$$|\mu(0,x_0)|\le M \quad (43)$$

and

$$\left|\mu\left(0, x_0^1\right)-\mu\left(0, x_0^2\right)\right| \leq F_2 F_3\left|x_0^1-x_0^2\right|, \tag{44}$$

where

$$F_1=2(1-\alpha)+\frac{\alpha T}{2}, \quad F_2=k_1+a_T L_{\vartheta_1} k_2+T L_{\vartheta_2} k_3, \quad F_3=\left(1-F_1 F_2\right)^{-1}.$$

Proof. From the properties of the function $x\left(\gamma, x_0\right)$ as in the Theorem 3.1, the function $\mu\left(0, \mathrm{x}_0\right), x_0 \in D$ is continuous and bounded by $\frac{1-\alpha}{\alpha T} \vartheta^T+M$ in the domain (25). From (36), we obtained that

$$\left|\mu\left(0, \mathrm{x}_0\right)\right| \leq \frac{1}{\mathrm{T}} \int_0^T\left|h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau)) d \tau, \int_0^T \vartheta_2(\tau, x(\tau)) d \tau\right)\right| d \nu \leq M. \tag{45}$$

Next, from inequality (36), we get

$$\begin{aligned} \mid \mu\left(0, x_0^1\right)-\mu\left(0, x_0^2\right) \mid & \leq\left(k_1+a_T L_{\vartheta_1} k_2+T L_{\vartheta_2} k_3\right) \mid x\left(\gamma, x_0^1\right)-x\left(\gamma, x_0^2\right) \mid \\ & \leq F_2\left|x\left(\gamma, x_0^1\right)-x\left(\gamma, x_0^2\right)\right|, \end{aligned} \tag{46}$$

where the functions $x\left(\gamma, x_0^1\right)$ and $x\left(\gamma, x_0^2\right)$ are solutions of the integral equation:-

$$\begin{aligned} & x\left(\gamma, \mathrm{x}_0^{\mathrm{k}}\right) \\ & =\mathrm{x}_0^{\mathrm{k}}+\frac{2(1-\alpha)}{(2-\alpha) N(\alpha)} h\left(\gamma, u\left(\gamma, \mathrm{x}_0^{\mathrm{k}}\right), \int_0^{a(\gamma)} \vartheta_1\left(\nu, x\left(\nu, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \nu, \int_0^T \vartheta_2\left(\nu, x\left(\nu, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \nu\right) \\ & -\frac{2(1-\alpha)}{(2-\alpha) N(\alpha)} \frac{1}{\mathrm{T}} \int_0^T h\left(\nu, x\left(\nu, \mathrm{x}_0^{\mathrm{k}}\right), \int_0^{a(\nu)} \vartheta_1\left(\tau, x\left(\tau, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \tau, \int_0^T \vartheta_2\left(\tau, x\left(\tau, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \tau\right) d \nu \\ & +\frac{2 \alpha}{(2-\alpha) N(\alpha)} \int_0^\gamma\left(h\left(\nu, x\left(\nu, \mathrm{x}_0^{\mathrm{k}}\right), \int_0^{a(\nu)} \vartheta_1\left(\tau, x\left(\tau, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \tau, \int_0^T \vartheta_2\left(\tau, x\left(\tau, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \tau\right)\right. \\ & \left.-\frac{1}{\mathrm{T}} \int_0^T h\left(\nu, x\left(\nu, \mathrm{x}_0^{\mathrm{k}}\right), \int_0^{a(\nu)} \vartheta_1\left(\tau, x\left(\tau, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \tau, \int_0^T \vartheta_2\left(\tau, x\left(\tau, \mathrm{x}_0^{\mathrm{k}}\right)\right) d \tau\right) d \nu\right) d \nu, \end{aligned} \tag{47}$$

where $\mathrm{k}=1,2$, from (47), we get

$$\begin{aligned} & \left|\mathrm{x}\left(\gamma, \mathrm{x}_0^1\right)-\mathrm{x}\left(\gamma, x_0^2\right)\right| \\ & \leq\mid \mathrm{x}_0^1-x_0^2]+\frac{4(1-\alpha)}{(2-\alpha) N(a)}\left(k_1+a_T L_{\vartheta_1} k_2+T L_{\vartheta_2} k_3\right)\left|\mathrm{x}\left(\gamma, \mathrm{x}_0^1\right)-\mathrm{x}\left(\gamma, x_0^2\right)\right| \\ & +\frac{\alpha T}{(2-\alpha) N(\alpha)}\left(k_1+a_T L_{\vartheta_1} k_2+T L_{\vartheta_2} k_3\right)\left|\mathrm{x}\left(\gamma, \mathrm{x}_0^1\right)-\mathrm{x}\left(\gamma, x_0^2\right)\right|. \end{aligned} \tag{48}$$

Therefore, we obtain that

$$\begin{aligned} \left|\mathrm{x}\left(\gamma, \mathrm{x}_0^1\right)-\mathrm{x}\left(\gamma, x_0^2\right)\right| & \leq\left|x_0^1-x_0^2\right|+F_1 F_2\left|u\left(\gamma, x_0^1\right)-u\left(\gamma, x_0^2\right)\right| \\ & \leq\left|\mathrm{x}_0^1-x_0^2\right|+F_1 F_2\left|\mathrm{x}\left(\gamma, \mathrm{x}_0^1\right)-\mathrm{x}\left(\gamma, x_0^2\right)\right|. \end{aligned} \tag{49}$$

From equations (49), we have

$$\left|\mathrm{x}\left(\gamma, \mathrm{x}_0^1\right)-\mathrm{x}\left(\gamma, x_0^2\right)\right| \leq F_3\left|\mathrm{x}_0^1-x_0^2\right|. \tag{50}$$

Substitutes (50) in (46), we get that (44).

Remark 3.1. [29]. *Theorem 6 confirms the stability of the solution of the system* (1)*, when a slight change happens in the points* x_0*, then a slight change will happen in the function* $\mu\left(0, x_0\right)$.

3.4. **Existence and uniqueness of periodic Solution of the system** (1)-(3)**.** In this section, we investigate the periodic solution of the system (1)-(3).

Theorem 3.6. *All assumptions of the Theorem 3.1 are satisfy, and the function $H(x(\nu))$ satisfies*

$$|H(x_1) - H(x_2)| \leq L_2 |x_1 - x_2| \tag{51}$$

then the system (1)-(3) *has unique solution if*

$$Q = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3) + \frac{(1-\alpha+\alpha T)}{\alpha} L_2 < 1. \tag{52}$$

Proof. We define an operator $P : C[0,T] \to C[0,T]$

$$\begin{aligned}
&P(x(\gamma)) \\
&= \mathrm{x}_0 - \frac{(1-\alpha+\alpha\gamma)}{\alpha T}\int_0^T H(x(\nu))d\gamma + \frac{2(1-\alpha)}{(2-\alpha)N(a)} \\
&\times \left[h\left(\gamma, x(\gamma), \int_0^{a(\gamma)} \vartheta_1(\nu, x(\nu))d\nu, \int_0^T \vartheta_2(\nu, x(\nu))d\nu\right)\right. \\
&\left. - \frac{1}{\mathrm{T}}\int_0^T h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right)d\nu\right] + \\
&+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^\gamma \left(h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right)\right. \\
&\left. - \frac{1}{T}\int_0^T h\left(\nu, x(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x(\tau))d\tau, \int_0^T \vartheta_2(\tau, x(\tau))d\tau\right)d\nu\right)d\nu.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&|P(x(\gamma)) - P(w(\gamma))| \\
&= \left(\left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3) + \frac{(1-\alpha+\alpha T)}{\alpha} L_2\right)|x(\gamma) - w(\gamma)|
\end{aligned}$$

From (52), the operator P satisfies contraction mapping, hence the system (1)-(3) has unique solution.

Theorem 3.7. *If the hypotheses and all the conditions of the theorem 3.1 and the inequality* (51) *are given, the following inequalities are satisfied:-*

$$|\sigma(0, \mathrm{x}_0) - \sigma_{\mathrm{m}}(0, \mathrm{x}_0)| \leq \left(k_1 + a_\Upsilon L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3 + \frac{L_1}{\alpha}\right) Q^{\mathrm{m}}(1-Q)^{-1} M_3, \tag{53}$$

where

$$\begin{aligned}
\sigma_{\mathrm{m}}(0, \mathrm{x}_0) = &\frac{1}{\mathrm{T}}\int_0^T h\left(\nu, x_m(\nu), \int_0^{a(\nu)} \vartheta_1(\tau, x_m(\tau))\, d\tau, \int_0^T \vartheta_2(\tau, x_m(\tau))\, d\tau\right) d\nu \\
&+ \frac{(2-\alpha)\mathrm{N}(\alpha)}{2\alpha\mathrm{T}}\int_0^T H(x_m(\nu))\, d\nu,
\end{aligned} \tag{54}$$

holds for all $\mathrm{m} \geq 0$*, here*

$$M_3 = M_1 + \frac{(1-\alpha+\alpha T)}{a} M_2 \tag{55}$$

and

$$M_2 \geq |H(x(\gamma))|. \tag{56}$$

Proof. The proof of this theorem is direct.

Theorem 3.8. *Let the functions $h(\nu, x(\nu), y(\gamma), z(\gamma))$ and $H(x(\gamma))$ be defined on the intervals $[c_1, d_1]$ on R and periodic in γ of period T, suppose that for all $\mathrm{m} \geq 0$, then the sequences of the functions $\sigma_{\mathrm{m}}(0, \mathrm{x}_0)$ which are defined in (54) satisfy the inequalities:-*

$$\left.\begin{array}{l} \min_{x_0\in[c_1,d_1]} \sigma_m(0,x_0) \leq -\left(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3 + \frac{L_2}{\alpha}\right) Q^m (1-Q)^{-1} M_3 \\ \max_{x_0\in[c_1,d_1 1} \sigma_m(0,x_0) \geq \left(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3 + \frac{L_2}{\alpha}\right) Q^m (1-Q)^{-1} M_3. \end{array}\right\} \tag{57}$$

Then the system (1)-(3) *has a periodic solution such that $x_0 \in [c_1 + M_3, d_1 - M_3]$ where M_3 defined in* (55).

Proof. This theorem's proof was similar to that of theorem 3.4.

Theorem 3.9. *Let the function $\sigma(0, \mathrm{x}_0)$ be defined by the equations* (54)*, then the following inequalities yield:-*

$$|\sigma(0, \mathrm{x}_0)| \leq M + \frac{M_2}{\alpha}, \tag{58}$$

and

$$\left|\sigma\left(0, x_0^1\right) - \sigma\left(0, x_0^2\right)\right| \leq E_2 E_3 \left|x_0^1 - x_0^2\right|, \tag{59}$$

where

$$E_1 = 2(1-\alpha) + \frac{\alpha T}{2}, \quad E_2 = k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3 + \frac{L_1}{\alpha}, \quad E_3 = \left(1 - \mathrm{E}_1 E_2\right)^{-1}.$$

Proof. The proof of this theorem was similar to the proof of theorem 3.5.

4. Examples

In this section contains two example to illustrate the previous theorems.

Example 1. Consider the following fractional integro-differential equation

$$\begin{aligned} &{}_0^{CF}D_\gamma^{0.7}(x(\gamma)) \\ &= \frac{1}{e^\gamma + 5} x(\gamma) + \int_0^{\gamma^2} \frac{1}{2(\nu+2)^3} \sin(x(\nu)) d\nu + \int_0^2 \frac{1}{2(\nu+2)^4} \cos(x(\nu)) d\nu, \end{aligned} \tag{60}$$

$$x(0) = 1, \tag{61}$$

where ${}_0^{CF}D_\gamma^\alpha$ denotes the fractional Caputo-Fabrizio derivative ($\alpha = 0.7$).
Here $T = 2, a(\gamma) = \gamma^2$,

$$h(\gamma, x(\gamma), y(\gamma), z(\gamma)) = \frac{1}{e^\gamma + 5} x(\gamma) + \int_0^{\gamma^2} \frac{1}{2(\nu+2)^3} \sin(x(\nu)) d\nu + \int_0^2 \frac{1}{2(\nu+2)^4} \cos(x(\nu)) d\nu,$$

$$\vartheta_1(\gamma, x(\gamma)) = \frac{1}{2(\nu+2)^3} \sin(x(\nu))$$

$$\vartheta_2(\gamma, x(\gamma)) = \frac{1}{2(\nu+2)^4} \cos(x(\nu)).$$

We obtain that $k_1 = 0.2, k_2 = 1, k_3 = 1, a_T = 4, L_{\vartheta_1} = 0.0625, L_{\vartheta_2} = 0.0312$, so that

$$\Lambda = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)\left(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_2} k_3\right) = 0.666 < 1$$

.

Therefore, by Theorem 3.1 and Theorem 3.2, the system (60)-(61) has exactly one periodic solution.

Example 2. Consider the following fractional integro-differential equation with integral boundary conditions

$$
\begin{aligned}
&{}_{0}^{CF}D_{\gamma}^{0.7}(x(\gamma))\\
&= \frac{1}{e^{\gamma}+5}x(\gamma) + \int_0^{\gamma^2} \frac{1}{2(\nu+2)^3}\sin(x(\nu))d\nu + \int_0^1 \frac{1}{2(\nu+2)^4}\cos(x(\nu))d\nu, \qquad (62)\\
&x(0) - x(1) = \int_0^1 \frac{1}{5}\cos(x(\gamma))d\gamma, \qquad (63)
\end{aligned}
$$

such that $\gamma \in (0,1]$, where ${}_{0}^{CF}D_{\gamma}^{\alpha}$ denotes the fractional Caputo-Fabrizio derivative ($\alpha = 0.7$). We obtain that $T = 1, k_1 = 0.2, k_2 = 1, k_3 = 1, a_T = 4, L_{\vartheta_1} = 0.0625, L_{\vartheta_2} = 0.0312$,

$$
h(\gamma, x(\gamma), y(\gamma), z(\gamma)) = \frac{1}{e^{\gamma}+5}x(\gamma) + \int_0^{\gamma^2} \frac{1}{2(s+2)^3}\sin(x(\nu))d\nu + \int_0^1 \frac{1}{2(\nu+2)^4}\cos(x(\nu))d\nu,
$$

$$
\vartheta_1(\gamma, x(\gamma)) = \frac{1}{2(\nu+2)^3}\sin(x(\nu))
$$

$$
\vartheta_2(\gamma, x(\gamma)) = \frac{1}{2(\nu+2)^4}\cos(x(\nu)),
$$

and

$$
H(x(\gamma)) = \frac{1}{2}\cos(x(\gamma)),
$$

we obtain that $a_T = 1$, and $L_2 = 0.2$, so that

$$
\mathrm{Q} = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_{\vartheta_1} k_2 + T L_{\vartheta_1} k_3) + \frac{(1-\alpha+\alpha T)}{\alpha} L_2 = 0.667 < 1.
$$

Therefore, by Theorem 3.6, the boundary value system (62)-(63) has exactly one periodic solution.

5. Conclusions

Caputo-Fabrizio FD, a general fractional operator, is of great use because of its wide freedom to cover many classical fractional operators. We have studied a fractional integro-differential equation involving Caputo-Fabrizio fractional derivative type nonlinearities together with the initial condition, periodic boundary conditions, and integral boundary conditions. In fact, we considered a more general situation by considering the fractional order nonlinear integral terms in the integro-differential equation at hand. Under appropriate assumptions, the existence, uniqueness, and stability results for the given system are proved by applying the standard tools of the fixed point theory and successive approximations technique. The results obtained in this paper are not only new, but they also lead to some new results associated with the particular choices of the parameters involved in the system. Thus, the work presented in this paper significantly contributes to the existing literature on the topic.

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