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FIXED POINT THEOREMS ON S-METRIC SPACES FOR GENERALIZED F-CONTRACTION AND APPLICATION

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ABSTRACT. Under the context of complete S-metric spaces, we establish certain fixed point theorems for generalized F-contraction of Khan type [4], Meir-Keeler type [10], and Meir-Keeler-Khan type [9]. We also provide some corollaries of the obtained findings. We also present an instance scenario to validate the findings. A novel fixed-circle solution is presented as an application on S-metric spaces through the use of generalized modified C-Khan type F-contraction. The findings presented in this work expand upon, broaden, and enhance a number of earlier findings from the body of current literature.

Keywords: Fixed point, fixed-circle, F-contraction, generalized F-contraction, S-metric space.

AMS Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The Banach contraction mapping principle (BCP), a significant metric fixed point result concerning a contraction mapping, was established by Polish mathematician Banach in 1992 [1]. This idea is widely regarded as one of the most notable outcomes in the field of analysis. This principle has been further examined and analysed through various methodologies. One of the employed methodologies involves the generalisation of the contraction conditions. Various contractions have been proposed in the literature, including the modified Hardy-Rogers type contraction [8], Khan type contraction [4], Meir-Keeler type contraction [10], and Meir-Keeler-Khan type contraction [9] (see references). The Banach contraction principle has been extensively examined by numerous mathematicians,

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who have applied it to various domains within mathematical disciplines such as approximation theory, game theory, quantum theory, economics, differential equations, and integral equations.

An alternative method involves the generalisation of the metric spaces utilised. The concept of a S-metric space was proposed as an extension of the metric space, with the aim of generalising its properties.

Definition 1.1. *([19]) Let $\Gamma \neq \emptyset$ be a set and let $\tilde{S}\colon \Gamma^3 \to [0,\infty)$ be a function satisfying the following conditions for all $\zeta, \eta, \theta, \vartheta \in \Gamma$:*

$(S1)$ $\tilde{S}(\zeta,\eta,\theta) = 0$ if and only if $\zeta = \eta = \theta$;

$(S2)$ $\tilde{S}(\zeta,\eta,\theta) \leq \tilde{S}(\zeta,\zeta,\vartheta) + \tilde{S}(\eta,\eta,\vartheta) + \tilde{S}(\theta,\theta,\vartheta)$.

Then the function $\tilde{S}$ is called an S-metric on Γ and the pair $(\Gamma,\tilde{S})$ is called an S-metric space (in short SMS).

Immediate examples of such S-metric spaces are as follows.

Example 1.1. *([19])*

(1) Let $\Gamma = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Γ, then $\tilde{S}(\zeta,\eta,\theta) = \|\eta + \theta - 2\zeta\| + \|\eta - \theta\|$ is an S-metric on Γ.

(2) Let $\Gamma = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Γ, then $\tilde{S}(\zeta,\eta,\theta) = \|\zeta - \theta\| + \|\eta - \theta\|$ is an S-metric on Γ.

Example 1.2. *([20]) Let $\Gamma = \mathbb{R}$ be the real line. Then $\tilde{S}(\zeta,\eta,\theta) = |\zeta - \theta| + |\eta - \theta|$ for all $\zeta,\eta,\theta \in \mathbb{R}$ is an S-metric on Γ. This S-metric on Γ is called the usual S-metric on Γ.*

Example 1.3. *([7]) Let $\Gamma \neq \emptyset$ be a set and d be an ordinary metric on Γ. Then $\tilde{S}(\zeta,\eta,\theta) = d(\zeta,\theta) + d(\eta,\theta)$ for all $\zeta,\eta,\theta \in \mathbb{R}$ is an S-metric on Γ.*

Example 1.4. *([21]) Let $\Gamma \neq \emptyset$ be a set and d_1, d_2 be two ordinary metrics on Γ. Then $\tilde{S}(\zeta,\eta,\theta) = d_1(\zeta,\theta) + d_2(\eta,\theta)$ for all $\zeta,\eta,\theta \in \Gamma$ is an S-metric on Γ.*

In the literature, there exist some examples of S-metric which is not generated by any metric (see, [6] and [13] for more details). Therefore it is important to investigate new fixed point theorems on S-metric spaces. Some fixed point results have been still established using different techniques to generalize some well-known fixed point theorems (for example, see [11], [12], [14] and [20] for more details).

Recently, a new approach is being studied to do geometric interpretations for fixed points which is called fixed-circle problem [15] (see, also [17], [18], [23]). The concept of a circle and a fixed circle were defined on S-metric spaces as follows:

Definition 1.2. *([18]) Let $(\Gamma,\tilde{S})$ be an S-metric space. Let $\mathcal{C}^{\tilde{S}}_{(x_0,r)} = \{\zeta \in \Gamma : \tilde{S}(\zeta,\zeta,x_0) = r\}$ be a circle centered at x_0 with radius r and let $\mathcal{T}\colon \Gamma \to \Gamma$ be a self-mapping. If $\mathcal{T}\zeta = \zeta$ for all $\zeta \in \mathcal{C}^{\tilde{S}}_{(x_0,r)}$, then the circle $\mathcal{C}^{\tilde{S}}_{(x_0,r)}$ is called as the fixed circle of $\mathcal{T}$.*

Using the above definitions, some fixed-circle results were obtained on S-metric spaces (see [16] and [18] for more details).

In [24], Wardowski introduced a new type of contraction called F-contraction and proved a new fixed point result regarding F-contraction. In this way, Wardowski [24] generalized the Banach contraction principle in a different manner from the well-known results in the literature.

Building upon the aforementioned research, the objective of this study is to develop fixed point theorems by employing various generalized F-contractive conditions inside the framework of S-metric spaces. In the second section of the paper, a novel fixed-point theorem is derived by employing the generalized modified Hardy-Rogers type F-contraction. Additionally, several well-established corollaries are developed. Section 3 of the paper presents a novel fixed-point theorem that is established by the utilisation of the generalized Khan type F-contraction. The theorem is accompanied by an illustrative case. The authors establish a fixed-point solution in Section 4 by employing the generalized Meir-Keeler-Khan type F-contraction. In Section 5, the obtained findings are applied to derive a new fixed-circle result on S-metric spaces. The findings of our study expand upon and provide a broader application of many findings already documented in the research literature.

Definition 1.3. *(*[19], [20]*) Let $(\Gamma, \tilde{S})$ be an S-metric space and $\mathcal{Z} \subset \Gamma$.*

(Δ_1) *The subset $\mathcal{Z}$ is said to be an open subset of Γ, if for every $\zeta \in \mathcal{Z}$ there exists $r > 0$ such that $B_{\tilde{S}}(\zeta, r) \subset \mathcal{Z}$.*

(Δ_2) *A sequence $\{\zeta_n\}$ in Γ converges to $\zeta \in \Gamma$ if $\tilde{S}(\zeta_n, \zeta_n, \zeta) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\tilde{S}(\zeta_n, \zeta_n, \zeta) < \varepsilon$. We denote this by $\lim_{n\to\infty} \zeta_n = \zeta$ or $\zeta_n \to \zeta$ as $n \to \infty$.*

(Δ_3) *A sequence $\{\zeta_n\}$ in Γ is called a Cauchy sequence if $\tilde{S}(\zeta_n, \zeta_n, \zeta_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $\tilde{S}(\zeta_n, \zeta_n, \zeta_m) < \varepsilon$.*

(Δ_4) *The S-metric space $(\Gamma, \tilde{S})$ is called complete if every Cauchy sequence in Γ is convergent in Γ.*

(Δ_5) *Let τ be the set of all $\mathcal{Z} \subset \Gamma$ with $\zeta \in \mathcal{Z}$ and there exists $r > 0$ such that $B_{\tilde{S}}(\zeta, r) \subset \mathcal{Z}$. Then τ is a topology on Γ (induced by the S-metric space).*

(Δ_6) *A nonempty subset $\mathcal{Z}$ of Γ is S-closed if closure of $\mathcal{Z}$ coincides with $\mathcal{Z}$.*

Definition 1.4. *(*[19]*) Let $(\Gamma, \tilde{S})$ and $(\Gamma', \tilde{S}')$ be two S-metric spaces. A function $f\colon \Gamma \to \Gamma'$ is said to be continuous at a point $\zeta_0 \in \Gamma$ if for every sequence $\{\zeta_n\}$ in Γ with $\tilde{S}(\zeta_n, \zeta_n, \zeta_0) \to 0$, $\tilde{S}'(f(\zeta_n), f(\zeta_n), f(\zeta_0)) \to 0$ as $n \to \infty$. We say that f is continuous on Γ if f is continuous at every point $\zeta_0 \in \Gamma$.*

In [24], Wardowski introduced a new concept of F-contraction on a complete metric space as follows.

Definition 1.5. *(*[24]*) Let $F\colon \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*

$(F1)$ F is strictly increasing, that is, $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$.

$(F2)$ For every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

$(F3)$ There exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In what follows, $\mathcal{F}$ stands for the family of all functions F which satisfies the above three conditions.

Let $F_1(\alpha) = ln(\alpha)$, $F_2(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_3(\alpha) = \alpha + ln(\alpha)$ for $\alpha > 0$, then $F_1, F_2, F_3 \in \mathcal{F}$.

Definition 1.6. *(*[2]*) Let $(\Gamma, \tilde{S})$ be an S-metric space. A mapping $\mathcal{T}\colon \Gamma \to \Gamma$ is said to be a F-contraction if there exists a number $\tau > 0$ such that*

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\eta, \mathcal{T}\theta) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\eta, \mathcal{T}\theta)) \leq F(\tilde{S}(\zeta, \eta, \theta)),$$

for all $\zeta, \eta, \theta \in \Gamma$.

Remark 1.1. *Clearly Definition 1.6 and $(F1)$ implies that $\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\eta, \mathcal{T}\theta) < \tilde{S}(\zeta, \eta, \theta)$ for all $\zeta, \eta, \theta \in \Gamma$ with $\mathcal{T}\zeta \neq \mathcal{T}\eta \neq \mathcal{T}\theta$. Hence every F-contraction mapping is continuous.*

We need the following lemmas in the sequel.

Lemma 1.1. *([19], Lemma 2.5) Let $(\Gamma, \tilde{S})$ be an S-metric space. Then, we have $\tilde{S}(\zeta, \zeta, \eta) = \tilde{S}(\eta, \eta, \zeta)$ for all $\zeta, \eta \in \Gamma$.*

Lemma 1.2. *([19], Lemma 2.12) Let $(\Gamma, \tilde{S})$ be an S-metric space. If $\zeta_n \to \zeta$ and $\zeta'_n \to \zeta'$ as $n \to \infty$ then $\tilde{S}(\zeta_n, \zeta_n, \zeta'_n) \to \tilde{S}(\zeta, \zeta, \zeta')$ as $n \to \infty$.*

Lemma 1.3. *([20]) The limit of the sequence $\{\zeta_n\}$ in an S-metric space $(\Gamma, \tilde{S})$ is unique.*

Lemma 1.4. *([19]) Let $(\Gamma, \tilde{S})$ be an S-metric space. Then the convergent sequence $\{\zeta_n\}$ in Γ is Cauchy.*

In the following lemma we see the relationship between a metric and S-metric.

Lemma 1.5. *([6])*
Let (Γ, d) be a metric space. Then the following properties are satisfied:
$(C1)$ $\tilde{S}_d(\zeta, \eta, \theta) = d(\zeta, \theta) + d(\eta, \theta)$ for all $\zeta, \eta, \theta \in \Gamma$ is an S-metric on Γ.
$(C2)$ $\zeta_n \to \zeta$ in (Γ, d) if and only if $\zeta_n \to \zeta$ in $(\Gamma, \tilde{S}_d)$.
$(C3)$ $\{\zeta_n\}$ is Cauchy in (Γ, d) if and only if $\{\zeta_n\}$ is Cauchy in $(\Gamma, \tilde{S}_d)$.
$(C4)$ (Γ, d) is complete if and only if $(\Gamma, \tilde{S}_d)$ is complete.

We call the function $\tilde{S}_d$ defined in Lemma 1.5 $(C1)$ as the S-metric generated by the metric d. It can be found an example of an S-metric which is not generated by any metric in [6, 13].

On the other hand, *Gupta* [5] claimed that Every S-metric on Γ defines a metric d_S on Γ as follows:

$$d_S(\zeta, \eta) = \tilde{S}(\zeta, \zeta, \eta) + \tilde{S}(\eta, \eta, \zeta),$$

for all $\zeta, \eta \in \Gamma$. However, the function $d_S(\zeta, \eta)$ defined as above does not always define a metric because a triangular inequality is not satisfied for all elements of Γ everywhere (see [13] for more details).

Corollary 1.1. *([20]) Let $\mathcal{T} \colon \Gamma \to \Gamma'$ be a map from an S-metric space Γ to an S-metric space Γ'. Then $\mathcal{T}$ is continuous at $\zeta \in \Gamma$ if and only if $\mathcal{T}\zeta_n \to \mathcal{T}\zeta$ whenever $\zeta_n \to \zeta$ as $n \to \infty$.*

In 2018, Tas [22] introduced the following definitions.

Modified Hardy-Rogers-Type Contraction

Let $(\Gamma, \tilde{S})$ be an S-metric space. A mapping $\mathcal{T} \colon \Gamma \to \Gamma$ is called modified Hardy-Rogers-type contraction if $\mathcal{T}$ satisfies the following condition:

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta) \leq \nu(\zeta, \zeta, \eta),$$

for all $\zeta, \eta \in \Gamma$, where

$$\begin{aligned} \nu(\zeta, \zeta, \eta) &= \lambda_1 \tilde{S}(\zeta, \zeta, \eta) + \lambda_2 \tilde{S}(\zeta, \zeta, \mathcal{T}\eta) + \lambda_3 \tilde{S}(\eta, \eta, \mathcal{T}\zeta) \\ &\quad + \lambda_4 \frac{\tilde{S}(\eta, \eta, \mathcal{T}\eta)[1 + \tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)]}{1 + \tilde{S}(\zeta, \zeta, \eta)} \\ &\quad + \lambda_5 \frac{\tilde{S}(\eta, \eta, \mathcal{T}\eta) + \tilde{S}(\eta, \eta, \mathcal{T}\zeta)}{1 + \tilde{S}(\eta, \eta, \mathcal{T}\eta)\tilde{S}(\eta, \eta, \mathcal{T}\zeta)} \\ &\quad + \lambda_6 \frac{\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)[1 + \tilde{S}(\eta, \eta, \mathcal{T}\zeta)]}{1 + \tilde{S}(\zeta, \zeta, \eta) + \tilde{S}(\eta, \eta, \mathcal{T}\eta)}, \end{aligned}$$

and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0$ with $\lambda_1 + 3\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 < 1$.

Khan-Type Contraction

Let $(\Gamma, \tilde{S})$ be an S-metric space. A mapping $\mathcal{T}\colon \Gamma \to \Gamma$ is called Khan-type contraction if $\mathcal{T}$ satisfies the following condition:

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta) \leq \xi(\zeta, \zeta, \eta),$$

for all $\zeta, \eta \in \Gamma$, where

$$\xi(\zeta,\zeta,\eta) = \begin{cases} h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\eta,\eta,\mathcal{T}\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta)}, & \text{if } \tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta) \neq 0, \\ 0, & \text{if } \tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta) = 0, \end{cases}$$

and $h \in [0, 1)$.

Meir-Keeler-Khan-Type Contraction

Let $(\Gamma, \tilde{S})$ be an S-metric space and let $\mathcal{T}\colon \Gamma \to \Gamma$ be a self-mapping. Then $\mathcal{T}$ is called a Meir-Keeler-Khan-Type contraction whenever for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \varepsilon &\leq h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\eta,\eta,\mathcal{T}\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta)} < \varepsilon + \delta(\varepsilon) \\ &\Rightarrow \tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta) < \varepsilon, \end{aligned}$$

where $h \in [0, 1)$.

Notice that if $\mathcal{T}$ is a Meir-Keeler-Khan-Type contraction on Γ, then we get

$$\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta) \leq h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\eta,\eta,\mathcal{T}\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta)}.$$

2. A fixed point theorem for generalized modified Hardy-Rogers-type F-contractive condition

In this section, we shall prove a new fixed point theorem using the generalized modified Hardy-Rogers-type F-contractive condition on complete S-metric spaces. First, we give the following definition.

Definition 2.1. *Let $(\Gamma, \tilde{S})$ be an S-metric space. A mapping $\mathcal{T}\colon \Gamma \to \Gamma$ is called generalized modified Hardy-Rogers type F-contraction if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that*

$$\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta)) \leq F(\nu(\zeta,\zeta,\eta)), \tag{1}$$

for all $\zeta, \eta \in \Gamma$, where

$$\begin{aligned} \nu(\zeta,\zeta,\eta) &= \lambda_1\tilde{S}(\zeta,\zeta,\eta) + \lambda_2\tilde{S}(\zeta,\zeta,\mathcal{T}\eta) + \lambda_3\tilde{S}(\eta,\eta,\mathcal{T}\zeta) \\ &\quad +\lambda_4\frac{\tilde{S}(\eta,\eta,\mathcal{T}\eta)[1+\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)]}{1+\tilde{S}(\zeta,\zeta,\eta)} \\ &\quad +\lambda_5\frac{\tilde{S}(\eta,\eta,\mathcal{T}\eta)+\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{1+\tilde{S}(\eta,\eta,\mathcal{T}\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)} \\ &\quad +\lambda_6\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)[1+\tilde{S}(\eta,\eta,\mathcal{T}\zeta)]}{1+\tilde{S}(\zeta,\zeta,\eta)+\tilde{S}(\eta,\eta,\mathcal{T}\eta)}, \end{aligned}$$

and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0$ with $\lambda_1 + 3\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 < 1$.

Theorem 2.1. *Let $(\Gamma, \tilde{S})$ be a complete S-metric space and let $\mathcal{T}\colon \Gamma \to \Gamma$ be a continuous self-mapping. If $\mathcal{T}$ satisfies the generalized modified Hardy-Rogers type F-contractive condition (1). Then $\mathcal{T}$ has a unique fixed point in Γ.*

Proof. Let $\zeta_0 \in \Gamma$ be an arbitrary point. We define the sequence $\{\zeta_n\}$ as follows:

$$\mathcal{T}\zeta_n = \zeta_{n+1}.$$

Assume that $\zeta_n \neq \zeta_{n+1}$ for all n. Set $\Theta_n = \tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})$. Now, using the generalized modified Hardy-Rogers type F-contractive condition (1), we have

$$\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1}) \;=\; \tilde{S}(\mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_n).$$

This implies that

$$\begin{aligned}
\tau + F(\Theta_n) &= \tau + F(\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})) \\
&= \tau + F\big(\tilde{S}(\mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_n)\big) \\
&\leq F\big(\nu(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)\big), \qquad (2)
\end{aligned}$$

where

$$\begin{aligned}
\nu(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) &= \lambda_1 \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + \lambda_2 \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \mathcal{T}\zeta_n) + \lambda_3 \tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_{n-1}) \\
&\quad + \lambda_4 \frac{\tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_n)[1 + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \mathcal{T}\zeta_{n-1})]}{1 + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)} \\
&\quad + \lambda_5 \frac{\tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_n) + \tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_{n-1})}{1 + \tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_n)\tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_{n-1})} \\
&\quad + \lambda_6 \frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \mathcal{T}\zeta_{n-1})[1 + \tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_{n-1})]}{1 + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + \tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_n)} \\
&= \lambda_1 \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + \lambda_2 \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_{n+1}) + \lambda_3 \tilde{S}(\zeta_n, \zeta_n, \zeta_n) \\
&\quad + \lambda_4 \frac{\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})[1 + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)]}{1 + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)} \\
&\quad + \lambda_5 \frac{\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1}) + \tilde{S}(\zeta_n, \zeta_n, \zeta_n)}{1 + \tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})\tilde{S}(\zeta_n, \zeta_n, \zeta_n)} \\
&\quad + \lambda_6 \frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)[1 + \tilde{S}(\zeta_n, \zeta_n, \zeta_n)]}{1 + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + \tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})} \\
&= (\lambda_1 + 2\lambda_2)\Theta_{n-1} + \lambda_2 \Theta_n + \lambda_4 \Theta_n + \lambda_5 \Theta_n + \lambda_6 \frac{\Theta_{n-1}}{1 + \Theta_{n-1} + \Theta_n} \\
&\leq (\lambda_1 + 2\lambda_2)\Theta_{n-1} + \lambda_2 \Theta_n + \lambda_4 \Theta_n + \lambda_5 \Theta_n + \lambda_6 \Theta_{n-1} \\
&= (\lambda_1 + 2\lambda_2 + \lambda_6)\Theta_{n-1} + (\lambda_2 + \lambda_4 + \lambda_5)\Theta_n.
\end{aligned}$$

Using the above value in equation (2), we obtain

$$\tau + F(\Theta_n) \;\leq\; F\Big((\lambda_1 + 2\lambda_2 + \lambda_6)\Theta_{n-1} + (\lambda_2 + \lambda_4 + \lambda_5)\Theta_n\Big). \qquad (3)$$

Since F is strictly increasing (by $(F1)$), we deduce that

$$\Theta_n < (\lambda_1 + 2\lambda_2 + \lambda_6)\Theta_{n-1} + (\lambda_2 + \lambda_4 + \lambda_5)\Theta_n,$$

and hence

$$(1 - \lambda_2 - \lambda_4 - \lambda_5)\Theta_n < (\lambda_1 + 2\lambda_2 + \lambda_6)\Theta_{n-1}, \text{ for all } n \in \mathbb{N}.$$

From assumption of the theorem $\lambda_1+3\lambda_2+\lambda_4+\lambda_5+\lambda_6<1$, we deduce that $1-\lambda_2-\lambda_4-\lambda_5>0$ and so

$$\begin{aligned}\Theta_n &< \Big(\frac{\lambda_1+2\lambda_2+\lambda_6}{1-\lambda_2-\lambda_4-\lambda_5}\Big)\Theta_{n-1}\\ &= d\Theta_{n-1}<\Theta_{n-1},\end{aligned}\tag{4}$$

for all $n\in\mathbb{N}$, where

$$d=\Big(\frac{\lambda_1+2\lambda_2+\lambda_6}{1-\lambda_2-\lambda_4-\lambda_5}\Big)<1,$$

since by hypothesis $\lambda_1+3\lambda_2+\lambda_4+\lambda_5+\lambda_6<1$.

Consequently, we have

$$\tau+F(\Theta_n)<F(\Theta_{n-1}),\ \text{for all}\, n\in\mathbb{N}.\tag{5}$$

This implies that

$$F(\Theta_n)\le F(\Theta_{n-1})-\tau\le F(\Theta_{n-2})-2\tau\le\cdots\le F(\Theta_0)-n\tau.\tag{6}$$

Then, it follows that

$$\lim_{n\to\infty}F(\Theta_n)\le\lim_{n\to\infty}[F(\Theta_0)-n\tau].$$

This implies that

$$\lim_{n\to\infty}F(\Theta_n)<-\infty.$$

By $F\in\mathcal{F}$ and $(F2)$, we have

$$\lim_{n\to\infty}\Theta_n=\lim_{n\to\infty}\tilde{S}(\zeta_n,\zeta_n,\zeta_{n+1})=0.\tag{7}$$

Now, by $F\in\mathcal{F}$ and $(F3)$, there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}(\Theta_n)^kF(\Theta_n)=0.\tag{8}$$

Now, using equation (6), we have

$$(\Theta_n)^k[F(\Theta_n)-F(\Theta_0)]\le-n(\Theta_n)^k\tau\le0.\tag{9}$$

Taking the limit as $n\to\infty$ on both sides of equation (9) and using equation (8), we obtain

$$\lim_{n\to\infty}n(\Theta_n)^k=0=\lim_{n\to\infty}n(\tilde{S}(\zeta_n,\zeta_n,\zeta_{n+1}))^k.\tag{10}$$

Therefore, there exists a positive integer $N_1\in\mathbb{N}$ such that $n(\Theta_n)^k<1$ for all $n>N_1$, or

$$\Theta_n=\tilde{S}(\zeta_n,\zeta_n,\zeta_{n+1})<\frac{1}{n^{1/k}}.\tag{11}$$

Let $m,n\in\mathbb{N}$ with $m>n>N_1$. Then, we have

$$\begin{aligned}\tilde{S}(\zeta_m,\zeta_m,\zeta_n) &\le \Theta_{m-1}+\Theta_{m-2}+\cdots+\Theta_n\\ &\le \sum_{p=n}^{m-1}\Theta_p\le\sum_{p=n}^{\infty}\Theta_p\\ &= \sum_{p=n}^{\infty}\frac{1}{p^{1/k}}.\end{aligned}\tag{12}$$

As $k\in(0,1)$ and the series $\sum_{p=n}^{\infty}\frac{1}{p^{1/k}}$ is convergent, so

$$\lim_{m,n\to\infty}\tilde{S}(\zeta_m,\zeta_m,\zeta_n)=0.\tag{13}$$

Thus $\{\zeta_n\}$ is a Cauchy sequence in an S-metric space $(\Gamma, \tilde{S})$. Since $(\Gamma, \tilde{S})$ is complete, so there exists a $q \in \Gamma$ such that $\lim_{n\to\infty} \zeta_n = q$. Since by hypothesis $\mathcal{T}$ is continuous, we have

$$q = \lim_{n\to\infty} \zeta_{n+1} = \lim_{n\to\infty} \mathcal{T}\zeta_n = \mathcal{T}\big(\lim_{n\to\infty} \zeta_n\big) = \mathcal{T}q.$$

This implies that

$$\mathcal{T}q = q.$$

This shows that $\mathcal{T}$ has a fixed point in Γ. Now, we prove the uniqueness of fixed point. Assume that $r \in \Gamma$ is another fixed point of $\mathcal{T}$ such that $q \neq r$. This means that $\tilde{S}(q, q, r) > 0$. Taking $\zeta = q$ and $\eta = r$ in the contractive condition (1), using Lemma 1.1 and $(S1)$, we have

$$\begin{aligned} \tau + F(\tilde{S}(q, q, r)) &= \tau + F(\tilde{S}(\mathcal{T}q, \mathcal{T}q, \mathcal{T}r)) \\ &\leq F(\nu(q, q, r)), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \nu(q, q, r) &= \lambda_1 \tilde{S}(q, q, r) + \lambda_2 \tilde{S}(q, q, \mathcal{T}r) + \lambda_3 \tilde{S}(r, r, \mathcal{T}q) \\ &\quad + \lambda_4 \frac{\tilde{S}(r, r, \mathcal{T}r)[1 + \tilde{S}(q, q, \mathcal{T}q)]}{1 + \tilde{S}(q, q, r)} \\ &\quad + \lambda_5 \frac{\tilde{S}(r, r, \mathcal{T}r) + \tilde{S}(r, r, \mathcal{T}q)}{1 + \tilde{S}(r, r, \mathcal{T}r)\tilde{S}(r, r, \mathcal{T}q)} \\ &\quad + \lambda_6 \frac{\tilde{S}(q, q, \mathcal{T}q)[1 + \tilde{S}(r, r, \mathcal{T}q)]}{1 + \tilde{S}(q, q, r) + \tilde{S}(r, r, \mathcal{T}r)} \\ &= \lambda_1 \tilde{S}(q, q, r) + \lambda_2 \tilde{S}(q, q, r) + \lambda_3 \tilde{S}(r, r, q) \\ &\quad + \lambda_4 \frac{\tilde{S}(r, r, r)[1 + \tilde{S}(q, q, q)]}{1 + \tilde{S}(q, q, r)} \\ &\quad + \lambda_5 \frac{\tilde{S}(r, r, r) + \tilde{S}(r, r, q)}{1 + \tilde{S}(r, r, r)\tilde{S}(r, r, q)} \\ &\quad + \lambda_6 \frac{\tilde{S}(q, q, q)[1 + \tilde{S}(r, r, q)]}{1 + \tilde{S}(q, q, r) + \tilde{S}(r, r, r)} \\ &= (\lambda_1 + \lambda_2 + \lambda_3)\tilde{S}(q, q, r). \end{aligned}$$

Using this in equation (14), we obtain

$$\tau + F(\tilde{S}(q, q, r)) \leq F\big((\lambda_1 + \lambda_2 + \lambda_3)\tilde{S}(q, q, r)\big),$$

which is a contradiction, since $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and hence $q = r$. This shows that the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

If we take $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ in Theorem 2.1, then we have the following result.

Corollary 2.1. *Let $(\Gamma, \tilde{S})$ be a complete S-metric space. Let $\mathcal{T} \colon \Gamma \to \Gamma$ be a self-mapping if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that*

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta)) \leq F(\tilde{S}(\zeta, \zeta, \eta)),$$

for all $\zeta, \eta \in \Gamma$. Then $\mathcal{T}$ has a unique fixed point.

Remark 2.1. *Corollary 2.1 generalizes Theorem 2.1 of Wardowski* [24] *from metric space to the setting of S-metric space.*

If we take $\lambda_2 = \lambda_3 = \frac{1}{2}$ and $\lambda_1 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ in Theorem 2.1, then we get the Chatterjae [3] version of fixed point theorem in complete S-metric space.

Corollary 2.2. $(\Gamma, \tilde{S})$ *be a complete S-metric space. Let* $\mathcal{T}\colon \Gamma \to \Gamma$ *be a self-mapping if there exist* $F \in \mathcal{F}$ *and a number* $\tau > 0$ *such that*

$$\tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta)) \leq F\Big(\frac{1}{2}\big[\tilde{S}(\zeta, \zeta, \mathcal{T}\eta) + \tilde{S}(\eta, \eta, \mathcal{T}\zeta)\big]\Big),$$

for all $\zeta, \eta \in \Gamma$, $\mathcal{T}\zeta \neq \mathcal{T}\eta$. *Then* $\mathcal{T}$ *has a unique fixed point in* Γ.

Remark 2.2. (1) *If we take* $\lambda_1 = L \in [0, 1)$ *and* $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ *in Theorem 2.1, then we get F-contraction version of Corollary 2.7 on page 118 in* [20].

(2) *If we take* $\lambda_2 = \lambda_3 = a \in [0, 1/3)$ *and* $\lambda_1 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ *in Theorem 2.1, then we get F-contraction version of Corollary 2.15 on page 121 in* [20].

(3) *If we take* $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = c$, $\lambda_1 + \lambda_2 + \lambda_3 < 1$, $\lambda_1 + 3\lambda_2 < 1$ *and* $\lambda_4 = \lambda_5 = \lambda_6 = 0$ *in Theorem 2.1, then we get F-contraction version of Corollary 2.17 on page 122 in* [20].

3. A fixed point theorem for generalized modified Khan-type F-contractive condition

In this section, we shall prove a new fixed point theorem using the generalized modified Khan-type F-contractive condition on complete S-metric spaces. First, we give the following definition.

Definition 3.1. *Let* $(\Gamma, \tilde{S})$ *be an S-metric space. A mapping* $\mathcal{T}\colon \Gamma \to \Gamma$ *is called generalized modified Khan-type F-contraction if there exist* $F \in \mathcal{F}$ *and a number* $\tau > 0$ *such that*

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta)) \leq F(\xi(\zeta, \zeta, \eta)), \tag{15}$$

for all $\zeta, \eta \in \Gamma$, *where*

$$\xi(\zeta,\zeta,\eta) = \begin{cases} h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\zeta,\zeta,\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta)}, & \text{if } \tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta) \neq 0, \\ 0, & \text{if } \tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta) = 0, \end{cases}$$

and $h \in [0, 1)$.

Theorem 3.1. *Let* $(\Gamma, \tilde{S})$ *be a complete S-metric space and let* $\mathcal{T}\colon \Gamma \to \Gamma$ *be a continuous self-mapping. If* $\mathcal{T}$ *satisfies the generalized modified Khan-type F-contractive condition (15). Then* $\mathcal{T}$ *has a unique fixed point in* Γ.

Proof. Let $\zeta_0 \in \Gamma$ be an arbitrary point and the sequence $\{\zeta_n\}$ be defined as in Theorem 2.1. Assume that $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbb{N}$. Set $\sigma_n = \tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})$. Now, using the generalized Khan-type F-contractive condition (15), we have the following cases:

Case I. Suppose that

$$\tilde{S}(\zeta_m, \zeta_m, \mathcal{T}\zeta_n) + \tilde{S}(\mathcal{T}\zeta_m, \mathcal{T}\zeta_m, \zeta_n) \neq 0,$$

for all $m \in \mathbb{N} - \{0\}$ and $n \in \mathbb{N}$. Then, we get

$$\begin{aligned}
\tau + F(\sigma_n) &= \tau + F\big(\tilde{S}(\zeta_n, \zeta_n, \zeta_{n+1})\big) \\
&= \tau + F\Big(\tilde{S}(\mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_n)\Big) \\
&\leq F\Big(h\frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \mathcal{T}\zeta_{n-1})\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \mathcal{T}\zeta_n) + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)\tilde{S}(\zeta_n, \zeta_n, \mathcal{T}\zeta_{n-1})}{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \mathcal{T}\zeta_n) + \tilde{S}(\mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_{n-1}, \zeta_n)}\Big) \\
&= F\Big(h\frac{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_{n+1}) + \tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)\tilde{S}(\zeta_n, \zeta_n, \zeta_n)}{\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_{n+1}) + \tilde{S}(\zeta_n, \zeta_n, \zeta_n)}\Big) \\
&= F\Big(h\,\tilde{S}(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)\Big),
\end{aligned}$$

that is

$$\tau + F(\sigma_n) \leq F(h\,\sigma_{n-1}). \tag{16}$$

Since F is strictly increasing (by $(F1)$), we deduce that

$$\sigma_n < h\,\sigma_{n-1}, \text{ for all } n \in \mathbb{N}.$$

From assumption of the theorem $h < 1$, we deduce that

$$\sigma_n < \sigma_{n-1}, \text{ for all } n \in \mathbb{N}. \tag{17}$$

Consequently, we have

$$\tau + F(\sigma_n) < F(\sigma_{n-1}), \text{ for all } n \in \mathbb{N}. \tag{18}$$

This implies that

$$F(\sigma_n) \leq F(\sigma_{n-1}) - \tau \leq F(\sigma_{n-2}) - 2\tau \leq \cdots \leq F(\sigma_0) - n\tau. \tag{19}$$

In the same fashion to proof Theorem 2.1, we can conclude that $\mathcal{T}$ has a fixed point. Now, we prove the uniqueness of fixed point. Assume that $q_1 \in \Gamma$ is another fixed point of $\mathcal{T}$ such that $q \neq q_1$. This means that $\tilde{S}(q, q, q_1) > 0$. Taking $\zeta = q$ and $\eta = q_1$ in the contractive condition (15), using Lemma 1.1 and condition $(S1)$, we have

$$\begin{aligned}
&\tau + F(\tilde{S}(q, q, q_1)) = \tau + F(\tilde{S}(\mathcal{T}q, \mathcal{T}q, \mathcal{T}q_1)) \\
\leq\ & F\Big(h\frac{\tilde{S}(q, q, \mathcal{T}q)\tilde{S}(q, q, \mathcal{T}q_1) + \tilde{S}(q, q, q_1)\tilde{S}(q_1, q_1, \mathcal{T}q)}{\tilde{S}(q, q, \mathcal{T}q_1) + \tilde{S}(\mathcal{T}q, \mathcal{T}q, q_1)}\Big) \\
=\ & F\Big(h\frac{\tilde{S}(q, q, q)\tilde{S}(q, q, q_1) + \tilde{S}(q, q, q_1)\tilde{S}(q_1, q_1, q)}{\tilde{S}(q, q, q_1) + \tilde{S}(q, q, q_1)}\Big) \\
=\ & F\Big(\frac{h}{2}\tilde{S}(q, q, q_1)\Big).
\end{aligned}$$

This implies that

$$\tau + F(\tilde{S}(q, q, q_1)) \leq F\Big(\frac{h}{2}\tilde{S}(q, q, q_1)\Big).$$

Since F is strictly increasing (by $(F1)$), we obtain

$$\begin{aligned}
\tilde{S}(q, q, q_1)) &\leq \frac{h}{2}\tilde{S}(q, q, q_1) \\
&< h\,\tilde{S}(q, q, q_1),
\end{aligned}$$

which is a contradiction, since $h < 1$. Hence, deduce that $\tilde{S}(q, q, q_1) = 0$ and hence $q = q_1$. Consequently, q is a unique fixed point of $\mathcal{T}$.

Case II. Suppose that

$$\tilde{S}(\zeta_m, \zeta_m, \mathcal{T}\zeta_n) + \tilde{S}(\mathcal{T}\zeta_m, \mathcal{T}\zeta_m, \zeta_n) = 0,$$

for all $m \in \mathbb{N} - \{0\}$ and $n \in \mathbb{N}$. Then it can be easily seen that $\mathcal{T}$ has a unique fixed point in Γ. This completes the proof. □

Example 3.1. *Let $\Gamma = \mathbb{R}$ and $\tilde{S}(\zeta, \eta, \theta) = |\zeta - \theta| + |\eta - \theta|$ for all $\zeta, \eta, \theta \in \mathbb{R}$ be an S-metric on Γ. Suppose that*

$$F(\zeta) = ln(\zeta) \text{ for all } \zeta > 0 \in \mathbb{R}^+ \text{ and } \mathcal{T}(\zeta) = \frac{\zeta}{2}.$$

Now, suppose $\zeta \geq \eta$, then for $\zeta = 3$ and $\eta = 2$, we have $\mathcal{T}(3) = 1.5$ and $\mathcal{T}(2) = 1$. Again, we have

$$\begin{aligned}
\tilde{S}(\zeta, \zeta, \eta) &= \tilde{S}(3, 3, 2) = |3 - 2| + |3 - 2| = 2,\\
\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta) &= \tilde{S}(3, 3, 1.5) = |3 - 1.5| + |3 - 1.5| = 3,\\
\tilde{S}(\zeta, \zeta, \mathcal{T}\eta) &= \tilde{S}(3, 3, 1) = |3 - 1| + |3 - 1| = 4,\\
\tilde{S}(\eta, \eta, \mathcal{T}\eta) &= \tilde{S}(2, 2, 1) = |2 - 1| + |2 - 1| = 2,\\
\tilde{S}(\eta, \eta, \mathcal{T}\zeta) &= \tilde{S}(2, 2, 1.5) = |2 - 1.5| + |2 - 1.5| = 1,\\
\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \eta) &= \tilde{S}(1.5, 1.5, 2) = |1.5 - 2| + |1.5 - 2| = 1.
\end{aligned}$$

1. **Verification of Theorem 2.1.**

We have to show that the equation (1) is satisfied.

Assume that $\zeta \geq \eta$. Let $\zeta = 3$, $\eta = 2$, $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \lambda_3 = \frac{1}{9}$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$ with $\lambda_1 + 3\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = \frac{7}{9} < 1$.

Now by equation (1), we have

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta)) \leq F(\nu(\zeta, \zeta, \eta)),$$

where

$$\begin{aligned}
\nu(\zeta, \zeta, \eta) &= \lambda_1 \tilde{S}(\zeta, \zeta, \eta) + \lambda_2 \tilde{S}(\zeta, \zeta, \mathcal{T}\eta) + \lambda_3 \tilde{S}(\eta, \eta, \mathcal{T}\zeta)\\
&\quad + \lambda_4 \frac{\tilde{S}(\eta, \eta, \mathcal{T}\eta)[1 + \tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)]}{1 + \tilde{S}(\zeta, \zeta, \eta)}\\
&\quad + \lambda_5 \frac{\tilde{S}(\eta, \eta, \mathcal{T}\eta) + \tilde{S}(\eta, \eta, \mathcal{T}\zeta)}{1 + \tilde{S}(\eta, \eta, \mathcal{T}\eta)\tilde{S}(\eta, \eta, \mathcal{T}\zeta)}\\
&\quad + \lambda_6 \frac{\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)[1 + \tilde{S}(\eta, \eta, \mathcal{T}\zeta)]}{1 + \tilde{S}(\zeta, \zeta, \eta) + \tilde{S}(\eta, \eta, \mathcal{T}\eta)}\\
&= \frac{1}{3}.(2) + \frac{1}{9}.(4) + \frac{1}{9}.1 + 0 + 0 + 0\\
&= \frac{2}{3} + \frac{4}{9} + \frac{1}{9} = \frac{11}{9} = 1.22.
\end{aligned}$$

Again, we have

$$\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta) = \tilde{S}(1.5, 1.5, 1) = 1,$$
$$F\big(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \mathcal{T}\eta)\big) = F(1) = ln(1) = 0,$$

and

$$F\big(\nu(\zeta, \zeta, \eta)\big) = F(1.22) = ln(1.22) = 0.19885.$$

Putting these values in equation (1), we obtain for taking $\tau = 0.001$

$$0 + 0.001 \leq 0.19885 \Rightarrow 0.001 \leq 0.19885.$$

This shows that $\mathcal{T}$ is a generalized modified Hardy-Rogers-type F-contraction. Also, $\mathcal{T}$ is continuous. Hence, $\mathcal{T}$ has a fixed point.

2. Verification of Theorem 3.1.

We have to show that the equation (15) is satisfied.
Assume that $\zeta \geq \eta$. Let $\zeta = 3$, $\eta = 2$ and $h = \frac{1}{2}$.
Now by equation (15), we have

$$\begin{aligned} \xi(\zeta,\zeta,\eta) &= h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\zeta,\zeta,\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta)} \\ &= \frac{1}{2}\Big(\frac{3\times 4+2\times 1}{4+1}\Big) = \frac{1}{2}.\frac{14}{5} = \frac{7}{5} = 1.4. \end{aligned}$$

Again, we have

$$\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta) = \tilde{S}(1.5,1.5,1) = 1,$$
$$F\big(\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta)\big) = F(1) = ln(1) = 0,$$

and

$$F\big(\xi(\zeta,\zeta,\eta)\big) = F(1.4) = ln(1.4) = 0.33647.$$

Putting these values in equation (15), we obtain for taking $\tau = 0.001$

$$0.0 + 0.001 \leq 0.33647 \ \Rightarrow\ 0.001 \leq 0.33647.$$

This shows that $\mathcal{T}$ is a generalized modified Khan-type F-contraction. Also, $\mathcal{T}$ is continuous. Hence, $\mathcal{T}$ has a unique fixed point. Here 0 is the unique fixed point of $\mathcal{T}$.

4. A fixed point theorem for generalized modified Meir-Keeler-Khan-type F-contractive condition

In this section, we shall prove a new fixed point theorem using the generalized modified Meir-Keeler-Khan-type F-contractive condition on complete S-metric spaces. First, we give the following definition.

We consider the condition that if $\mathcal{T}\colon \Gamma \to \Gamma$ is a self-mapping, then for all $\zeta,\eta \in \Gamma$,

$$\zeta \neq \eta \ \Rightarrow\ \tilde{S}(\zeta,\zeta,\mathcal{T}\eta) + \tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta) \neq 0. \tag{20}$$

Assume that

$$\Psi(\zeta,\zeta,\eta) = h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\zeta,\zeta,\eta)\tilde{S}(\eta,\eta,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\eta)+\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\eta)}, \tag{21}$$

where $h \in [0,1)$ is a constant.

Definition 4.1. *Let $(\Gamma,\tilde{S})$ be an S-metric space. A mapping $\mathcal{T}\colon \Gamma \to \Gamma$ is called generalized modified Meir-Keeler-Khan-type F-contraction if there exist $F \in \mathcal{F}$ and a number $\tau > 0$ such that*

$$\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta) > 0 \Rightarrow \tau + F(\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\mathcal{T}\eta)) \leq F(\Psi(\zeta,\zeta,\eta)), \tag{22}$$

for all $\zeta,\eta \in \Gamma$, where $\Psi(\zeta,\zeta,\eta)$ is defined as in equation (21).

Theorem 4.1. *Let $(\Gamma,\tilde{S})$ be a complete S-metric space and let $\mathcal{T}\colon \Gamma \to \Gamma$ be a continuous self-mapping. If $\mathcal{T}$ satisfies the generalized modified Meir-Keeler-Khan-type F-contractive condition (22). Then $\mathcal{T}$ has a unique fixed point in Γ.*

Proof. By the similar arguments used in the proof of Theorem 3.1, it can be easily seen that $\mathcal{T}$ has a unique fixed point in Γ. Indeed, if we consider the condition (20) and the inequality (22), then we can use the techniques of **Case I** in the proof of Theorem 3.1. □

5. An application to fixed-circle problem on S-metric spaces

In this section, we obtain a fixed-circle result using generalized modified C-Khan type F-contractive condition on S-metric spaces.

Now we consider the condition (20).

Definition 5.1. *Let $(\Gamma, \tilde{S})$ be an S-metric space and let $\mathcal{T}\colon \Gamma \to \Gamma$ be a self-mapping. Then $\mathcal{T}$ is called a generalized modified C-Khan type F-contractive condition if there exist $F \in \mathcal{F}$, a number $\tau > 0$ and $\zeta_0 \in \Gamma$ such that*

$$\begin{aligned} \tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta) > 0 \quad &\Rightarrow \quad \tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)) \\ &\leq \quad F\Big(h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta_0) + \tilde{S}(\zeta_0,\zeta_0,\mathcal{T}\zeta_0)\tilde{S}(\zeta_0,\zeta_0,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta_0) + \tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\zeta_0)}\Big), \end{aligned} \tag{23}$$

for all $\zeta \in \Gamma$, where $h \in [0, 1)$ is a constant.

Theorem 5.1. *Let $(\Gamma, \tilde{S})$ be a complete S-metric space, $\mathcal{T}\colon \Gamma \to \Gamma$ be a continuous self-mapping and $\mathcal{C}^{\tilde{S}}_{(\zeta_0,r)}$ be a circle on Γ. If $\mathcal{T}$ is a generalized modified C-Khan type F-contraction for all $\zeta \in \mathcal{C}^{\tilde{S}}_{(\zeta_0,r)}$ with $\mathcal{T}\zeta_0 = \zeta_0$, then $\mathcal{T}$ fixes the circle $\mathcal{C}^{\tilde{S}}_{(\zeta_0,r)}$.*

Proof. Let $\zeta \in \mathcal{C}^{\tilde{S}}_{(\zeta_0,r)}$. Assume that $\zeta \neq \mathcal{T}\zeta$. Hence using the generalized modified C-Khan type F-contractive condition (23) with $\mathcal{T}\zeta_0 = \zeta_0$, we obtain

$$\tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta))$$

$$\begin{aligned} &\leq \quad F\Big(h\frac{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta_0) + \tilde{S}(\zeta_0,\zeta_0,\mathcal{T}\zeta_0)\tilde{S}(\zeta_0,\zeta_0,\mathcal{T}\zeta)}{\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta_0) + \tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\zeta_0)}\Big) \\ &= \quad F\Big(\frac{hr\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)}{r + \tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\zeta_0)}\Big). \end{aligned}$$

Now using the above inequality and Lemma 1.1, we get the following cases:

Case 1. If $\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0) = r$, then we have

$$\begin{aligned} \tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)) \quad &\leq \quad F\Big(\frac{hr\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\zeta)}{2r}\Big) \\ &= \quad F\Big(\frac{h\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\zeta)}{2}\Big). \end{aligned}$$

Since F is strictly increasing (by $(F1)$), we deduce that

$$\begin{aligned} \tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta) \quad &< \quad \frac{h\tilde{S}(\zeta,\zeta,\mathcal{T}\zeta)}{2} \\ &< \quad h\tilde{S}(\mathcal{T}\zeta,\mathcal{T}\zeta,\zeta), \end{aligned}$$

which is a contradiction since $h < 1$. Hence, we conclude that $\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta) = 0$ and so $\mathcal{T}\zeta = \zeta$.

Case 2. If $\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0) > r$, then we have

$$\begin{aligned}\tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)) &\leq F\Big(\frac{hr\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)}{2r}\Big) \\ &= F\Big(\frac{h\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)}{2}\Big).\end{aligned}$$

Since F is strictly increasing (by $(F1)$), we deduce that

$$\begin{aligned}\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta) &< \frac{h\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)}{2} \\ &< h\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta),\end{aligned}$$

which is a contradiction since $h < 1$. Hence, we conclude that $\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta) = 0$ and so $\mathcal{T}\zeta = \zeta$. This is a contradiction since $\zeta \in \mathcal{C}^{\tilde{S}}_{(\zeta_0, r)}$.

Case 3. If $\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0) < r$, then

$$\frac{1}{\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0)} > \frac{1}{r},$$

$$\frac{1}{r + \tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0)} > \frac{1}{2r},$$

and

$$F\Big(\frac{hr\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)}{r + \tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0)}\Big) > F\Big(\frac{hr\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)}{2r}\Big) = F\Big(\frac{h\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)}{2}\Big).$$

Now, we have

$$\tau + F(\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)) \leq F\Big(\frac{hr\tilde{S}(\zeta, \zeta, \mathcal{T}\zeta)}{r + \tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta_0)}\Big) > F\Big(\frac{h\tilde{S}(\mathcal{T}\zeta, \mathcal{T}\zeta, \zeta)}{2}\Big).$$

In this case, we can not obtained any conclusion, since if $a \leq b$ and $b \geq c$, so we can not say that either $a \geq c$ or $a \leq c$.

Consequently, $\mathcal{C}^{\tilde{S}}_{(\zeta_0, r)}$ is a fixed circle of $\mathcal{T}$. This completes the proof. □

6. Conclusion

In this article, some new fixed-point theorems are proved using various types of generalized F-contractive conditions such as generalized modified Hardy- Rogers type, generalized Khan-type etc. These obtained results generalize and extend some well-known fixed-point theorems on S-metric spaces. Also, we furnish an illustrative example in support of the results. Furthermore, as an application of our results, a fixed-circle theorem is given with generalized modified C-Khan type F-contraction. Our results extend, generalize and enrich several results from the existing literature.

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