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SOLVING A NONLINEAR INVERSE PROBLEM OF THE CAMASSA-HOLM EQUATION

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Abstract. In order to solve the nonlinear inverse Camassa-Holm equations, a numerical method is developed by applying finite difference formula to time discrimination and collocation of polynomial scaling functions for spatial variable. Using operational matrix of derivative, the problem is reduced to a set of algebraic equation. An estimation of error bound is investigated for presented method. Also, to show the accuracy of the proposed method, it is applied on two test problems. One of the most important advantages of this work, compared to previous works, is the implementation simplicity.

Keywords: Polynomial scaling functions, Operational matrix of derivative, Collocation method, Inverse problems, Convergence analysis, Noisy data.

AMS Subject Classification: 65M32, 35K05

1. Introduction

Nonlinear phenomena are of great importance in applied mathematics, physics and engineering [1]. As mentioned in [2], nonlinear evolution equations are used to model many phenomena in engineering and applied sciences, such as: solid state physics [3], fluid mechanics [4], chemical kinetics [5], plasma physics, population models [6], nonlinear optics and etc. Analytical exact solutions to nonlinear partial differential equation play a significant role in nonlinear science, particularly they may provide us with much physical information and more insights into the physical aspects of the problem and even may lead to further applications. Recently, a variety of powerful methods, such as inverse scattering method [7], Exp-function method [8], Homotopy perturbation method [9], $(\frac{G}{G'})$-expansion method [10] were used to obtain explicit travelling and solitary wave solutions

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of nonlinear evolutions equation. Investigating the exact solution for nonlinear partial differential equations is very challenging. Therefore, numerical methods are very useful to solve these equations. Inverse problems are encountered in many branches of science and engineering. For example, the inverse problem has been used to evaluate the thermal properties of solids in the field of heat transfer. Moreover, using the inverse problem, several functions and parameters such as static and moving heating sources, material properties, initial conditions, boundary conditions, optimal shape etc., can be estimated [11, 12, 13, 14, 15, 16, 17, 18].

Mathematically, the inverse problems belong to the class of problems called the ill-posed problems [19, 20, 21, 22, 26], i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solutions do not satisfy the general requirement of existence, uniqueness and stability under small changes in the initial parameters. In order to simplify the inverse problem, a variety of techniques resulted from mathematical fields such as partial differential equations, numerical analysis, harmonic analysis, functional analysis, Fourier analysis and etc. have been proposed. Tikhonov regularization [23], iterative regularization [24], base function [25] and the function specification methods [26] were used as solutions to the inverse problem.

Polynomial scaling function (PSF) as expansion functions has some advantages that one of them is the good representation of smooth functions by finite Chebyshev expansion. In this paper, by expanding the approximate solution in terms of PSF with unknown coefficients, the nonlinear inverse problem is reduced to a set of algebraic equations. Then, to evaluate the unknown coefficients of the solution at each time step, the operational matrix of derivative is constructed and the collocation method is utilized.

The rest of this manuscript is as follows: In Section 2, inverse problem is formulated. In Sections 3 and 4, description of the Polynomial scaling and wavelet functions and procedure for implementation of the present method are illustrated, respectively. The convergence analysis is discussed in Section 5. In Section 6, to illustrate the effectiveness and compare the presented method with quintic B-spline (QBS) method, some examples with analytical solution are given. Section 7 is the final section that ends the paper with a brief conclusion.

2. Nonlinear inverse problem

In this work, our focus is on the nonlinear inverse problem and determining the boundary conditions as follows:

$$u_t - u_{xxt} + \alpha u_x + \beta u u_x = \lambda u_x u_{xx} + u u_{xxx}, \ \ 0 \le x \le 1, \ \ 0 \le t \le T, \tag{1}$$

the initial condition is

$$u(x,0) = f_1(x), \qquad 0 \le x \le 1, \tag{2}$$

the boundary conditions are

$$\begin{aligned} u(0,t) &= p_1(t), \qquad u_x(0,t) = p_2(t), \qquad 0 \le t \le T, \\ u(1,t) &= q_1(t), \qquad u_x(1,t) = q_2(t), \qquad 0 \le t \le T, \end{aligned} \tag{3}$$

and the overspecified data are

$$u(x^*,t) = g_1(t), \qquad u(x^{**},t) = g_2(t), \qquad 0 < x^*, x^{**} < 1, \qquad 0 \le t \le T, \tag{4}$$

where α, β and λ are arbitrary constants. Also, T represents the final time, $f_1(x), p_2(t), q_2(t)$ and $g_1(t), g_2(t)$ are given continuous functions. The boundary conditions $p_1(t), q_1(t)$ are unknown and are determined from overspecified data. We want to find the functions $u(x,t)$ and $p_1(t), q_1(t)$. For two unknown boundary conditions $p_1(t), q_1(t)$ then we must provide additional information (4) to find a solution $\big(u(x,t), p_1(t), q_1(t)\big)$ to the inverse problem.

3. Description of the PSFs

Let's start with a brief introduction of PSF. Suppose ψ_j is defined as [27]:

$$\psi_j(y) = (1-y^2)U_{2^j-1}(y) = \frac{1-y^2}{2^j}T'_{2^j}(y), \qquad j = 0,1,2,\ldots.$$

where T_{2^j} and U_{2^j} are chebyshev polynomials of the first and second kind, respectively,

$$T_{2^j}(y) = \cos(2^j\theta), \quad \textit{and} \quad U_{2^j}(y) = \frac{\sin((2^j+1)\theta)}{\sin(2^j\theta)}, \qquad y = \cos(\theta).$$

The zeros of $\psi_j(y)$ are $y_k = \cos(\frac{k\pi}{2^j})$ for $k = 0,1,2,\ldots,2^j$.

Definition 3.1. *For all $j \in \mathbb{N}\cup\{0\}$, the space of polynomial scaling functions in $[-1,1]$ is defined by $V_j = span\{\phi_{j,l} : l = 0,1,...,2^j\}$, where*

$$\phi_{j,l}(y) = \frac{\psi_j(y)}{2^j(-1)^{l+1}(y-y_l)}\epsilon_{j,l}, \tag{5}$$

where

$$\epsilon_{j,l} = \begin{cases} \frac{1}{2}, & for\ l = 0\ or\ l = 2^j, \\ 1 & for\ l = 1,2,\ldots,2^j-1. \end{cases} \tag{6}$$

It can be easily seen that the spaces $V_j = \Pi_{2^j}$, where Π_{2^j} denotes the set of all polynomials of degree at most 2^j. The important property of this functions which accelerates the computations is:

$$\phi_{j,l}(y_k) = \delta_{k,l}, \qquad k,l = 0,1,...,2^j. \tag{7}$$

Where $\delta_{k,l}$ is Kronecker delta function which has a value of 1 for $k = l$ and a value of 0 for $k \neq l$. It is essential to change the variable $y = \frac{(x+1)}{2}$ to use polynomial scaling functions on $[0,1]$. For any $j \in \mathbb{N}_0$, the operator L_j is mapping any real-value function $u(x)$ on $[0,1]$ into the space V_j by the Lagrange formula

$$L_j u(x) = \sum_{l=0}^{2^j} u(x_l)\phi_{j,l}(x) = C^T\Phi(x), \tag{8}$$

where $x_l = 2\cos(\frac{l\pi}{2^j}) - 1$; also, C and Φ are vectors with 2^j+1 components as:

$$C = [u(x_0), u(x_1), \ldots, u(x_{2^j})]^T, \tag{9}$$

$$\Phi(x) = [\phi_{j,0}(x), \phi_{j,1}(x), \ldots, \phi_{j,2^j}(x)]^T. \tag{10}$$

3.1. Operational matrices of PSFs.

Theorem 3.1. *The derivative of the vector $\Phi(x)$ can be expressed by*

$$\frac{d}{dx}\Phi(x) = D\Phi(x), \tag{11}$$

where D is the $(2^j+1)\times(2^j+1)$ operational matrix of derivative given by

$$d_{k,l} = \begin{cases} \sum_{\substack{i=0\\ i\neq k}}^{2^j} \frac{1}{x_l - x_i}, & if \quad l = k, \\ \\ 2^{2^j-j-1}(-1)^k\epsilon_{j,k}\prod_{\substack{r=0\\ r\neq l,k}}^{2^j}(x_l - x_r), & if \quad l \neq k. \end{cases} \tag{12}$$

Proof. See [27]. □

4. Mixed finite difference and collocation method

In this section, we solve nonlinear partial differential equation (1) on a bounded region. For this purpose, we use finite difference method for one variable to reduce these equations to a system of ordinary differential equations, [28] then we solve this system in order to find the solution of the given equation at the points $t_n = n\Delta t$ for $\Delta t = \frac{T}{N}$, $n = 0, 1, ..., N$.

4.1. Discretization.

We discretize (1) according to following θ-weighted scheme

$$\frac{u^{n+1}-u^n}{\Delta t} - \frac{u^{n+1}_{xx}-u^n_{xx}}{\Delta t} + \theta\left(\alpha u^{n+1}_x + \beta u^{n+1}u^{n+1}_x - \lambda u^{n+1}_x u^{n+1}_{xx} - u^{n+1}u^{n+1}_{xxx}\right) + (1-\theta)(\alpha u^n_x + \beta u^n u^n_x - \lambda u^n_x u^n_{xx} - u^n u^n_{xxx}) = 0 \tag{13}$$

where Δt is the time step size and u^{n+1} is used to show $u(x, t_n + \Delta t)$. To linearize the nonlinear term $u^{n+1}u^{n+1}_x$, we use the linearization form applied in [29, Page 10].

$$(uu_x)^{n+1} = u^{n+1}u^n_x + u^n u^{n+1}_x - u^u u^n_x. \tag{14}$$

Using the linearization form (14) in (13), we obtain

$$\begin{aligned}\frac{u^{n+1}-u^n}{\Delta t} - \frac{u^{n+1}_{xx}-u^n_{xx}}{\Delta t} + \theta\Big(\alpha u^{n+1}_x &+ \beta\big(u^{n+1}u^n_x + u^n u^{n+1}_x - u^n u^n_x\big)\\ &- \lambda\big(u^{n+1}_x u^n_{xx} + u^n_x u^{n+1}_{xx} - u^n_x u^n_{xx}\big)\\ &- \big(u^{n+1}u^n_{xxx} + u^n u^{n+1}_{xxx} - u^n u^n_{xxx}\big)\Big)\\ &+ \big(1-\theta\big)\big(\alpha u^n_x + \beta u^n u^n_x - \lambda u^n_x u^n_{xx} - u^n u^n_{xxx}\big) = 0.\end{aligned} \tag{15}$$

By choosing $\theta = \frac{1}{2}$ we have

$$\begin{aligned}\frac{u^{n+1}-u^n}{\Delta t} - \frac{u^{n+1}_{xx}-u^n_{xx}}{\Delta t} + \frac{1}{2}\Big(\alpha u^{n+1}_x &+ \beta\big(u^{n+1}u^n_x + u^n u^{n+1}_x - u^n u^n_x\big)\\ &- \lambda\big(u^{n+1}_x u^n_{xx} + u^n_x u^{n+1}_{xx} - u^n_x u^n_{xx}\big)\\ &- \big(u^{n+1}u^n_{xxx} + u^n u^{n+1}_{xxx} - u^n u^n_{xxx}\big)\Big)\\ &+ \frac{1}{2}\big(\alpha u^n_x + \beta u^n u^n_x - \lambda u^n_x u^n_{xx} - u^n u^n_{xxx}\big) = 0.\end{aligned} \tag{16}$$

After simplifying we have

$$\begin{aligned}u^{n+1} - u^{n+1}_{xx} + \frac{\Delta t}{2}\Big(\alpha u^{n+1}_x &+ \beta\big(u^{n+1}u^n_x + u^n u^{n+1}_x\big)\\ &- \lambda\big(u^{n+1}_x u^n_{xx} + u^n_x u^{n+1}_{xx}\big)\\ &- \big(u^{n+1}u^n_{xxx} + u^n u^{n+1}_{xxx}\big)\Big)\\ &= u^n - u^n_{xx} - \frac{\Delta t}{2}\big(\alpha u^n_x\big).\end{aligned} \tag{17}$$

4.2. The PSF collocation method.

Using equation (8), the approximate solution for $u^n(x)$ is represented as follows:

$$L_j u^n(x) = U^T_n \Phi(x), \tag{18}$$

where vectors $U_n = [u^n(x_0), u^n(x_1), \dots, u^n(x_{2^j})]^T$ and $\Phi(x)$ is defined as (10). Using (11), one can write the following relations for various derivatives of $u^n(x)$,

$$L_j \frac{d^r}{dx^r} u^n(x) = U_n^T \frac{d^r}{dx^r} \Phi(x) = U_n^T D^{(r)} \Phi(x), \qquad r = 1, 2, 3. \tag{19}$$

Substituting equations (19) in (17), one can get

$$\begin{aligned}
\Phi^T(x)U_{n+1} - \Phi^T(x)(D^2)^T U_{n+1} + \frac{\Delta t}{2}\Big(&\alpha \Phi^T(x)(D)^T U_{n+1} \\
&+ \beta\big(u_x^n \Phi^T(x) U_{n+1} + u^n \Phi^T(x)(D)^T U_{n+1}\big) \\
&- \lambda\big(u_{xx}^n \Phi^T(x)(D)^T U_{n+1} + u_x^n \Phi^T(x)(D^2)^T U_{n+1}\big) \\
&- \big(u_{xxx}^n \Phi^T(x) U_{n+1} + u^n \Phi^T(x)(D^3)^T U_{n+1}\big)\Big) \\
= \Phi^T(x)U_n - \Phi^T(x)(D^2)^T U_n - \frac{\Delta t}{2}&\alpha \Phi^T(x)(D)^T U_n.
\end{aligned} \tag{20}$$

Also considering equation (8), one can approximate $u_x^n \Phi^T(x)$, $u_{xx}^n \Phi^T(x)$, $u_{xxx}^n \Phi^T(x)$ and $u^n \Phi^T(x)$ in the following forms

$$u_x^n \Phi^T(x) = \Phi^T(x) M_1, \tag{21}$$

$$u_{xx}^n \Phi^T(x) = \Phi^T(x) M_2, \tag{22}$$

$$u_{xxx}^n \Phi^T(x) = \Phi^T(x) M_3, \tag{23}$$

$$u^n \Phi^T(x) = \Phi^T(x) N, \tag{24}$$

where M_1, M_2, M_3 and N are matrices with dimension $(2^j+1)\times(2^j+1)$ as follows,

$$M_1 = \begin{pmatrix} u_x^n(x_0) & 0 & 0 & \dots & 0 \\ 0 & u_x^n(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & u_x^n(x_{2^j-1}) & 0 \\ 0 & 0 & \dots & 0 & u_x^n(x_{2^j}) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} u_{xx}^n(x_0) & 0 & 0 & \dots & 0 \\ 0 & u_{xx}^n(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & u_{xx}^n(x_{2^j-1}) & 0 \\ 0 & 0 & \dots & 0 & u_{xx}^n(x_{2^j}) \end{pmatrix},$$

$$M_3 = \begin{pmatrix} u_{xxx}^n(x_0) & 0 & 0 & \dots & 0 \\ 0 & u_{xxx}^n(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & u_{xxx}^n(x_{2^j-1}) & 0 \\ 0 & 0 & \dots & 0 & u_{xxx}^n(x_{2^j}) \end{pmatrix},$$

$$N=\begin{pmatrix} u^n(x_0) & 0 & 0 & \dots & 0 \\ 0 & u^n(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & u^n(x_{2^j-1}) & 0 \\ 0 & 0 & \dots & 0 & u^n(x_{2^j}) \end{pmatrix}.$$

Substituting equations (21)-(24) in (20), we have

$$\begin{aligned}\Phi^T(x)\Bigg(&U_{n+1}-(D^2)^TU_{n+1}+\frac{\Delta t}{2}\Big(\alpha(D)^TU_{n+1}+\beta\big(M_1U_{n+1}+N(D)^TU_{n+1}\big)\\ &-\lambda\big(M_2(D)^TU_{n+1}+M_1(D^2)^TU_{n+1}\big)-\big(M_3U_{n+1}+N(D^3)^TU_{n+1}\big)\Big)\Bigg)\\ &=\Phi^T(x)\Big(U_n-(D^2)^TU_n-\frac{\Delta t}{2}\alpha(D)^TU_n\Big).\end{aligned}\tag{25}$$

So by collocating equation (25) in the points x_k, $k=0,1,\dots,2^j$, and property (7) we get

$$\begin{aligned}\Bigg(I-(D^2)^T&+\frac{\Delta t}{2}\Big(\alpha(D)^T+\beta\big(M_1+N(D)^T\big)-\lambda\big(M_2(D)^T+M_1(D^2)^T\big)\\ &-\big(M_3+N(D^3)^T\big)\Big)\Bigg)U_{n+1}\\ &=\Big(I-(D^2)^T-\frac{\Delta t}{2}\alpha(D)^T\Big)U_n,\end{aligned}\tag{26}$$

representing a system of $(2^j+1)\times(2^j+1)$ linear equations. Using (18) and (21) in (3), we have

$$\Phi^T(x^*)U_{n+1}=g_1(t_{n+1}),\tag{27}$$

$$\Phi^T(x^{**})U_{n+1}=g_2(t_{n+1}),\tag{28}$$

$$\Phi^T(0)D^TU_{n+1}=p_2(t_{n+1}),\tag{29}$$

$$\Phi^T(1)D^TU_{n+1}=q_2(t_{n+1}).\tag{30}$$

Associating (27)-(30) with the linear system (26) we finally obtain a following matrix form of the system,

$$A_nU_{n+1}=B_n,\qquad n=1,2,\dots.\tag{31}$$

where A_n is a matrix with dimension $(2^j+1)\times(2^j+1)$ as,

$$A_n=\begin{pmatrix}\Phi^T(0)\\ \Phi^T(0)D^T\\ \Lambda_3\\ \vdots\\ \Lambda_{2^j-2}\\ \Phi^T(1)\\ \Phi^T(1)D^T\end{pmatrix},\quad B_n=\begin{pmatrix}g_1(t_{n+1})\\ p_2(t_{n+1})\\ \lambda_3\\ \vdots\\ \lambda_{2^j-2}\\ g_2(t_{n+1})\\ q_2(t_{n+1})\end{pmatrix},$$

where

$$\Lambda_i = \left(I-(D^2)^T+\frac{\Delta t}{2}\Big(\alpha(D)^T+\beta\big(M_1+N(D)^T\big)-\lambda\big(M_2(D)^T+M_1(D^2)^T\big)-\big(M_3+N(D^3)^T\big)\Big)\right)_i,$$

and

$$\lambda_i = \left\{ \Big(I-(D^2)^T-\frac{\Delta t}{2}\alpha(D)^T\Big)U_n \right\}_i.$$

Using (1) as the starting point, equation (31) gives a linear system of equations with 2^j+1 unknowns and equations, which can be solved to find U_{n+1} in any step $n = 0, 1, 2, \ldots$. So the unknown functions $u(x, t_n)$ in any time $t = t_n$, $n = 0, 1, 2, \ldots$ can be found.

5. Convergence analysis

The aim of this section is to present the error analysis of the method presented in the previous section. Suppose that

$$BV = \{\mathcal{P} : \mathbb{R} \to \mathbb{R} | \mathcal{P} \text{ is bounded variation on } [0,1]\}.$$

The value $V(\mathcal{P}(x))$ is defined as total variation of $\mathcal{P}(x)$ on $[0,1]$. For the given weight function $w(x) = (1-(2x-1)^2)^{\frac{-1}{2}}$ and $2 \le p \le \infty$, one can define

$$\|\mathcal{P}\|_p := \Big(\int_0^1 w(x)|\mathcal{P}(x)|^p\Big)^{\frac{1}{p}}.$$

We recall two corollaries from [30, 31].

Corollary 5.1. *Let $p \neq 2$, $\mathcal{P}^s \in BV$ and $0 \le s \le 2^j$ then,*

$$\|\mathcal{P} - L_j\mathcal{P}\|_p < \xi 2^{-j(s+\frac{1}{p})} V(\mathcal{P}^s). \tag{32}$$

Corollary 5.2. *Let $p \neq 2$, $0 \le l \le s$ and $\mathcal{P}^{(s)} \in BV$, then for the interpolatory polynomial based on the zeros of the Jacobi polynomial we have,*

$$\|(\mathcal{P} - L_j\mathcal{P})^{(l)}\|_p < \xi 2^{-j(s+\frac{1}{p}-max\{l,2l-\frac{1}{p}\})} V(\mathcal{P}^s). \tag{33}$$

In the above corollaries ξ is a constant that depends on s. Now, the Crank-Nikolson scheme (17) can be represented in an operator equation in the following form

$$Hu^{n+1} = (I + \frac{\Delta t}{2}\mathcal{D})u^{n+1} = F, \tag{34}$$

where I is an identity operator and

$$\mathcal{D} = -\frac{2}{\Delta t}\frac{d^2}{dx^2} + \beta u_x^n + \frac{d}{dx} + \beta u^n\frac{d}{dx} - \lambda u_{xx}^n\frac{d}{dx} - \lambda u_x^n\frac{d^2}{dx^2} - u_{xxx}^n - u^n\frac{d^3}{dx^3},$$
$$F = u^n - u_{xx}^n - \frac{\Delta t}{2}\big(\alpha u_x^n\big).$$

The operator Equation (34) can be approximated by the following solution

$$L_j(H)u_j^{n+1} = L_j(I + \frac{\Delta t}{2}\mathcal{D})u_j^{n+1} = F_j. \tag{35}$$

System (35) can be solved numerically to give an approximate solution for equation (1) at each level of time given by the expression $u_j^{n+1} = U_{n+1}^T\Phi(x)$.

Theorem 5.1. *If* $(u^{n+1})^{(s)} \in BV$, $s \neq 0$ *and by considering*

$$\eta = \max\Big\{|\beta|, |\lambda|, \|u^n\|_p, \|u_x^n\|_p, \|u_{xx}^n\|_p, \|u_{xxx}^n\|_p\Big\},$$

then

$$\|\mathcal{D}u^{n+1} - L_j\mathcal{D}u^{n+1}\|_p \leq C_1\eta 2^{-j(s+\frac{2}{p}-6)}V\Big((u^{n+1})^{(s)}\Big). \tag{36}$$

where C_1 *is a constant.*

Proof. We have

$$\begin{aligned}
\|\mathcal{D}u^{n+1} - L_j\mathcal{D}u^{n+1}\|_p \leq \|&-\frac{2}{\Delta t}u_{xx}^{n+1} + \beta u_x^n u^{n+1} + u_x^{n+1} + \beta u^n u_x^{n+1}\\
&- \lambda u_{xx}^n u_x^{n+1} - \lambda u_x^n u_{xx}^{n+1} - u_{xxx}^n u^{n+1} - u^n u_{xxx}^{n+1}\\
&+ \frac{2}{\Delta t}L_j u_{xx}^{n+1} - \beta L_j u_x^n u^{n+1} - L_j u_x^{n+1} - \beta L_j u^n u_x^{n+1}\\
&+ \lambda L_j u_{xx}^n u_x^{n+1} + \lambda L_j u_x^n u_{xx}^{n+1} + L_j u_{xxx}^n u^{n+1} + L_j u^n u_{xxx}^{n+1}\|_p\\
&\leq \frac{2}{\Delta t}\|u_{xx}^{n+1} - L_j u_{xx}^{n+1}\|_p + |\beta|\,\|u_x^n\|_p\|u^{n+1} - L_j u^{n+1}\|_p\\
&+ |\beta|\|u^n\|_p\|u_x^{n+1} - L_j u_x^{n+1}\|_p + \|u_x^{n+1} - L_j u_x^{n+1}\|_p\\
&+ |\lambda|\|u_{xx}^n\|_p\|u_x^{n+1} - L_j u_x^{n+1}\|_p + |\lambda|\|u_x^n\|_p\|u_{xx}^{n+1} - L_j u_{xx}^{n+1}\|_p\\
&+ \|u_{xxx}^n\|_p\|u^{n+1} - L_j u^{n+1}\|_p + \|u^n\|_p\|u_{xxx}^{n+1} - L_j u_{xxx}^{n+1}\|_p.
\end{aligned}$$

Using (33)

$$\begin{aligned}
\|\mathcal{D}u^{n+1} - L_j\mathcal{D}u^{n+1}\|_p \leq &\frac{2}{\Delta t}\xi 2^{-j(s+\frac{2}{p}-4)}V\Big((u^{n+1})^{(s)}\Big) + |\beta|\eta\xi 2^{-j(s+\frac{1}{p})}V\Big((u^{n+1})^{(s)}\Big)\\
&+ |\beta|\eta\xi 2^{-j(s+\frac{2}{p}-2)}V\Big((u^{n+1})^{(s)}\Big) + \xi 2^{-j(s+\frac{2}{p}-2)}V\Big((u^{n+1})^{(s)}\Big)\\
&+ |\lambda|\eta\xi 2^{-j(s+\frac{2}{p}-2)}V\Big((u^{n+1})^{(s)}\Big) + |k|\eta\xi 2^{-j(s+\frac{2}{p}-4)}V\Big((u^{n+1})^{(s)}\Big)\\
&+ \eta 2^{-j(s+\frac{1}{p})}V\Big((u^{n+1})^{(s)}\Big) + \eta 2^{-j(s+\frac{2}{p}-6)}V\Big((u^{n+1})^{(s)}\Big)\\
&\leq C_1\eta 2^{-j(s+\frac{2}{p}-6)}V\Big((u^{n+1})^{(s)}\Big).
\end{aligned}$$

□

Theorem 5.2. *If* u^{n+1} *and* u_j^{n+1} *be the exact and approximate solutions of (1) at each level of time* $n+1$*, respectively, also assume that the operator* $H = I + \frac{\Delta t}{2}\mathcal{D}$ *has bounded inverse and* $\Big(u^{n+1}\Big)^{(s)}$, $F^{(s)} \in BV$, $s \neq 0$, *then*

$$\|E_j^{n+1}\|_p = \|u^{n+1} - u_j^{n+1}\|_p \leq C\mu\|(L_jH)^{-1}\|_p 2^{-j(s+\frac{2}{p}-6)},$$

where

$$\mu = \max\Big\{V\Big((u^{n+1})^{(s)}\Big), V\Big(F^{(s)}\Big)\Big\}.$$

So for $s \geq 6$ *we ensure the convergence when* $j \to \infty$.

Proof. Subtracting equation (35) from (34) gives

$$-L_jH(u^{n+1}-u_j^{n+1}) = (H-L_jH)(u^{n+1}) - (F-F_j).$$

Provided that H^{-1} exists and bounded, we obtain the error bound as follows

$$\|E_j^{n+1}\|_p = \|(L_jH)^{-1}\|_p\|(H-L_jH)(u^{n+1}) - (F-F_j)\|_p, \tag{37}$$

where

$$(L_jH)^{-1} = \left(I+\frac{\Delta t}{2}\Big(-\frac{\Delta t}{2}D^2+\alpha D+\beta\big(M_1+ND\big)-\lambda\big(M_2D+M_1D^2\big)-\big(M_3+ND^3\big)\Big)\right)^{-1}.$$

Furthermore, by considering (37) and using Theorem 5.1 and Corollary (5.2) we have

$$\begin{aligned}\|(H-L_jH)(u^{n+1})\|_p &= \|(I-L_jI)(u^{n+1}) + \frac{\Delta t}{2}(\mathcal{D}-L_j\mathcal{D})(u^{n+1})\|_p \\ &\le \|(I-L_jI)(u^{n+1})\|_p + \frac{\Delta t}{2}\|(\mathcal{D}-L_j\mathcal{D})(u^{n+1})\|_p \\ &\le C_1 2^{-j(s+\frac{1}{p})}V\Big((u^{n+1})^{(s)}\Big) + \frac{\Delta t}{2}C_2\eta 2^{-j(s+\frac{2}{p}-6)}V\Big((u^{n+1})^{(s)}\Big),\end{aligned} \tag{38}$$

and

$$\|(F-F_j)\|_p \le C_3 2^{-j(s+\frac{1}{p})}V\Big((F)^{(s)}\Big). \tag{39}$$

By substituting equations (38) and (39) in (37), we have

$$\begin{aligned}\|E_j^{n+1}\|_p \le \|(L_jH)^{-1}\|_p\Bigg(&C_1 2^{-j(s+\frac{1}{p})}V\Big((u^{n+1})^{(s)}\Big) + \frac{\Delta t}{2}C_1\eta 2^{-j(s+\frac{2}{p}-6)}V\Big((u^{n+1})^{(s)}\Big) \\ &+ C_1 2^{-j(s+\frac{1}{p})}V\Big((F)^{(s)}\Big)\Bigg).\end{aligned} \tag{40}$$

By choosing $C = \max\{C_1, \frac{\Delta t}{2}C_2\eta, C_3\}$, finally we can obtain

$$\|E_j^{n+1}\|_p \le C\mu 2^{-j(s+\frac{2}{p}-6)}\|(L_jH)^{-1}\|_p. \tag{41}$$

□

6. Numerical results

In this section, we study the inverse problem (1)-(4) with the unknown boundary conditions. The main aim here is to show the applicability of the present method for solving inverse problems. As we know, the inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.

Remark 6.1. *In an inverse problem, two sources of error in the estimation exist. The first source is the unavoidable bias deviation or deterministic error, and the second one is the variance due to the amplification of measurement errors or stochastic error. The global effect of deterministic and stochastic errors is considered in the root mean square or total error* [32]. *Therefore, we compute total error RMS by using following formula*

$$RMS = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\Big(p(t_i)^{exact} - p(t_i)^{numerical}\Big)^2},$$

where N is the total number of estimated values.

The comparison between the exact solutions of $p_1(t)$, $q_1(t)$ and numerical solutions of the PSF method and quintic B-spline (QBS) method by noisy are presented. Also, we consider $T = 1$, $2^j = 16$, $\Delta t = \frac{1}{1000}$, $x^* = 0.375$, and $x^{**} = 0.75$ with noisy data (input data+0.001×rand(1)).

Example 6.1. *We consider the Equation (1) with $\alpha = -2$, $\beta = 4$ and $\lambda = 3$. The exact solution is given by*

$$u(x,t) = \frac{4}{3 + 3\tanh\left(t - \frac{1}{2}x\right)},$$

with initial condition

$$u(x,0) = \frac{4}{3 + 3\tanh\left(-\frac{1}{2}x\right)}, \qquad 0 \leq x \leq 1,$$

and boundary conditions as follows:

$$u_x(0,t) = -\frac{4\left(\frac{-3}{2} + \frac{3}{2}\tanh^2(t)\right)}{\left(3 + 3\tanh(t)\right)^2}, \qquad u_x(1,t) = -\frac{4\left(\frac{-3}{2} + \frac{3}{2}\tanh^2\left(t - \frac{1}{2}\right)\right)}{\left(3 + 3\tanh(t - \frac{1}{2})\right)^2}, \qquad 0 \leq t \leq T.$$

time	*Exact*		*PSF method*		*QBS method*	
t	$p_1(t)$	$q_1(t)$	$p_1^*(t)$	$q_1^*(t)$	$p_1^*(t)$	$q_1^*(t)$
0.1	1.212487	2.150360	1.213579	2.153330	1.210208	2.150439
0.2	1.113546	1.881412	1.114441	1.883844	1.104048	1.881272
0.3	1.032541	1.661216	1.033272	1.663207	1.012710	1.661062
0.4	0.966219	1.480935	0.966818	1.482562	0.940206	1.480125
0.5	0.911919	1.333333	0.912410	1.334667	0.890089	1.330239
0.6	0.867462	1.212487	0.867863	1.213579	0.857181	1.204925
0.7	0.831064	1.113546	0.831395	1.114440	0.839722	1.103717
0.8	0.801264	1.032451	0.801531	1.033273	0.835555	1.019539
0.9	0.776865	0.966219	0.777086	0.966817	0.835600	0.956719
1	0.756890	0.911919	0.757082	0.912412	0.841953	0.908802
RMS			6.588×10^{-4}	1.790×10^{-3}	3.104×10^{-2}	6.563×10^{-3}

TABLE 1. The comparison between exact and numerical solutions of Example 6.1 at $x = 0$ and $x = 1$ with noisy data.

Example 6.2. *As the second test problem, consider the equation (1) with $\alpha = 0$, $\beta = 3$ and $\lambda = 2$. The exact solution is given by*

$$u(x,t) = 2\exp(-2t + x),$$

with initial condition

$$u(x,0) = 2\exp(x) \qquad 0 \leq x \leq 1,$$

and boundary conditions as follows:

$$u_x(0,t) = 2\exp(-2t), \qquad u_x(1,t) = 2\exp(-2t+1), \qquad 0 \leq t \leq T.$$

Also, the applied noise is randomly applied from the order of one thousandth

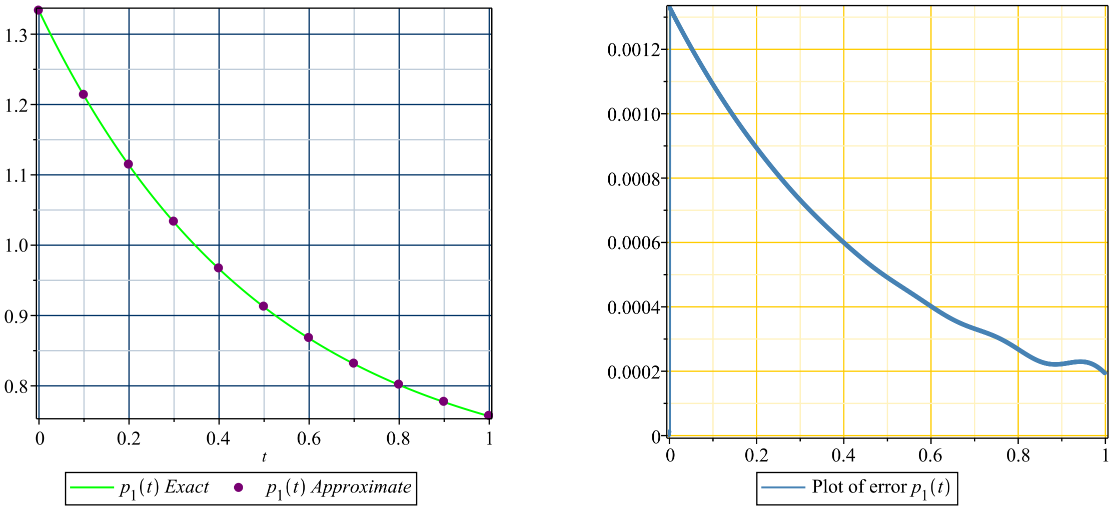

FIGURE 1. The plots of approximate and exact solutions of $p_1(t)$ and error of approximation $|p_1(t) - p_1^*(t)|$ for Example 6.1 with the noisy data.

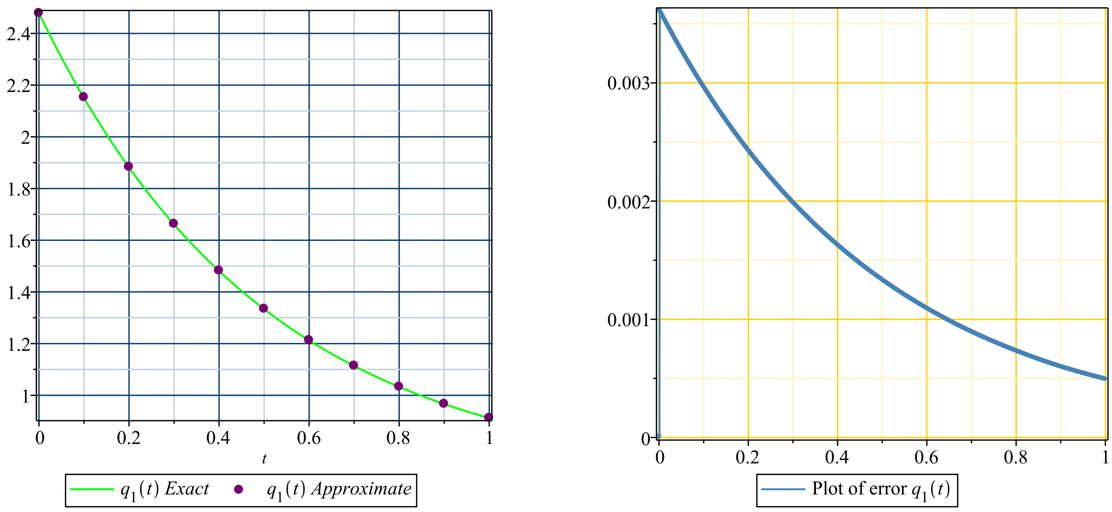

FIGURE 2. The plots of approximate and exact solutions of $q_1(t)$ and error of approximation $|q_1(t) - q_1^*(t)|$ for Example 6.1 with the noisy data.

7. CONCLUSION

In order to estimate unknown boundary conditions, a numerical method is proposed and the following results are obtained:

- The present study successfully applies the numerical method to inverse problems.
- Simplicity of implementation and less computational cost are main advantages of the proposed scheme compared to previous proposals.
- Unlike some previous techniques that use various transformations to simplify the equation, the current method does not require extra effort to deal with the non-linear terms. Therefore, the equations can be solved easily and elegantly using the present method.

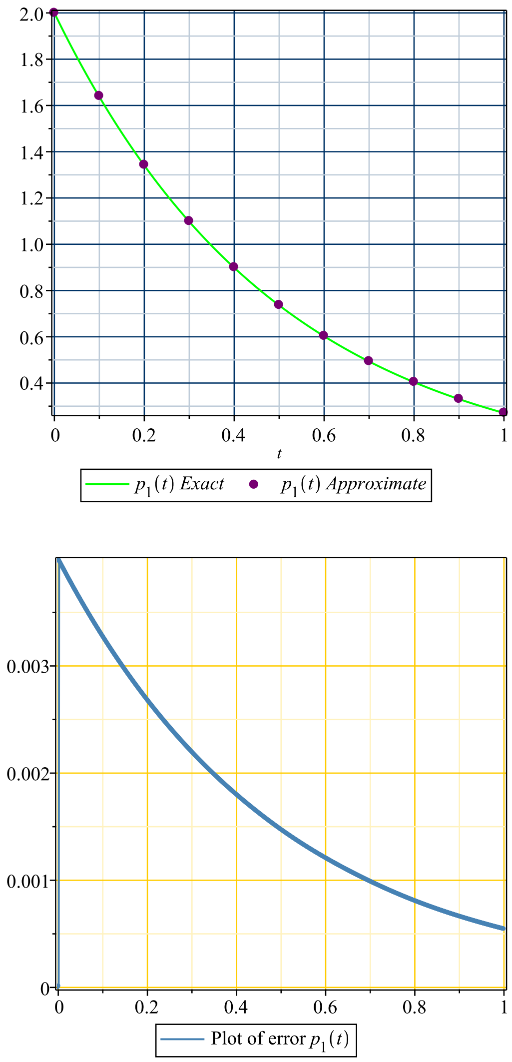

FIGURE 3. The plots of approximate and exact solutions of $p_1(t)$ and error of approximation $|p_1(t) - p_1^*(t)|$ for Example 6.2 with the noisy data.

- Numerical examples also verified the efficiency and accuracy of method that can be obtained within a couple of minutes of CPU time at Core($i5$)–2.67 GHz PC.
- The present method has been found to be stable with respect to the small perturbation in the input data.

REFERENCES

[1] Ganji, Z. Z., Ganji, D. D., Bararnia, H., (2009), Approximate general and explicit solutions of nonlinear BBMB equations by Exp-Function method, Appl. Math. Model., 33(4), pp.1836-1841.
[2] Noor, M. A., Noor, K. I., Waheed, A., Al-Said, E. A., (2011), Some new solitonary solutions of the modified Benjamin–Bona–Mahony equation, Comput. Math. Appl. , 62(4), pp.2126-2131.
[3] Eilenberger G., (1983), Solitons, Springer-Verlag, Berlin.
[4] Whitham, G., (1974), Linear and Nonlinear Waves, Wiley, New York.
[5] Gray, P., Scott, S., (1990), Chemical Oscillations and Instabilities, Clarendon, Oxford.
[6] Hasegawa, A., (1975), Plasma Instabilities and Nonlinear Effects, Springer-Verlag, Berlin.

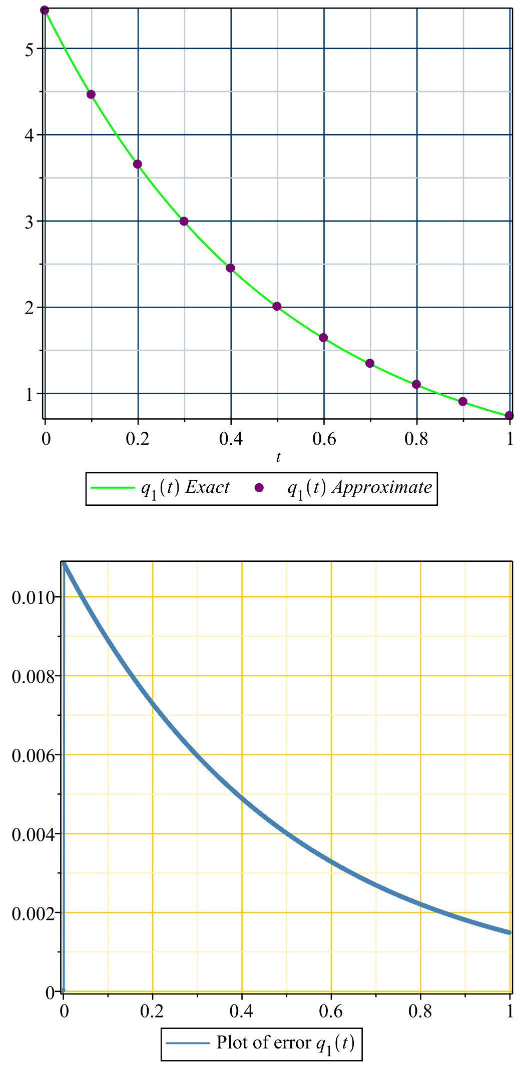

FIGURE 4. The plots of approximate and exact solutions of $q_1(t)$ and error of approximation $|q_1(t) - q_1^*(t)|$ for Example 6.2 with the noisy data.

[7] Ablowitz, M. J., Segur, H., (1981), Solitons and the inverse scattering transform, Society for Industrial and Applied Mathematics.

[8] Wu, X. H. B., He, J. H., (2008), Exp-function method and its application to nonlinear equations, Chaos, Solitons & Fractals, 38(3), pp.903-910.

[9] Ganji, D. D., Afrouzi, G. A., Talarposhti, R.A., (2007), Application of variational iteration method and homotopy–perturbation method for nonlinear heat diffusion and heat transfer equations, Physics Letters A, 368(6), pp.450-457.

[10] Abdollahzadeh, M., Ghanbarpour, M., Hosseini, A., Kashani, S., (2010), Exact travelling solutions for Benjamin-Bona-Mahony-Burgers equations by $\frac{G}{G'}$-expansion method, Int. J. Appl. Math. Comput. Sci., 3(1), pp.70-76.

[11] Beck, J. V., Blackwell, B., Haji-Sheikh, A., (1996), Comparison of some inverse heat conduction methods using experimental data, Internat. J. Heat Mass Transfer, 39(17), pp.3649-3657.

[12] Pourgholi, R., Azizi, N., Gasimov, Y. S., Aliev, F., Khalafi, H. K., (2009), Removal of numerical instability in the solution of an inverse heat conduction problem, Commun. Nonlinear Sci. Numer. Simul., 14(6), pp.2664-2669.

[13] Pourgholi, R., Saeedi, A., (2017), Applications of cubic B-splines collocation method for solving nonlinear inverse parabolic partial differential equations, Numer. Methods Partial Differ. Equ. , 33(1), pp.88-104.

[14] Shenefelt, J. R., Luck, R., Taylor, R. P., Berry, J. T., (2002), Solution to inverse heat conduction problems employing singular value decomposition and model-reduction, Int. J. Heat Mass Transf., 45(1), pp.67-74.

[15] Aliyari Boroujeni, A., Pourgholi, R., Tabasi, S. H., (2023). A new improved teaching–learning-based optimization (ITLBO) algorithm for solving nonlinear inverse partial differential equation problems, Comput. Appl. Math., 42(2), p.99, https://doi.org/10.1007/s40314-023-02247-4.

[16] Aliyari Boroujeni, A., Pourgholi, R., Tabasi, S. H. Solving inverse partial differential equations problems by using teaching learning based optimization algorithm, TWMS J. App. Eng. Math. (Accepted)

[17] Darehmiraki, M. and Rezazadeh, A., (2022), A Solution for Sparse PDE-Constrained Optimization by the Partition of Unity and RBFs, Glob. anal. discret. math., 7(2), pp.179-192.

[18] Azimi, R., Mohagheghy Nezhad, M. and Foadian, S., (2021), Shifted Legendre Tau method for solving the fractional stochastic integro-differential equations, Glob. anal. discret. math., 6(2), pp.221-241.

[19] Pourgholi, R., Foadian, S., Esfahani, A., (2013), Haar basis method to solve some inverse problems for two-dimensional parabolic and hyperbolic equations, TWMS J. App. Eng. Math., 3(1), pp.10-32.

[20] Azimi, R., Pourgholi, R., Tahmasbi, A., (2021), An efficient method for a class of integro-differential equations with a weakly singular kernel, TWMS J. App. and Eng. Math., 11(4), pp.1036-1049.

[21] Pourgholi, R., Molai, A. A., Houlari, T., (2013), Resolution of an inverse parabolic problem using sinc-galerkin method, TWMS J. App. Eng. Math., 3(2), pp.160-181.

[22] Pourgholi, R., Abtahi, M., Tabasi, S. H., (2012), A numerical solution of an inverse parabolic problem, TWMS J. App. Eng. Math., 2(2), pp.195-209.

[23] Tikhonov, A. N.,Arsenin, V. Y., (1977), Solution of Ill-Posed Problems, V.H. Winston and Sons, Washington.

[24] Alifanov, O. M., (2012), Inverse heat transfer problems. Springer Science & Business Media.

[25] Foadian, S., Pourgholi, R. and Tabasi, S. H., (2018), Cubic B-spline method for the solution of an inverse parabolic system. Applicable Analysis, 97(3), pp.438-465.

[26] Beck, J. V., Blackwell, B. and Clair, C. R. S., (1985), Inverse Heat Conduction: Ill Posed Problems, Wiley, New York.

[27] Rashidinia, J. and Jokar, M., (2016), Application of polynomial scaling functions for numerical solution of telegraph equation, Applicable Analysis, 95(1), pp.105-123.

[28] Rashidinia, J., Barati, A. and Nabati, M., (2014), Application of Sinc-Galerkin method to singularly perturbed parabolic convection-diffusion problems, Numer. Algor., 66, pp.643-662.

[29] Rubin, S. G. and Graves Jr, R. A., (1975), Cubic spline approximation for problems in fluid mechanics, NASA TR R-436, Washington, DC.

[30] Prestin, J., (1989), Mean convergence of Lagrange interpolation, Seminar Analysis, Operator Equat. and Numer. Anal., pp.75-86.

[31] Prestin, J. and Tasche, M., (1989), Trigonometric interpolation for bivariate functions of bounded variation, Banach Center Publications, 1(22), pp.309-321.

[32] Cabeza, J. M. G., García, J. A. M. and Rodríguez, A. C., (2005), A sequential algorithm of inverse heat conduction problems using singular value decomposition, Int. J. Therm. Sci. , 44(3), pp.235-244.

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