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DISTINGUISHING NUMBER IN SOME ASPECTS OF DENDRIMERS

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Abstract. The main purpose of the present study is to determine the distinguishing number of some dendrimers. The most known dendrimers such as Polyamidoamine (PAMAM), polylysine (PLL), Poly propyl ether imine (PETIM) and zinc porphyrin (DPZn) have a great impact in drug delivery systems, biomedical, etc. Based on structure of these dendrimers, some new graphs have been defined which resemble with the structure of dendrimers.The n-generation of hexa-cyclic dendrimer $HC_{n,d}$ and hexa-star dendrimer $HS_{n,d}$ have been introduced. Further, the distinguishing number of $n-$generations of graphs; C_6^d, hexa-cyclic dendrimer $HC_{n,d}$ and hexa-star dendrimer $HS_{n,d}$ have been determined successfully.

Keywords: Automorphism of graph, Distinguishing number, Dendrimer.

AMS Subject Classification: 05C78, 05C92.

1. Introduction

The dendrimers have a great impact in various feild like chemistry, nanomedicine research, etc. Molecules and molecular compounds are often represented by molecular graphs.

In terms of graph theory, the chemical compounds can be represented as a molecular graph, where the vertices and edges represent the atoms and chemical bonds of compound, respectively. It is evident from the literature that dendrimers are primarily associated with chemistry and materials science. In graph theory, dendrimers can be used to model and analyze complex networks or graphs with intricate connectivity patterns.

Dendrimers are symmetric molecules with well-defined shapes and highly ordered branched structures. The main three architectural parts of a dendrimer are (i) central core, (ii) branches, and (iii) terminal groups which have been illustrated in Figure 1.

In view of the molecular structure of zinc porphyrin (DPZn), the n-generations of hexa-cyclic dendrimer $HC_{n,d}$ has been introduced initially. The term "hexa-cyclic" refers to

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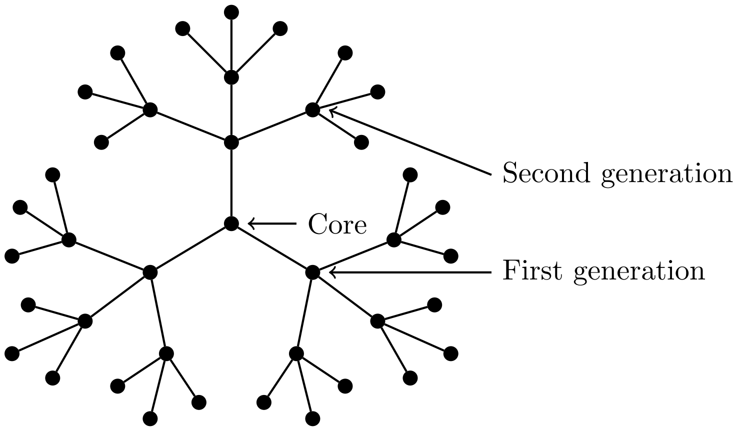

FIGURE 1. Structure of dendrimer with architectural parts.

the fact that the dendrimer consists of multiple cycles of length 6. The term "dendrimer" is appropriate since dendrimers are typically branched, tree-like molecules with a highly symmetric structure, which is similar to the hexa-cyclic dendrimer $HC_{n,d}$.

Further, a hexa-star dendrimer $HS_{n,d}, d \geq 2$ has been defined taking into account the structure of nanostar dendrimer. The hexa-star dendrimer $HS_{n,d}$ obtained from star graph S_d, where $d \geq 2$ which consists $d(d-1)^{n-1}$ branches of a graph C_6^{d-1} in the n^{th}-generation.

Definition 1.1. *A graph $C_6^d, d \geq 2$ is obtained from d-copies of cycle C_6 by joining their single vertex with an isolated vertex. For example, C_6^4 is shown in Figure 2.*

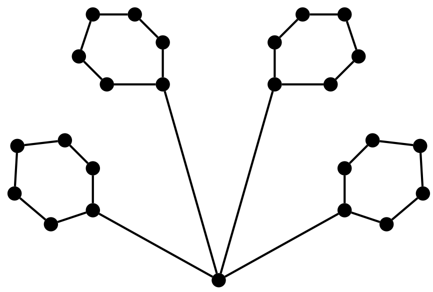

FIGURE 2. A graph C_6^4.

Definition 1.2. $1-$*generation of hexa-cyclic dendrimer $HC_{1,d}, d > 1$ is obtained from cycle C_6 by replacing each vertex of C_6 with C_6^d. For example, $HC_{1,2}$ is shown in Figure 3.*

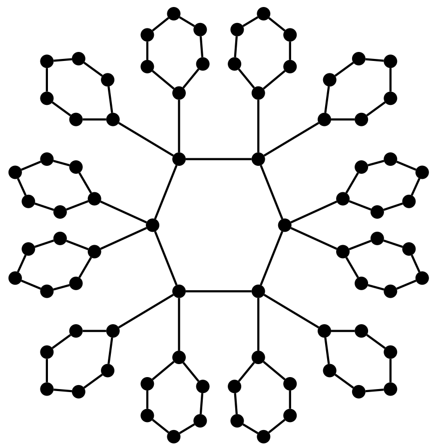

FIGURE 3. A first generation of hexa-cyclic dendrimer $HC_{1,2}$.

Definition 1.3. 2-*generations of hexa-cyclic dendrimer* $HC_{2,d}$ *is obtained from* 1-*generation of hexa-cyclic dendrimer* $HC_{1,d}$ *by replacing the top vertex of each cycle* C_6 *in* 1-*generation with* C_6^d. *Similarly, the* n-*generations of hexa-cyclic dendrimer* $HC_{n,d}$ *is obtained from the* $(n-1)$-*generations of hexa-cyclic dendrimer* $HC_{n-1,d}$ *by replacing the fourth vertex of each cycle* C_6 *of* $(n-1)$-*generations with* C_6^d. *For example,* $HC_{2,2}$ *is shown in Figure 4.*

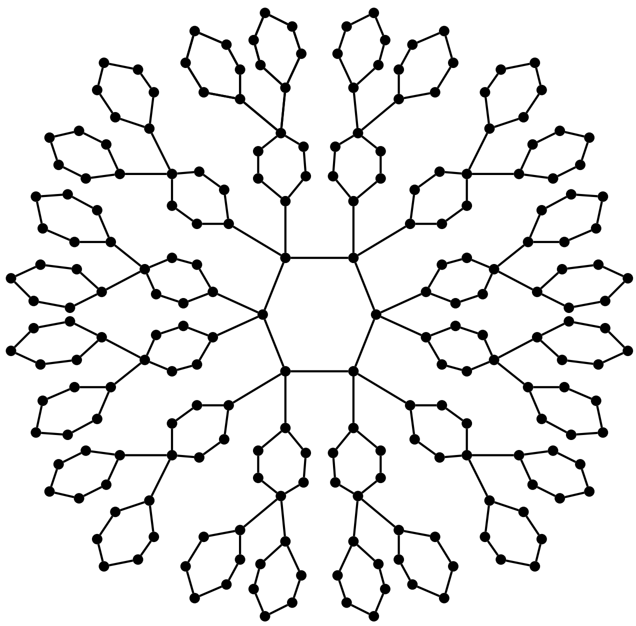

FIGURE 4. A second generation of hexa-cyclic dendrimer $HC_{2,2}$.

Definition 1.4. 1-*generation of a hexa-star dendrimer* $HS_{1,d}, d \geq 2$ *is obtained from star graph* S_d *replacing each pendant vertex of* S_d *with* C_6^{d-1}. *For example,* $HS_{1,3}$ *is shown in Figure 5.*

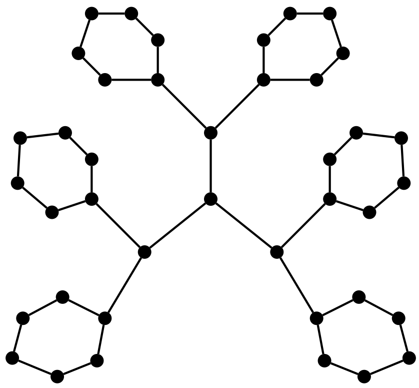

FIGURE 5. A first generation of a hexa-star dendrimer $HS_{1,3}$.

Definition 1.5. *2-generations of a hexa-star dendrimer* $HS_{2,d}, d \geq 2$ *is a graph obtained from* 1*-generation of a hexa-star dendrimer* $HS_{1,d}$ *in which every vertex whose distance from the centre vertex of a star graph* S_d *is* 5 *joined by* C_6^{d-1} *with an edge. Similarly, the* n*-generations of a hexa-star dendrimer* $HS_{n,d}$*, where* $d \geq 2$ *is a graph obtained from the* $(n-1)$*-generations of a hexa-star dendrimer* $HS_{n-1,d}$ *in which every vertex whose distance from the centre vertex of a star graph* S_d *is* $5n$ *joined by* C_6^{d-1} *with an edge. For example,* $HS_{2,3}$ *is shown in Figure 6.*

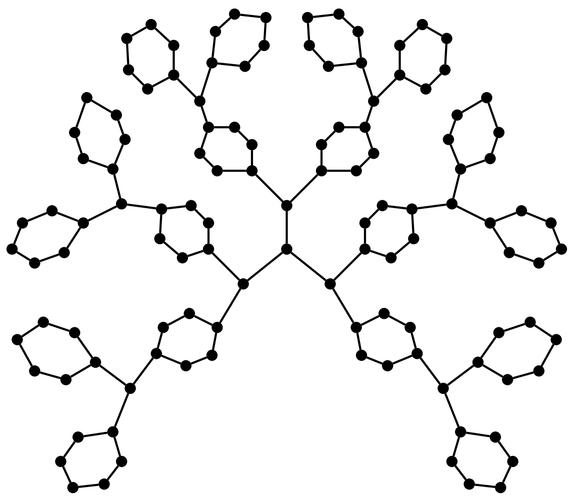

FIGURE 6. A second generation of a hexa-star dendrimer $HS_{2,3}$.

1.1. **Motivation.** Some research work related to chromatic polynomial, Zagreb indices, domination number, Nirmala index, power domination number, multiplicative sum connectivity, multiplicative randic and multiplicative harmonic index have been reported in literature [12, 4, 6, 5, 9, 11, 8, 10]. So, it is full of zest to determine the distinguishing number of dendrimers.

Distinguishing numbers of dendrimers are important because it affect their physical and chemical properties, including their size, shape, solubility, and reactivity. For example, higher-generation dendrimers are generally larger and more complex with more surface functional groups that can be tailored for specific applications.

The study of the distinguishing number of dendrimers is the keen interest where researchers can explore their structural properties. The uniqueness in the structures of dendrimers provides the better insight to unravel their complexity and it helps to understand their behavior in various applications.

1.2. **Review of literature.** The distinguishing number of graph G is the minimum number of labels to the vertices of G those are only preserved by the trivial automorphism of G [1].

The illustration for distinguishing number of some graph G is demonstrated in following Figure 7. The Figure 7(a) shows the vertex labeling for the graph G while the Figure 7(b) shows the distinguishing labeling for the graph G. Since the following labels are not preserved by any non-trivial automorphism of the graph G, so the distinguishing number for the graph G is 2.

(a) The graph G. (b) The distinguishing labeling for G.

FIGURE 7. The graph G and it's corresponding distinguishing labeling.

Some important results on the distinguishing numbers of graphs have been carried out as follows [7]:

(1) If G is a graph with trivial automorphism, then the distinguishing number of G is 1.
(2) For a path P_n, the distinguishing number of P_n is 2.
(3) For a cycle C_n, the distinguishing number of C_n is 3 if $n = 3, 4, 5$, and is 2 if $n \geq 6$.
(4) For a line graph $L(K_n)$ of complete graph K_n, the distinguishing number of $L(K_n)$ is 2 if $n \geq 6$.
(5) For a complete bipartite graph $K_{n,n}$, the distinguishing number of $L(K_n)$ is $n + 1$ if $n \geq 4$.

Alikhani and Soltani [2] determined the distinguishing numbers of friendship graph F_n and book graph B_n as follows:

(1) For a friendship graph F_n, the distinguishing number of F_n is $\lceil \frac{1+\sqrt{8n+1}}{2} \rceil$ if $n \geq 2$.
(2) For a book graph B_n, the distinguishing number of B_n is $\lceil \sqrt{n} \rceil$ if $n \geq 2$.

Alikhani and Soltani [3] proposed important results on the distinguishing numbers of some significant families of graphs that are helpful in chemistry as follows:

(1) For a graph $Q(p,q)$, the distinguishing number $D\left(Q(p,q)\right) = \min \left\{ k : k\binom{k}{q-1} \geq p \right\}$.
(2) For a dutch windmill graph D_n^m, $D\left(D_n^m\right) = \min \left\{ k : \frac{m^{k-1} - m^{\lceil \frac{k-1}{2} \rceil}}{2} \geq n \right\}$ when $n \geq 2, m \geq 3$.
(3) If G is the link of the graphs $G_1, G_2, \ldots, G_n$, then $D\left(G\right) \leq \max \left\{ D\left(G_i\right) : 1 \leq i \leq n \right\}$.

(4) If G is the bouquet of the graphs $G_1, G_2, \ldots, G_n$, then $D(G) \leq \sum_{k=1}^{n} D(G_k)$.
(5) For a spiro-chain $S_{p,q,r}$, $D(S_{p,q,r}) = 2$ except $p = 3, q = 2, r = 1$, and $D(S_{3,2,1}) = 3$.
(6) For a polyphenylenes $L_{p,q,r}$, $D(L_{p,q,r}) = 2$.
(7) For a nanostar dendrimer ND_n, $D(ND_n) = 2$.

2. Main Results

Theorem 2.1. *The distinguishing number of a graph C_6^d is,* $D\left(C_6^d\right) = \begin{cases} 2 & \text{if } 2 \leq d \leq 24 \\ 3 & \text{if } 24 < d \leq 324 \\ k & \text{if } d_{k-1} < d \leq d_k, \end{cases}$,
where $k \geq 4$ and

$$d_k = k^2 \left[\left(\frac{k(k-1)}{2} \times 6 \right) + \left(\frac{k(k-1)(k-2)}{2} \times 6 \right) + \left(\frac{k(k-1)(k-2)(k-3)}{4!} \times 12 \right) \right].$$

Proof. As $D(C_6) = 2$, so $D\left(C_6^d\right) \geq 2$. For $1 \leq i \leq d$, let $u_1^i, u_2^i, u_3^i, u_4^i, u_5^i, u_6^i$ be the consecutive vertices of C_6 in i^{th} cycle of C_6^d. The vertices of C_6^d can be mapped by the automorphism ϕ of C_6^d as follows:

(1) $\phi\left(u_1^i\right) = u_1^j$, $\phi\left(u_4^i\right) = u_4^j$, $\phi\left(u_2^i\right) = u_2^j$ or u_6^j and $\phi\left(u_3^i\right) = u_3^j$ or u_5^j, for all $1 \leq i, j \leq d$.
(2) $\phi\left(u_1^i\right) = u_1^i$, $\phi\left(u_4^i\right) = u_4^i$, $\phi\left(u_2^i\right) = u_6^i$, $\phi\left(u_6^i\right) = u_2^i$, $\phi\left(u_3^i\right) = u_5^i$ and $\phi\left(u_5^i\right) = u_3^i$, for all $1 \leq i \leq d$.

Let $u_1, u_2, u_3, u_4, u_5, u_6$ be the consecutive vertices of cycle C_6 as shown in Figure 8, where u_1 and u_6 are the top and bottom vertices of C_6, respectively.

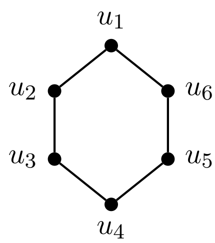

Figure 8. A cycle C_6 with vertex labeling.

Case 1: If $D\left(C_6^d\right) = 2$, then our motive to know how many copies of cycle C_6 in C_6^d can be found so that the distinguishing number for C_6^d is 2. Fix the top and bottom vertices of C_6 and assign u_2, u_3, u_5, u_6 with labels. We have the following different assignments for u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d.

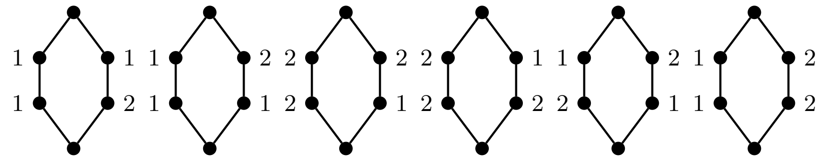

FIGURE 9. Assignments for the vertices u_2, u_3, u_5, u_6 having two different labels

Further, top and bottom vertices can be assigned by 1 and 2. So, there are $2^2.6 = 24$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d if they are assigned with two labels. If $d > 24$, then we have to assign at-least one vertex of 25^{th}-copy of C_6 with label 3. Hence, $D\left(C_6^d\right) = 2$ when $1 \leq d \leq 24$.

Case 2: If $D\left(C_6^d\right) = 3$, then our motive to know how many copies of cycle C_6 in C_6^d can be found so that the distinguishing number for C_6^d is 3. Fix the top and bottom vertices of C_6 and assign the vertices u_2, u_3, u_5, u_6 with labels. The cycles C_6 can be assigned as follows:

(1) Assign the vertices u_2, u_3, u_5, u_6 with two different labels. There are $\dfrac{3 \times 2}{2} =$ 3 pairs of u_2, u_3, u_5, u_6 having two different labels. For each pair, there are 6 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d as discussed in case 1. Also, top and bottom vertices can be assigned by three labels. So, there are $3^2 \times 6 = 24$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d. Hence, there are $3^2 \times 3 \times 6$ copies of C_6 in which u_2, u_3, u_5, u_6 are assigned by two different labels.

(2) Assign the vertices u_2, u_3, u_5, u_6 with three different labels. There are $\dfrac{3 \times 2 \times 1}{2} =$ 3 pairs of u_2, u_3, u_5, u_6 having three different labels. We have the following assignments for u_2, u_3, u_5, u_6 having three different labels which are not fixed by any non-trivial automorphism of C_6^d.

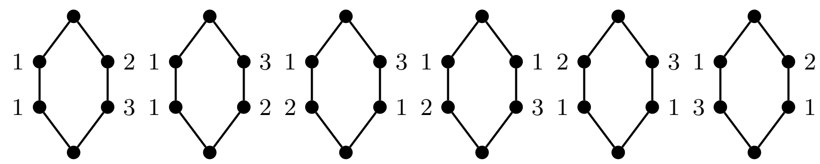

FIGURE 10. Assignments for the vertices u_2, u_3, u_5, u_6 having three different labels.

For each pair, there are 6 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d shown in Figure 9. As top and bottom vertices can be assigned by three labels, so there are $3^2 \times 3 \times 6$ copies of C_6 in which u_2, u_3, u_5, u_6 are assigned by three different labels. Therefore, the total $3^2 \times 3 \times 6 + 3^2 \times 3 \times 6 = 324$ copies of C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d if they are assigned with three labels. Hence, $D\left(C_6^d\right) = 3$ when $24 < d \leq 324$.

Case 3: If $D\left(C_6^d\right) = 4$, then our motive to know how many copies of cycle C_6 in C_6^d

can be found so that the distinguishing number for C_6^d is 4. Fix the top and bottom vertices of C_6 and assign the vertices u_2, u_3, u_5, u_6 with labels. The cycles C_6 can be assigned as follows:

(1) Assign the vertices u_2, u_3, u_5, u_6 with two different labels. There are $\frac{4 \times 3}{2}$ pairs of u_2, u_3, u_5, u_6 having two different labels. For each pair, there are 6 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d as discussed in case 1. Also, top and bottom vertices can be assigned by four labels. So, there are $4^2 \times 6$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d. Hence, there are $4^2 \times \left(\frac{4 \times 3}{2}\right) \times 6$ copies of C_6 in which u_2, u_3, u_5, u_6 are assigned by two different labels.

(2) Assign the vertices u_2, u_3, u_5, u_6 with three different labels. There are $\frac{4 \times 3 \times 2}{2}$ pairs of u_2, u_3, u_5, u_6 having three different labels. For each pair, there are 6 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d as discussed in case 2. Also, top and bottom vertices can be assigned by four labels. So, there are $4^2 \times \left(\frac{4 \times 3 \times 2}{2}\right) \times 6$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d. Hence, there are $4^2 \times \left(\frac{4 \times 3 \times 2}{2}\right) \times 6$ copies of C_6 in which u_2, u_3, u_5, u_6 are assigned by three different labels.

(3) Assign the vertices u_2, u_3, u_5, u_6 with four different labels. We have the following assignments for u_2, u_3, u_5, u_6 having four different labels which are not fixed by any non-trivial automorphism of C_6^d.

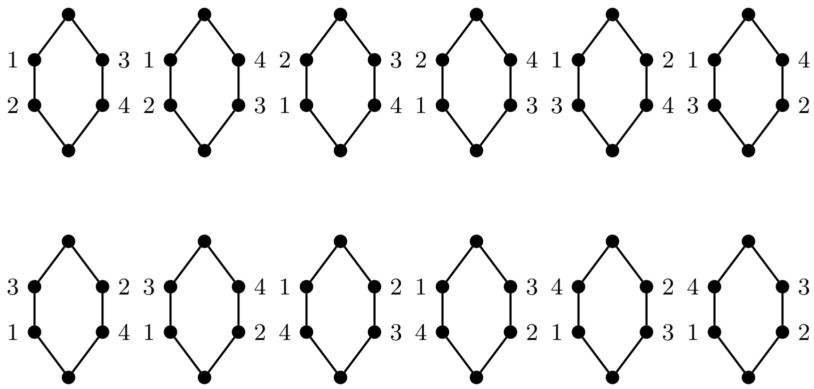

FIGURE 11. Assignments for the vertices u_2, u_3, u_5, u_6 having four different labels.

Also, top and bottom vertices can be assigned by four labels. So, there are $4^2 \times 12$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d in which u_2, u_3, u_5, u_6 are assigned by four different labels. Therefore, the total $\left[4^2 \times \left(\frac{4 \times 3}{2}\right) \times 6\right] + \left[4^2 \times \left(\frac{4 \times 3 \times 2}{2}\right) \times 6\right] + \left(4^2 \times 12\right)$ copies of C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d if they are assigned with four labels. Hence, $D\left(C_6^d\right) = 4$ when $324 < d \leq \left[4^2 \times \left(\frac{4 \times 3}{2}\right) \times 6\right] + \left[4^2 \times \left(\frac{4 \times 3 \times 2}{2}\right) \times 6\right] + \left(4^2 \times 12\right)$.

Case 4: If $D\left(C_6^d\right) = 5$, then our motive to know how many copies of cycle C_6 in C_6^d can be found so that the distinguishing number for C_6^d is 4. Fix the top and bottom vertices of C_6 and assign the vertices u_2, u_3, u_5, u_6 with labels. The cycles C_6 can be assigned as follows:

(1) Assign the vertices u_2, u_3, u_5, u_6 with two different labels. There are $\dfrac{5\times 4}{2}$ pairs of u_2, u_3, u_5, u_6 having two different labels. For each pair, there are 6 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d as discussed in case 1. Also, top and bottom vertices can be assigned by four labels. So, there are $5^2 \times 6$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d. Hence, there are $5^2 \times \left(\dfrac{5\times 4}{2}\right) \times 6$ copies of C_6 in which u_2, u_3, u_5, u_6 are assigned by two different labels.

(2) Assign the vertices u_2, u_3, u_5, u_6 with three different labels. There are $\dfrac{5\times 4\times 3}{2}$ pairs of u_2, u_3, u_5, u_6 having three different labels. For each pair, there are 6 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d as discussed in case 2. Also, top and bottom vertices can be assigned by four labels. So, there are $4^2 \times \left(\dfrac{5\times 4\times 3}{2}\right) \times 6$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d. Hence, there are $5^2 \times \left(\dfrac{5\times 4\times 3}{2}\right) \times 6$ copies of C_6 in which u_2, u_3, u_5, u_6 are assigned by three different labels.

(3) Assign the vertices u_2, u_3, u_5, u_6 with four different labels. There are $\dfrac{5\times 4\times 3\times 2}{4!}$ pairs of u_2, u_3, u_5, u_6 having four different labels. We have the following assignments for u_2, u_3, u_5, u_6 having four different labels which are not fixed by any non-trivial automorphism of C_6^d. For each pair, there are 12 assignments of u_2, u_3, u_5, u_6 which are not fixed by any non-trivial automorphism of C_6^d as discussed in case 3.

Also, top and bottom vertices of C_6^d can be assigned by four labels. So, there are $5^2 \times \left(\dfrac{5\times 4\times 3\times 2}{4!}\right) \times 12$ copies of cycle C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d in which u_2, u_3, u_5, u_6 are assigned by four different labels. Therefore, the total $\left[5^2 \times \left(\dfrac{5\times 4}{2}\right) \times 6\right] + \left[5^2 \times \left(\dfrac{5\times 4\times 3}{2}\right) \times 6\right] + \left[5^2 \times \left(\dfrac{5\times 4\times 3\times 2}{4!}\right) \times 12\right]$ copies of C_6 in C_6^d which are not fixed by any non-trivial automorphism of C_6^d if they are assigned with four labels. Hence, $D\left(C_6^d\right) = 5$ when $4^2\left[\left(\dfrac{4\times 3}{2} \times 6\right) + \left(\dfrac{4\times 3\times 2}{2} \times 6\right) + 12\right] < d \leq 5^2\left[\left(\dfrac{5\times 4}{2} \times 6\right) + \left(\dfrac{5\times 4\times 3}{2} \times 6\right) + \left(\dfrac{5\times 4\times 3\times 2}{4!} \times 12\right)\right]$. Based on above mentioned steps, it can be observed that $D\left(C_6^d\right) = k$ when $d_{k-1} < d \leq d_k$, where $d_k = k^2\left[\left(\dfrac{k(k-1)}{2} \times 6\right) + \left(\dfrac{k(k-1)(k-2)}{2} \times 6\right) + \left(\dfrac{k(k-1)(k-2)(k-3)}{4!} \times 12\right)\right]$.

□

Theorem 2.2. *The distinguishing number of n-generations of C_6^d is same as the distinguishing number of a graph C_6^d. That means, $D\left(C_6^d\right)^n = D\left(C_6^d\right)$.*

Proof. Let $\left(C_6^d\right)^n$ be the n-generations of C_6^d obtained from the $(n-1)^{\text{th}}$-generation of C_6^d by replacing the fourth vertex of each cycle C_6 of $(n-1)$-generations with C_6^d. First we determine $D\left(C_6^d\right)^2 = D\left(C_6^d\right)$.

The vertices of $d-$cycles C_6 in each branch of C_6^d in the 2^{nd}-generation can be mapped to each other, so we have to assign each branch of C_6^d in a distinguishing way. As $D\left(C_6^d\right) = k$, so d-copies of cycle C_6 can be assigned in distinguishing way with labels $1, 2, \ldots, k$ when $d_{k-1} < d \leq d_k$, where $d_k = k^2\left[\left(\frac{k(k-1)}{2} \times 6\right) + \left(\frac{k(k-1)(k-2)}{2} \times 6\right) + \left(\frac{k(k-1)(k-2)(k-3)}{4!} \times 12\right)\right]$. Therefore, $D\left(C_6^d\right)^2 \geq k$. First, we assign the vertices of C_6^d in the 1^{st}-generation as mentioned in Theorem 2.1. Since fourth vertex of each cycle the 1^{st}-generation is joined by C_6^d in 2^{nd}-generation of $\left(C_6^d\right)^n$.

For $1 \leq i \leq d$, let $u_1^i, u_2^i, u_3^i, u_4^i, u_5^i, u_6^i$ be the consecutive vertices of C_6 in i^{th}-cycle of C_6^d in first generation. For $1 \leq l \leq d$, let $u_1^{(i,l)}, u_2^{(i,l)}, u_3^{(i,l)}, u_4^{(i,l)}, u_5^{(i,l)}, u_6^{(i,l)}$ be the consecutive vertices of l^{st}-copy of cycle C_6 in the i^{th}-branch of C_6^d in the second generation of C_6^d corresponding to i^{th}-cycle of C_6^d of the first generation. The vertices of C_6^d can be mapped by the automorphism ϕ of C_6^d as follows:

(1) $\phi\left(u_1^{(i,l)}\right) = u_1^{(i,l)}$, $\phi\left(u_4^{(i,l)}\right) = u_4^{(i,l)}$, $\phi\left(u_2^{(i,l)}\right) = u_6^{(i,l)}$, $\phi\left(u_6^{(i,l)}\right) = u_2^{(i,l)}$, $\phi\left(u_3^{(i,l)}\right) = u_5^{(i,l)}$ and $\phi\left(u_5^{(i,l)}\right) = u_3^{(i,l)}$, for all $1 \leq i, l \leq d$.

(2) $\phi\left(u_1^{(i,l)}\right) = u_1^{(j,k)}$, $\phi\left(u_4^{(i,l)}\right) = u_4^{(j,k)}$, $\phi\left(u_2^{(i,l)}\right) = u_2^{(j,k)}$ or $u_6^{(j,k)}$, $\phi\left(u_3^{(i,l)}\right) = u_3^{(j,k)}$ or $u_5^{(j,k)}$, $\phi\left(u_1^i\right) = u_1^j$, $\phi\left(u_4^i\right) = u_4^j$, $\phi\left(u_2^i\right) = u_2^j$ or u_6^j, and $\phi\left(u_3^i\right) = u_3^j$ or u_5^j, for all $1 \leq i, j, k, l \leq d$.

If we assign the vertices of each branch of C_6^d in the 2^{nd}-generation as mentioned in Theorem 2.1, this assignment is not fixed by automorphism of above forms because the vertices of 1^{st}-generation of $\left(C_6^d\right)^2$ are already assigned in a distinguishing way. Therefore, as many labels are needed for distinguishing labeling of C_6^d as many labels are needed for distinguishing labeling of $\left(C_6^d\right)^2$. Hence, $D\left[\left(C_6^d\right)^2\right] = D\left(C_6^d\right)$.

Similarly, If we assign the vertices of each branch of C_6^d in $\left(C_6^d\right)^n$ as mentioned in Theorem 2.1, this assignment is not fixed by automorphism of $\left(C_6^d\right)^n$ because the vertices of $(n-1)^{\text{th}}$-generation of $\left(C_6^d\right)^2$ are already assigned in a distinguishing way. Hence, $D\left[\left(C_6^d\right)^n\right] = D\left(C_6^d\right)$.

□

Theorem 2.3. *The distinguishing number of a hexa-cyclic dendrimer $HC_{n,d}$ is same as the distinguishing number of a graph C_6^d. That means, $D\left(HC_{n,d}\right) = D\left(C_6^d\right)$.*

Proof. Let $HC_{n,d}$ be the n-generations of hexa cyclic dendrimer. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the consecutive vertices of central cycle C_6 of $HC_{n,d}$. Since $D\left(C_6\right) = 2$, so assign the vertices of central cycle C_6 of $HC_{n,d}$ as shown in Figure 5.

For $1 \leq l \leq d$, $1 \leq i \leq n$, let $u_1^{(m_i,l)}, u_2^{(m_i,l)}, u_3^{(m_i,l)}, u_4^{(m_i,l)}, u_5^{(m_i,l)}, u_6^{(m_i,l)}$ be the consecutive vertices m_i^{th}-branch of C_6^d in the last generation of $HC_{n,d}$ corresponding to v_1. The vertices of the n^{th}-generation of $HC_{n,d}$ can be mapped by the automorphism ϕ of $HC_{n,d}$ as follows:

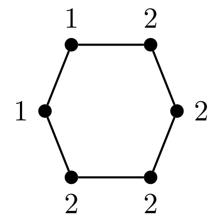

FIGURE 12. Distingushing labeling for cycle C_6.

(1) $\phi\left(u_1^{(m_i,k)}\right) = u_1^{(m_i,l)}$, $\phi\left(u_4^{(m_i,k)}\right) = u_4^{(m_i,l)}$, $\phi\left(u_2^{(m_i,k)}\right) = u_2^{(m_i,l)}$or $u_6^{(m_i,l)}$, and $\phi\left(u_3^{(m_i,k)}\right) = u_3^{(m_i,l)}$or $u_5^{(m_i,l)}$, for all $1 \le i \le n$ & $1 \le k, l \le d$.

Further, the n^{th}-generation of $HC_{n,d}$ has n-branches of C_6^d, so the vertices of n-branches of C_6^d can be also mapped by the automorphism ϕ of $HC_{n,d}$ as follows:

(2) $\phi\left(u_1^{(m_i,k)}\right) = u_1^{(m_j,l)}$, $\phi\left(u_4^{(m_i,k)}\right) = u_4^{(m_j,l)}$, $\phi\left(u_2^{(m_i,k)}\right) = u_2^{(m_j,l)}$or $u_6^{(m_j,l)}$, and $\phi\left(u_3^{(m_i,k)}\right) = u_3^{(m_j,l)}$or $u_5^{(m_j,l)}$, for all $1 \le i, j \le n$ & $1 \le k, l \le d$.

The automorphisms of each branch of the n^{th}-generation of C_6^d are the automorphisms C_6^d, so assign the vertices of each branch of C_6^d as the vertices of C_6^d given in Theorem 2.1. Further, assign the vertices of each branch of C_6^d in each generation of $HC_{n,d}$ as the vertices of C_6^d given in Theorem 2.1, this assignment is not fixed by automorphism of above forms because the vertices of 1^{st}-generation of $HC_{n,d}$ are not fixed by automorphism of above forms. As the vertices of central cycle C_6 are assigned in distinguishing way as shown in Figure 5, so this assignment of labeling is also not fixed by any rotation or reflection of $HC_{n,d}$. Therefore, as many labels are needed for distinguishing labeling of C_6^d as many labels are needed for distinguishing labeling of $HC_{n,d}$. Hence, $D(HC_{n,d}) = D\left(C_6^d\right)$. □

Theorem 2.4. *The distinguishing number of a hexa-star graph $HS_{1,d}$ is same as the distinguishing number of a graph C_6^d. That means, $D(HS_{1,d}) = D\left(C_6^d\right)$.*

Proof. Let $HS_{1,d}, d \ge 2$ be the 1−generation of a hexa-star dendrimer tree obtained from star graph S_d by replacing each pendant vertex with C_6^{d-1}. Since $D\left(C_6^d\right) = k$ when $d_{k-1} < d \le d_k$, where $d_k = k^2\left[\left(\frac{k(k-1)}{2} \times 6\right) + \left(\frac{k(k-1)(k-2)}{2} \times 6\right) + \left(\frac{k(k-1)(k-2)(k-3)}{4!} \times 12\right)\right]$, so first we want to know how many copies of C_6^{d-1} in $HS_{1,d}$ can be found so that the distinguishing number for $C_{1,d}$ is k. It means that we want to find d such that $D(HS_{1,d}) = k$. It can be seen that the vertices of one copy of C_6^{d-1} can be mapped to the vertices of another copy of C_6^{d-1} by a non-trivial automorphism of $HS_{1,d}$. As we want to assign each copy of C_6^{d-1} in a distinguishing way with the labels $1, 2, \ldots, k$, we need to assign at least one copy of cycle C_6 in each branch of C_6^{d-1} which differ from labels of each cycle C_6 in the other branches of C_6^{d-1}. According to Theorem 1, there are d-copies of cycle C_6 in C_6^{d-1} which are already assigned in distinguishing way with the labels $1, 2, \ldots, k$, where $d_{k-1} < d \le d_k$. There are d-branches of C_6^{d-1} in $HS_{1,d}$ and we need to assign $d-1$ copies cycles C_6 in each branch of C_6^{d-1} such that at least one copy of cycle C_6 in each branch of C_6^{d-1} which differ from labels of each cycle C_6 in the

other branches of C_6^{d-1}. There are $\binom{d}{d-1} = d$ such possible pairs of C_6^{d-1}. The vertices which are not part of any cycle can be assigned by any one of $1, 2, \ldots k$. It means that there are d-copies of C_6^{d-1} in $C_{1,d}$ such that $D\left(HS_{1,d}\right) = k$ when $d_{k-1} < d \leq d_k$, where $d_k = k^2\left[\left(\frac{k(k-1)}{2} \times 6\right) + \left(\frac{k(k-1)(k-2)}{2} \times 6\right) + \left(\frac{k(k-1)(k-2)(k-3)}{4!} \times 12\right)\right]$.
Hence, $D\left(HS_{1,d}\right) = D\left(C_6^d\right)$. □

3. Conclusion

In this paper, the distinguishing numbers of some dendrimer have been determined. Some graphs have been defined considering most well-known dendrimers useful in chemistry that resemble dendrimers. The distingushing number of graph C_6^d has been carried out successefully. It has been shown that the distingushing number of n-generations of C_6^d is same as the distingushing number of C_6^d. Further, the distingushing number of n-generation of hexa-cyclic dendrimer and 1-generation of hexa-star dendrimer have been determined which are same as the distingushing number of C_6^d. As the number of generations in the dendrimers mentioned above increases, they generally become larger, more complex, and more branched, but their distinguishing number does not increase or decrease shown in Theorem 2.2, 2.3 and 2.4. The distinguishing number of these dendrimers can be applied to analyze the structure and properties of dendrimers.

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