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THE EXTENDED LEGENDRE WAVELETS OPERATIONAL MATRIX OF INTEGRATION AND ITS APPLICATIONS

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ABSTRACT. In this paper, the general operational matrix of integration P based on the extended Legendre wavelets has been developed which generalizes the idea of the operational matrix of integration for $\mu = 2$ given in [30]. A brief procedure for forming this matrix has been discussed. Also, we have solved Bessel differential equation of order zero by using extended Legendre wavelets method for different values of μ, M and k. The results show the better accuracy of the proposed method, which is justified through the illustrative examples.

Keywords: Extended Legendre wavelet, operational matrix of integration, Legendre polynomials, orthonormal set.

AMS Subject Classification: 42C40, 42C15, 65T60, 65L10, 65L60, 65R20.

1. INTRODUCTION

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In recent years, orthogonal polynomials like the Legendre, Chebyshev, Laguerre, Gegenbauer, Jacobi, Hermite etc. played a very vital role in solving differential and integral equations. Using these polynomials, researchers have developed different types of wavelets like the Legendre wavelet, Chebyshev wavelet, Laguerre wavelet, Gegenbauer wavelet, Jacobi wavelet, Hermite wavelet etc. The application of these wavelets play a very crucial role in solving many differential and integral equations. Wavelets are an excellent alternative to the present traditional methods. In recent years, both mathematics and physicists have devoted considerable effort to find robust and stable numerical and analytical methods for solving differential equations of physical interest. Wavelets also have a great role in dealing with different types of problems arising in mathematical statistics, medical sciences, heat conduction, electromagnetic theory etc.

System of differential equations with different type of initial and boundary conditions play a very important role in the discussion of various types of mathematical problems in different fields of science and technology such as physics, chemistry, control system, statistics, robotics, medical science, biological science, economics etc. Hence, for getting the solutions of these types of problems such as above, there is the necessity of numerical solutions of differential equations. As we know that, in most of the cases analytical solution of the differential equation is not possible. So, in such situation we try to develop numerical solution of differential equations. There are various types of numerical method have been discussed by the researchers as given in [11, 18, 31, 28, 35, 36, 37, 38]. Various authors have developed different methods for solving differential equations. Chen and Hsiao [7] developed the Haar wavelet method for solving the lumped and distribution parameter system. Lepik [26, 27, 25] discussed the solution of different kinds of nonlinear partial differential equations. Hariharan [15] used the Haar wavelet approach to solve a well known Fisher's equation. Non-linear *Schrödinger* equation solved by Behzad Ghanbari et al. [10] using conformal derivative technique. Chauhan et al. [5] discussed nonlinear fractional differential equations by Benoulli wavelet method, Srivastava et al. [32] developed dual phase-lag heat transfer model by using Fibonacci wavelet method, Srivastava et al. [33] solved fractional Bagley-Torvik equation by Gegenbauer wavelet method.

Researchers like Agarwal [1], Dieudonné [12] have used and explained the applications of orthogonal polynomials in various fields of mathematical sciences. Gülsu et al. [14] has introduced a new collocation method for finding the solution of mixed linear integro - differential - differences equations. Chang et al. [4] proposed a method based on shifted Legendre polynomials for solving variational problems. Chen et al. [6] have also investigated a Walsh series method for finding the solution of some special kinds of problems arises in the calculus of variations. Gu & Jiang [13] has discussed the Haar wavelet operational matrix of integration, which is very useful in solving different kinds of problems in mathematical analysis. Horng et al. [17] applied a method based on shifted Chebyshev wavelet for solving variational problems. Hwang et al. [19] proposed the Laguerre polynomials method to solve integral equations. Razzaghi et al. [30] introduced the Legendre wavelet operational matrix of integration to solve differential and integral equations.

H. Jafari et al. [21] used the Legendre wavelet technique to solve fractional differential equations. Fractional order differential equations are used in the modeling of several engineering problems such as viscoelastic damping [2, 9, 3]. In control theory researcher's like [8] applied fractional order derivatives to solve some important class of problems. Authors [29, 16] provide the present applications of this techniques in different fields. Lal and Yadav [23] have used extended Haar wavelet to solve Bessel's differential equation.

Working in this direction, in this paper, we have generalized the idea of Razzaghi et al. [30] using the extended Legendre wavelet and obtained the extended Legendre wavelet operational matrix of integration which works for any positive integer $\mu \geq 2$.

2. Definitions and Preliminaries

The Extended Legendre Wavelets and Its Properties. The extended Legendre wavelets on the interval $[0, 1)$ are defined by

$$\psi^{(\mu)}_{n,m}(t) = \begin{cases} \sqrt{2m+1}\mu^{\frac{k}{2}} L_m(2\mu^k t - 2n + 1), & \frac{n-1}{\mu^k} \leq t < \frac{n}{\mu^k}; \\ 0, & \textit{otherwise}, \end{cases}$$

where $n = 1, 2, ..., \mu^k$ and m is order of the Legendre polynomial, $k = 1, 2, 3, ...$ and t is normalized time. In the above definition, the polynomials $L_m(t)$ are the Legendre polynomials of degree m over the interval $[-1, 1]$ which are defined as (i) $L_0(t) = 1$ (ii) $L_1(t) = t$ (iii) $(m+1)L_{m+1}(t) = (2m+1)tL_m(t) - mL_{m-1}(t), \quad m = 1, 2, 3, ...$
The set of extended Legendre wavelets is an orthonormal set.

Extended Legendre Wavelet Expansion. A function $f \in L^2(\mathbb{R})$ defined over $[0, 1)$ is expanded in terms of extended Legendre wavelets series as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c^{(\mu)}_{n,m} \psi^{(\mu)}_{n,m}(t), \tag{1}$$

where $c^{(\mu)}_{n,m} = \langle f, \psi^{(\mu)}_{n,m} \rangle$ on $L^2[0, 1)$ (Razzaghi[30]).
If the above infinite series is truncated then Eq. (1) is written as

$$S_{\mu^k,M}(t) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c^{(\mu)}_{n,m} \psi^{(\mu)}_{n,m}(t) = C^T \psi^{(\mu)}(t), \tag{2}$$

where

$$C = \left[c^{(\mu)}_{1,0}, c^{(\mu)}_{1,1}, ..., c^{(\mu)}_{1,M-1}, c^{(\mu)}_{2,0}, ..., c^{(\mu)}_{2,M-1}, ..., c^{(\mu)}_{\mu^k,0}, ..., c^{(\mu)}_{\mu^k,M-1}\right]^T \tag{3}$$

and

$$\psi^{(\mu)}(t) = \left[\psi^{(\mu)}_{1,0}, \psi^{(\mu)}_{1,1}, ..., \psi^{(\mu)}_{1,M-1}, \psi^{(\mu)}_{2,0}, ..., \psi^{(\mu)}_{2,M-1}, ..., \psi^{(\mu)}_{\mu^k,0}, ..., \psi^{(\mu)}_{\mu^k,M-1}\right]^T. \tag{4}$$

Theorems on Extended Legendre Wavelet.

Theorem 2.1. *If $f \in L^2[0, 1)$ such that $0 < |f'(t)| < \infty$ and its extended Legendre wavelets expansion is given by*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c^{(\mu)}_{n,m} \psi^{(\mu)}_{n,m}(t).$$

Then the extended Legendre wavelets approximation of f by $(\mu^k, M)^{th}$ partial sums

$$S_{\mu^k,M}(t) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c^{(\mu)}_{n,m} \psi^{(\mu)}_{n,m}(t)$$

under the norm $\|.\|_2$ satisfies

$$E_{\mu^k,M}(f) = \begin{cases} O\left(\frac{1}{\mu^k}\right), & M = 0; \\ O\left(\frac{1}{\mu^k\sqrt{2M+1}}\right), & M \geq 1. \end{cases}$$

Proof of theorem (2.1)

See the proof of theorems (5.1) and (5.2) of [24].

Theorem 2.2. *If* $0 < |f''(t)| < \infty \quad \forall \quad t \in [0,1)$ *and its extended Legendre wavelets expansion is given by*

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}^{(\mu)}\psi_{n,m}^{(\mu)}(t).$$

Then the extended Legendre wavelets approximation of f *by* $(\mu^k, M)^{th}$ *partial sums*

$$S_{\mu^k,M}(t) = \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{n,m}^{(\mu)}\psi_{n,m}^{(\mu)}(t)$$

under the norm $\|.\|_2$ *is calculated as*

$$E_{\mu^k,M}(f) = \begin{cases} O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^k}\right)\right), & M=0; \\ O\left(\frac{1}{\mu^{2k}}\right), & M=1; \\ O\left(\frac{1}{\mu^{2k}(2M-1)^{\frac{3}{2}}}\right), & M \geq 2. \end{cases}$$

Proof of theorem (2.2)

See proof of theorems (5.3), (5.4) and (5.5) of [24].

3. Extended Legendre Wavelets operational matrix of integration

In this section, the operational matrix of integration P for extended Legendre wavelets has been derived. First, we find the matrix P for $\mu = 4$, $M = 3$ and $k = 1$. Then the twelve basis functions are given by

$$\left.\begin{aligned} \psi_{1,0}^{(4)}(t) &= 2, \\ \psi_{1,1}^{(4)}(t) &= 2\sqrt{3}(8t-1), \\ \psi_{1,2}^{(4)}(t) &= 2\sqrt{5}\left(\tfrac{3(8t-1)^2-1}{2}\right), \end{aligned}\right\} \quad 0 \leq t < \frac{1}{4}, \tag{5}$$

,

$$\left.\begin{aligned} \psi_{2,0}^{(4)}(t) &= 2, \\ \psi_{2,1}^{(4)}(t) &= 2\sqrt{3}(8t-3), \\ \psi_{2,2}^{(4)}(t) &= 2\sqrt{5}\left(\tfrac{3(8t-3)^2-1}{2}\right), \end{aligned}\right\} \quad \frac{1}{4} \leq t < \frac{2}{4}, \tag{6}$$

$$\left.\begin{aligned} \psi_{3,0}^{(4)}(t) &= 2, \\ \psi_{3,1}^{(4)}(t) &= 2\sqrt{3}(8t-5), \\ \psi_{3,2}^{(4)}(t) &= 2\sqrt{5}\left(\tfrac{3(8t-5)^2-1}{2}\right), \end{aligned}\right\} \quad \frac{2}{4} \leq t < \frac{3}{4}, \tag{7}$$

$$\left.\begin{aligned} \psi_{4,0}^{(4)}(t) &= 2, \\ \psi_{4,1}^{(4)}(t) &= 2\sqrt{3}(8t-7), \\ \psi_{4,2}^{(4)}(t) &= 2\sqrt{5}\left(\tfrac{3(8t-7)^2-1}{2}\right), \end{aligned}\right\} \quad \frac{3}{4} \leq t < 1. \tag{8}$$

Integrating Eqs. (5) - (8) from 0 to t and expanding in terms of $\psi^{(4)}_{n,m}(t)$, we get

$$\begin{aligned}
\int_0^t \psi^{(4)}_{1,0}(x)\,dx &= \begin{cases} 2t, & 0 \le t < \frac{1}{4}; \\ \frac{1}{2}, & otherwise, \end{cases} \\
&= \frac{1}{8}\psi^{(4)}_{1,0}(t) + \frac{1}{8\sqrt{3}}\psi^{(4)}_{1,1}(t) + \frac{1}{4}\psi^{(4)}_{2,0}(t) + \frac{1}{4}\psi^{(4)}_{3,0}(t) + \frac{1}{4}\psi^{(4)}_{4,0}(t), \\
&= \left[\frac{1}{8}, \frac{1}{8\sqrt{3}}, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, 0\right]\psi^{(4)}(t). \\
\int_0^t \psi^{(4)}_{1,1}(x)\,dx &= \begin{cases} \frac{\sqrt{3}}{8}[(8t-1)^2 - 1], & 0 \le t < \frac{1}{4}; \\ 0, & otherwise, \end{cases} \\
&= -\frac{\sqrt{3}}{24}\psi^{(4)}_{1,0}(t) + \frac{\sqrt{3}}{24\sqrt{5}}\psi^{(4)}_{1,2}(t), \\
&= \left[-\frac{\sqrt{3}}{24}, 0, \frac{\sqrt{3}}{24\sqrt{5}}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]\psi^{(4)}(t). \\
\int_0^t \psi^{(4)}_{1,2}(x)\,dx &= \begin{cases} \frac{\sqrt{5}}{8}[(8t-1)^3 - (8t-1)], & 0 \le t < \frac{1}{4}; \\ 0, & otherwise, \end{cases} \\
&= -\frac{\sqrt{5}}{40\sqrt{3}}\psi^{(4)}_{1,1}(t), \\
&= \left[0, -\frac{\sqrt{5}}{40\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]\psi^{(4)}(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_0^t \psi^{(4)}_{2,0}(x)\,dx &= \frac{1}{8}\psi^{(4)}_{2,0}(t) + \frac{1}{8\sqrt{3}}\psi^{(4)}_{2,1}(t) + \frac{1}{4}\psi^{(4)}_{3,0}(t) + \frac{1}{4}\psi^{(4)}_{4,0}(t), \\
\int_0^t \psi^{(4)}_{2,1}(x)\,dx &= -\frac{\sqrt{3}}{24}\psi^{(4)}_{2,0}(t) + \frac{\sqrt{3}}{24\sqrt{5}}\psi^{(4)}_{2,2}(t), \\
\int_0^t \psi^{(4)}_{2,2}(x)\,dx &= -\frac{\sqrt{5}}{40\sqrt{3}}\psi^{(4)}_{2,1}(t), \\
\int_0^t \psi^{(4)}_{3,0}(x)\,dx &= \frac{1}{8}\psi^{(4)}_{3,0}(t) + \frac{1}{8\sqrt{3}}\psi^{(4)}_{3,1}(t) + \frac{1}{4}\psi^{(4)}_{4,0}(t), \\
\int_0^t \psi^{(4)}_{3,1}(x)\,dx &= -\frac{\sqrt{3}}{24}\psi^{(4)}_{3,0}(t) + \frac{\sqrt{3}}{24\sqrt{5}}\psi^{(4)}_{3,2}(t), \\
\int_0^t \psi^{(4)}_{3,2}(x)\,dx &= -\frac{\sqrt{5}}{40\sqrt{3}}\psi^{(4)}_{3,1}(t), \\
\int_0^t \psi^{(4)}_{4,0}(x)\,dx &= \frac{1}{8}\psi^{(4)}_{4,0}(t) + \frac{1}{8\sqrt{3}}\psi^{(4)}_{4,1}(t), \\
\int_0^t \psi^{(4)}_{4,1}(x)\,dx &= -\frac{\sqrt{3}}{24}\psi^{(4)}_{4,0}(t) + \frac{\sqrt{3}}{24\sqrt{5}}\psi^{(4)}_{4,2}(t), \\
\int_0^t \psi^{(4)}_{4,2}(x)\,dx &= -\frac{\sqrt{5}}{40\sqrt{3}}\psi^{(4)}_{4,1}(t).
\end{aligned}$$

Thus

$$\int_0^t \psi^{(4)}(x)\,dx = P\psi^{(4)}(t), \tag{9}$$

where

$$\psi^{(4)}(t) = \left[\psi^{(4)}_{1,0}, \psi^{(4)}_{1,1}, \psi^{(4)}_{1,2}, \psi^{(4)}_{2,0}, \psi^{(4)}_{2,1}, \psi^{(4)}_{2,2}, \psi^{(4)}_{3,0}, \psi^{(\mu)}_{3,1}, \psi^{(4)}_{3,2}, \psi^{(4)}_{4,0}, \psi^{(4)}_{4,1}, \psi^{(4)}_{4,2}\right]^T,$$

and the extended Legendre wavelets operational matrix of integration is given by

$$P = \frac{1}{8}\begin{bmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
-\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{\sqrt{3}} & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{3}} & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0
\end{bmatrix}.$$

The operational matrix P can be written as

$$P = \frac{1}{8}\begin{bmatrix}
L_{3\times3} & F_{3\times3} & F_{3\times3} & F_{3\times3} \\
0_{3\times3} & L_{3\times3} & F_{3\times3} & F_{3\times3} \\
0_{3\times3} & 0_{3\times3} & L_{3\times3} & F_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & L_{3\times3}
\end{bmatrix},$$

where L and F are 3×3 matrices given by

$$L_{3\times3} = \begin{bmatrix}
1 & \frac{1}{\sqrt{3}} & 0 \\
-\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} \\
0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0
\end{bmatrix}, \quad and \quad F_{3\times3} = \begin{bmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$

In general, we have

$$\int_0^t \psi^{(\mu)}(x)\,dx = P\psi^{(\mu)}(t), \tag{10}$$

where $\psi^{(\mu)}(t)$ is given in (4) and P is a $(\mu^k M) \times (\mu^k M)$ matrix given by

$$P = \frac{1}{2\mu^k}\begin{bmatrix} L & F & F & \dots & F \\ 0 & L & F & \dots & F \\ 0 & 0 & L & \dots & F \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F \\ 0 & 0 & \dots & 0 & L \end{bmatrix},$$

and L & F are $(M \times M)$ matrices given as

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \ddots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\frac{\sqrt{(2M-3)}}{(2M-3)\sqrt{(2M-5)}} & 0 & \frac{\sqrt{(2M-3)}}{(2M-3)\sqrt{(2M-1)}} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{\sqrt{(2M-1)}}{(2M-1)\sqrt{(2M-3)}} & 0 \end{bmatrix},$$

and

$$F = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

4. Extended Legendre Wavelets product operational matrix

The following property of the product of two extended Legendre wavelets function vectors will also be used :

$$\psi^{(\mu)}(t)(\psi^{(\mu)}(t))^T C \approx \tilde{C}\psi^{(\mu)}(t) \tag{11}$$

Here C is given in Eq. (3), $\psi^{(\mu)}(t)$ can be obtained similar to equation (4) and $\tilde{C}$ is a $(\mu^k M) \times (\mu^k M)$ matrix. To illustate the calculation procedure, we choose $\mu = 4$, $M = 3$ and $k = 1$. Using $\psi^{(4)}(t)$ we get

$\psi^{(4)}(\psi^{(4)})^T =$

$$\begin{bmatrix}
\psi^{(4)}_{1,0}\psi^{(4)}_{1,0} & \psi^{(4)}_{1,0}\psi^{(4)}_{1,1} & \psi^{(4)}_{1,0}\psi^{(4)}_{1,2} & \psi^{(4)}_{1,0}\psi^{(4)}_{2,0} & \psi^{(4)}_{1,0}\psi^{(4)}_{2,1} & \psi^{(4)}_{1,0}\psi^{(4)}_{2,2} & \psi^{(4)}_{1,0}\psi^{(4)}_{3,0} & \psi^{(4)}_{1,0}\psi^{(4)}_{3,1} & \psi^{(4)}_{1,0}\psi^{(4)}_{3,2} & \psi^{(4)}_{1,0}\psi^{(4)}_{4,0} & \psi^{(4)}_{1,0}\psi^{(4)}_{4,1} & \psi^{(4)}_{1,0}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{1,1}\psi^{(4)}_{1,0} & \psi^{(4)}_{1,1}\psi^{(4)}_{1,1} & \psi^{(4)}_{1,1}\psi^{(4)}_{1,2} & \psi^{(4)}_{1,1}\psi^{(4)}_{2,0} & \psi^{(4)}_{1,1}\psi^{(4)}_{2,1} & \psi^{(4)}_{1,1}\psi^{(4)}_{2,2} & \psi^{(4)}_{1,1}\psi^{(4)}_{3,0} & \psi^{(4)}_{1,1}\psi^{(4)}_{3,1} & \psi^{(4)}_{1,1}\psi^{(4)}_{3,2} & \psi^{(4)}_{1,1}\psi^{(4)}_{4,0} & \psi^{(4)}_{1,1}\psi^{(4)}_{4,1} & \psi^{(4)}_{1,1}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{1,2}\psi^{(4)}_{1,0} & \psi^{(4)}_{1,2}\psi^{(4)}_{1,1} & \psi^{(4)}_{1,2}\psi^{(4)}_{1,2} & \psi^{(4)}_{1,2}\psi^{(4)}_{2,0} & \psi^{(4)}_{1,2}\psi^{(4)}_{2,1} & \psi^{(4)}_{1,2}\psi^{(4)}_{2,2} & \psi^{(4)}_{1,2}\psi^{(4)}_{3,0} & \psi^{(4)}_{1,2}\psi^{(4)}_{3,1} & \psi^{(4)}_{1,2}\psi^{(4)}_{3,2} & \psi^{(4)}_{1,2}\psi^{(4)}_{4,0} & \psi^{(4)}_{1,2}\psi^{(4)}_{4,1} & \psi^{(4)}_{1,2}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{2,0}\psi^{(4)}_{1,0} & \psi^{(4)}_{2,0}\psi^{(4)}_{1,1} & \psi^{(4)}_{2,0}\psi^{(4)}_{1,2} & \psi^{(4)}_{2,0}\psi^{(4)}_{2,0} & \psi^{(4)}_{2,0}\psi^{(4)}_{2,1} & \psi^{(4)}_{2,0}\psi^{(4)}_{2,2} & \psi^{(4)}_{2,0}\psi^{(4)}_{3,0} & \psi^{(4)}_{2,0}\psi^{(4)}_{3,1} & \psi^{(4)}_{2,0}\psi^{(4)}_{3,2} & \psi^{(4)}_{2,0}\psi^{(4)}_{4,0} & \psi^{(4)}_{2,0}\psi^{(4)}_{4,1} & \psi^{(4)}_{2,0}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{2,1}\psi^{(4)}_{1,0} & \psi^{(4)}_{2,1}\psi^{(4)}_{1,1} & \psi^{(4)}_{2,1}\psi^{(4)}_{1,2} & \psi^{(4)}_{2,1}\psi^{(4)}_{2,0} & \psi^{(4)}_{2,1}\psi^{(4)}_{2,1} & \psi^{(4)}_{2,1}\psi^{(4)}_{2,2} & \psi^{(4)}_{2,1}\psi^{(4)}_{3,0} & \psi^{(4)}_{2,1}\psi^{(4)}_{3,1} & \psi^{(4)}_{2,1}\psi^{(4)}_{3,2} & \psi^{(4)}_{2,1}\psi^{(4)}_{4,0} & \psi^{(4)}_{2,1}\psi^{(4)}_{4,1} & \psi^{(4)}_{2,1}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{2,2}\psi^{(4)}_{1,0} & \psi^{(4)}_{2,2}\psi^{(4)}_{1,1} & \psi^{(4)}_{2,2}\psi^{(4)}_{1,2} & \psi^{(4)}_{2,2}\psi^{(4)}_{2,0} & \psi^{(4)}_{2,2}\psi^{(4)}_{2,1} & \psi^{(4)}_{2,2}\psi^{(4)}_{2,2} & \psi^{(4)}_{2,2}\psi^{(4)}_{3,0} & \psi^{(4)}_{2,2}\psi^{(4)}_{3,1} & \psi^{(4)}_{2,2}\psi^{(4)}_{3,2} & \psi^{(4)}_{2,2}\psi^{(4)}_{4,0} & \psi^{(4)}_{2,2}\psi^{(4)}_{4,1} & \psi^{(4)}_{2,2}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{3,0}\psi^{(4)}_{1,0} & \psi^{(4)}_{3,0}\psi^{(4)}_{1,1} & \psi^{(4)}_{3,0}\psi^{(4)}_{1,2} & \psi^{(4)}_{3,0}\psi^{(4)}_{2,0} & \psi^{(4)}_{3,0}\psi^{(4)}_{2,1} & \psi^{(4)}_{3,0}\psi^{(4)}_{2,2} & \psi^{(4)}_{3,0}\psi^{(4)}_{3,0} & \psi^{(4)}_{3,0}\psi^{(4)}_{3,1} & \psi^{(4)}_{3,0}\psi^{(4)}_{3,2} & \psi^{(4)}_{3,0}\psi^{(4)}_{4,0} & \psi^{(4)}_{3,0}\psi^{(4)}_{4,1} & \psi^{(4)}_{3,0}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{3,1}\psi^{(4)}_{1,0} & \psi^{(4)}_{3,1}\psi^{(4)}_{1,1} & \psi^{(4)}_{3,1}\psi^{(4)}_{1,2} & \psi^{(4)}_{3,1}\psi^{(4)}_{2,0} & \psi^{(4)}_{3,1}\psi^{(4)}_{2,1} & \psi^{(4)}_{3,1}\psi^{(4)}_{2,2} & \psi^{(4)}_{3,1}\psi^{(4)}_{3,0} & \psi^{(4)}_{3,1}\psi^{(4)}_{3,1} & \psi^{(4)}_{3,1}\psi^{(4)}_{3,2} & \psi^{(4)}_{3,1}\psi^{(4)}_{4,0} & \psi^{(4)}_{4,1}\psi^{(4)}_{4,1} & \psi^{(4)}_{3,1}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{3,2}\psi^{(4)}_{1,0} & \psi^{(4)}_{3,2}\psi^{(4)}_{1,1} & \psi^{(4)}_{3,2}\psi^{(4)}_{1,2} & \psi^{(4)}_{3,2}\psi^{(4)}_{2,0} & \psi^{(4)}_{3,2}\psi^{(4)}_{2,1} & \psi^{(4)}_{3,2}\psi^{(4)}_{2,2} & \psi^{(4)}_{3,2}\psi^{(4)}_{3,0} & \psi^{(4)}_{3,2}\psi^{(4)}_{3,1} & \psi^{(4)}_{3,2}\psi^{(4)}_{3,2} & \psi^{(4)}_{3,2}\psi^{(4)}_{4,0} & \psi^{(4)}_{3,2}\psi^{(4)}_{4,1} & \psi^{(4)}_{3,2}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{4,0}\psi^{(4)}_{1,0} & \psi^{(4)}_{4,0}\psi^{(4)}_{1,1} & \psi^{(4)}_{4,0}\psi^{(4)}_{1,2} & \psi^{(4)}_{4,0}\psi^{(4)}_{2,0} & \psi^{(4)}_{4,0}\psi^{(4)}_{2,1} & \psi^{(4)}_{4,0}\psi^{(4)}_{2,2} & \psi^{(4)}_{4,0}\psi^{(4)}_{3,0} & \psi^{(4)}_{4,0}\psi^{(4)}_{3,1} & \psi^{(4)}_{4,0}\psi^{(4)}_{3,2} & \psi^{(4)}_{4,0}\psi^{(4)}_{4,0} & \psi^{(4)}_{4,0}\psi^{(4)}_{4,1} & \psi^{(4)}_{4,0}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{4,1}\psi^{(4)}_{1,0} & \psi^{(4)}_{4,1}\psi^{(4)}_{1,1} & \psi^{(4)}_{4,1}\psi^{(4)}_{1,2} & \psi^{(4)}_{4,1}\psi^{(4)}_{2,0} & \psi^{(4)}_{4,1}\psi^{(4)}_{2,1} & \psi^{(4)}_{4,1}\psi^{(4)}_{2,2} & \psi^{(4)}_{4,1}\psi^{(4)}_{3,0} & \psi^{(4)}_{4,1}\psi^{(4)}_{3,1} & \psi^{(4)}_{4,1}\psi^{(4)}_{3,2} & \psi^{(4)}_{4,1}\psi^{(4)}_{4,0} & \psi^{(4)}_{4,1}\psi^{(4)}_{4,1} & \psi^{(4)}_{4,1}\psi^{(4)}_{4,2} \\
\psi^{(4)}_{4,2}\psi^{(4)}_{1,0} & \psi^{(4)}_{4,2}\psi^{(4)}_{1,1} & \psi^{(4)}_{4,2}\psi^{(4)}_{1,2} & \psi^{(4)}_{4,2}\psi^{(4)}_{2,0} & \psi^{(4)}_{4,2}\psi^{(4)}_{2,1} & \psi^{(4)}_{4,2}\psi^{(4)}_{2,2} & \psi^{(4)}_{4,2}\psi^{(4)}_{3,0} & \psi^{(4)}_{4,2}\psi^{(4)}_{3,1} & \psi^{(4)}_{4,2}\psi^{(4)}_{3,2} & \psi^{(4)}_{4,2}\psi^{(4)}_{4,0} & \psi^{(4)}_{4,2}\psi^{(4)}_{4,1} & \psi^{(4)}_{4,2}\psi^{(4)}_{4,2}
\end{bmatrix}. \quad (12)$$

In Eq. (12) we have

$$\psi^{(4)}_{i,j}\psi^{(4)}_{k,l} = 0 \quad if \quad i \neq k.$$

Also,

$$\psi^{(4)}_{i,0}\psi^{(4)}_{i,j} = 2\psi^{(4)}_{i,j},$$

$$\psi^{(4)}_{i,1}\psi^{(4)}_{i,1} = 2\psi^{(4)}_{i,0} + \frac{4}{\sqrt{5}}\psi^{(4)}_{i,2},$$

$$\psi^{(4)}_{i,1}\psi^{(4)}_{i,2} = \frac{4}{\sqrt{5}}\psi^{(4)}_{i,1},$$

$$\psi^{(4)}_{i,2}\psi^{(4)}_{i,2} = 2\psi^{(4)}_{i,0} + \frac{4\sqrt{5}}{7}\psi^{(4)}_{i,2}.$$

If we retain only the elements of $\psi^{(4)}(t)$, then we have

$$\psi^{(4)}(\psi^{(4)})^T = \begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix},$$

where

$$B_i = \begin{bmatrix} 2\psi^{(4)}_{i,0} & 2\psi^{(4)}_{i,1} & 2\psi^{(4)}_{i,2} \\ 2\psi^{(4)}_{i,1} & (\psi^{(4)}_{i,0} + \frac{4}{\sqrt{5}}\psi^{(4)}_{i,2}) & \frac{4}{\sqrt{5}}\psi^{(4)}_{i,1} \\ 2\psi^{(4)}_{i,2} & \frac{4}{\sqrt{5}}\psi^{(4)}_{i,1} & (2\psi^{(4)}_{i,0} + \frac{4\sqrt{5}}{7}\psi^{(4)}_{i,2}) \end{bmatrix}.$$

By using the vector C in (3), the 12 × 12 matrix $\tilde{C}$ in (11) can be written as

$$\tilde{C} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

where,

$$A_i = \begin{bmatrix} 2C_{i,0} & 2C_{i,1} & 2C_{i,2} \\ 2C_{i,1} & (2C_{i,0} + \frac{4}{\sqrt{5}}C_{i,2}) & \frac{4}{\sqrt{5}}C_{i,1} \\ 2C_{i,2} & \frac{4}{\sqrt{5}}C_{i,1} & (2C_{i,0} + \frac{4\sqrt{5}}{7}C_{i,2}) \end{bmatrix},$$

and 0 is 3 × 3 zero matrix.

5. Applications

Example 1. Consider the following differential equation,

$$0.25y'(t) + y(t) = 1, \tag{13}$$

with initial condition,

$$y(0) = 0. \tag{14}$$

The exact solution of (13) - (14) is $y(t) = 1 - exp(-4t)$.
Let

$$y(t) = C^T \psi^{(\mu)}(t). \tag{15}$$

Suppose

$$1 = d^T \psi^{(\mu)}(t), \tag{16}$$

integrating Eq. (13) from 0 to t, using Eqs. (10) & (14) - (16), and after simplification, we get

$$C^T(0.25I + P) = d^T p. \tag{17}$$

Eq. (17) represents the set of algebraic equations which is solvable for C. In the case of $\mu = 2, 4, 5$ the approximate solution by the presented method is shown in Figures (1), (2), and (3) to make it easier to compare with the exact solution. In Table (1), the approximate solutions for $\mu = 2, 4, 5$; $k = 1$, and $M = 3$ have been compared with Legendre wavelets method (LWM) and exact solution, which is shown in Figures (4).

Example 2.

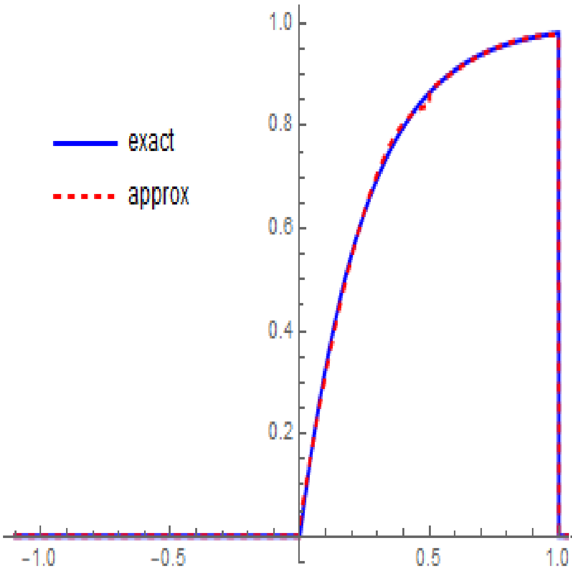

FIGURE 1. Graphical representation of exact sol. (dark line) and approx. sol. (dashed line) of example (1) for $\mu = 2$.

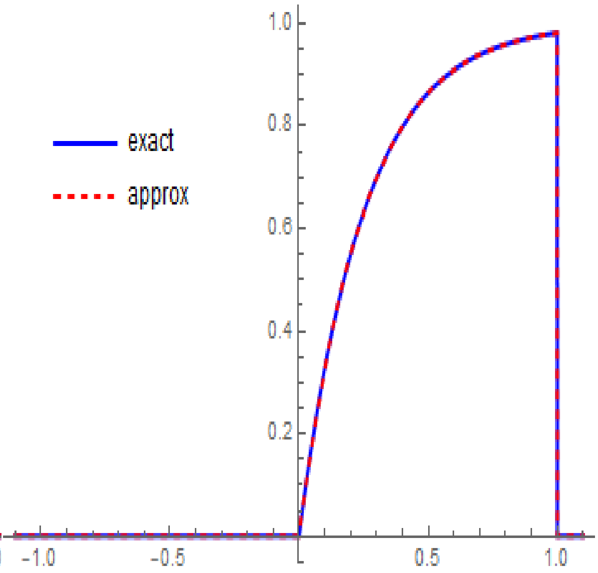

FIGURE 2. Graphical representation of exact sol. (dark line) and approx. sol. (dashed line) of example (1) for $\mu = 4$.

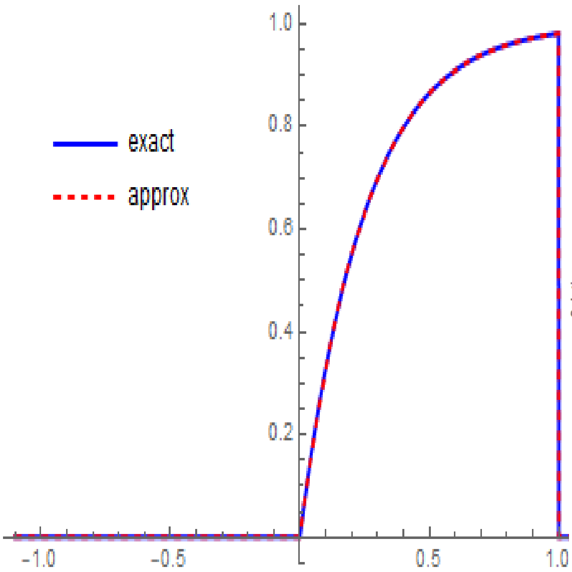

FIGURE 3. Graphical representation of exact sol. (dark line) and approx. sol. (dashed line) of example (1) for $\mu = 5$.

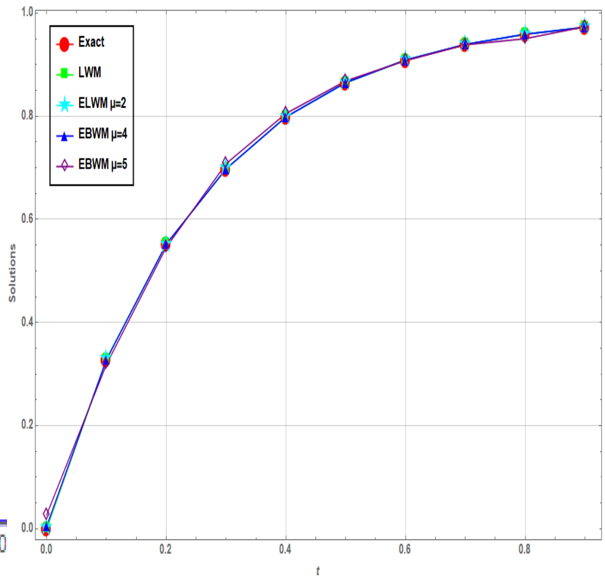

FIGURE 4. Numerical results obtained from LWM, ELWM for $\mu = 2, 4, 5; k = 1, M = 3$ of example (1).

t	Solution for $\mu = 2$, $k = 1$, $M = 3$	Solution for $\mu = 4$, $k = 1$, $M = 3$	Solution for $\mu = 5$, $k = 1$, $M = 3$	LWM for $k = 1$, $M = 3$	Exact solution
0.0	0.0270	0.0051	0.0029	0.0002	0.0000
0.1	0.3189	0.3284	0.3298	0.3284	0.3286
0.2	0.5459	0.5523	0.5519	0.5523	0.5521
0.3	0.7081	0.6980	0.6988	0.6980	0.6984
0.4	0.8054	0.7987	0.7986	0.7987	0.7986
0.5	0.8685	0.8653	0.8646	0.8653	0.8650
0.6	0.9079	0.9091	0.9095	0.9091	0.9091
0.7	0.9386	0.9394	0.9392	0.9394	0.9393
0.8	0.9605	0.9591	0.9593	0.9591	0.9591
0.9	0.9737	0.9727	0.9726	0.9727	0.9726

TABLE 1. Comparison between exact, LWM and ELWM solution for $\mu = 2, 4, 5$; $k = 1$, $M = 3$ of example (1).

Bessel differential equation of order zero : Consider the following Bessel differential equation of order zero

$$ty^{''} + y^{'} + ty = 0, \tag{18}$$

with initial conditions,

$$y(0) = 1, \quad y^{'}(0) = 0. \tag{19}$$

The solution of Eqs. (18) - (19) known as the Bessel function of first kind of order zero denoted by $J_0(t)$ is given by

$$J_0(t) = \sum_{i=1}^{\infty} \frac{(-1)^i}{(i!)^2} \left(\frac{t}{2}\right)^{2i}. \tag{20}$$

Here we solve same problem using extended Legendre wavelets method,
let us consider

$$y^{''}(t) = C^T \psi^{(\mu)}(t). \tag{21}$$

Integrating Eq. (21) and using Eqs. (10), (16), (18) & (19), we have

$$y^{'}(t) = C^T P \psi^{(\mu)}(t), \quad y(t) = C^T P^2 \psi^{(\mu)}(t) + d^T \psi^{(\mu)}(t), \tag{22}$$

we can write

$$t = e^T \psi^{(\mu)}(t). \tag{23}$$

Now substituting Eqs. (21) - (23) in (18), we obtain

$$e^T\psi^{(\mu)}(t)(\psi^{(\mu)}(t))^T C + (\psi^{(\mu)}(t))^T P^T C + e^T\psi^{(\mu)}(t)(\psi^{(\mu)}(t))^T (P^2)^T C + e^T\psi^{(\mu)}(t)(\psi^{(\mu)}(t))^T d = 0. \tag{24}$$

Let

$$\psi^{(\mu)}(t)(\psi^{(\mu)}(t))^T e = \tilde{E}\psi^{(\mu)}(t), \tag{25}$$

using Eq. (25) in (24) and simplifying, we have

$$\tilde{E}C + P^T C + \tilde{E}(P^2)^T C + \tilde{E}d = 0. \tag{26}$$

Equation (26) again represents the system of algebraic equations which is solvable for C. Table (2) shows the comparison between the exact solution, Legendre wavelet method (LWM) and approximate solutions for $\mu = 2, 4, 5$, $k = 1$, $M = 3$, which is presented in Figure (5).

t	Solution for $\mu = 2,\ k = 1,\ M = 3$	Solution for $\mu = 4,\ k = 1,\ M = 3$	Solution for $\mu = 5,\ k = 1,\ M = 3$	LWM for $k = 1,\ M = 3$	Exact solution
0.0	1.000096	1.000006	1.000002	1.000000	1.000000
0.1	0.997472	0.997499	0.997501	0.997502	0.997501
0.2	0.989988	0.990027	0.990032	0.990024	0.990024
0.3	0.977644	0.977620	0.977626	0.977625	0.977626
0.4	0.960439	0.960402	0.960410	0.960396	0.960398
0.5	0.938739	0.938498	0.938469	0.938468	0.938469
0.6	0.911913	0.911996	0.912021	0.912004	0.912004
0.7	0.881118	0.881211	0.881201	0.881200	0.881200
0.8	0.846356	0.846274	0.846307	0.846285	0.846287
0.9	0.807626	0.807534	0.807524	0.807524	0.807523

TABLE 2. Comparison between exact, LWM and ELWM solution for $\mu = 2, 4, 5;\ k = 1,\ M = 3$ of example (2).

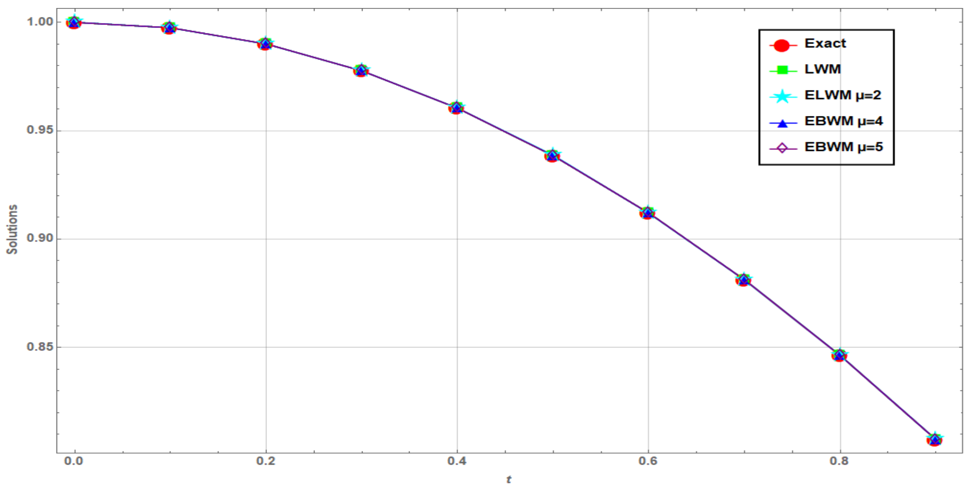

FIGURE 5. Numerical results obtained from LWM, ELWM for $\mu = 2, 4, 5; k = 1, M = 3$ of example (2).

6. Conclusions

In this paper, we have obtained the operational matrix of integration for extended Legendre wavelets (ELW) which generalizes the Legendre wavelets (LW) operational matrix of integration given in [30]. From our discussion, it is clear that the obtained extended Legendre wavelets operational matrix of integration is valid for any positive integer $\mu \geq 2$. We have also solved two differential equations using this extended Legendre wavelets operational matrix of integration and observe that if we use larger values of μ, then the numerical accuracy of the solution increases. In the figures $(1, 2, 3, 4$ & $5)$ the exact solutions and the approximate solutions for $\mu = 2, 4, 5;\ k = 1, M = 3$ have been ploted and we observe that the solutions become very close to the exact solution which is also justified in Table (1) & (2).

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7. Declaration

Data availability statement. The data from the previous studies are used to support the findings of the paper and they are cited at the relevant places in the paper according as the reference list of the article.

Conflict of interest. There is no conflict of interest.

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