TWMS J. App. and Eng. Math. V.15, N.6, 2025, pp. 1325-1346

# SOMBOR INDEX OF FUZZY GRAPH

B. SOME[1*], A. PAL[2], §

Abstract. Topological indices (TI) play a crucial role across various research domains, including network theory, spectral graph theory, and molecular chemistry. These indices are created primarily in the context of crisp graphs, but they can also be applied to fuzzy graphs, which are a more generalized version of crisp graphs. This article presents the Sombor index for fuzzy graphs (SOF(G)) and explores how it can be applied to diverse categories of fuzzy graphs, such as cycles, stars, complete graphs, and fuzzy subgraphs. Results are obtained by this index after vertices and edges are eliminated and we also proved that it holds for isomorphic fuzzy graphs and established interesting bounds for SOF(G). Along with several theorems and examples, the Sombor index is introduced and explored for fuzzy directed graphs (FDG), regular fuzzy graphs (RFGs), and fuzzy cycles. Furthermore, a connection between the Sombor index and other fuzzy graph indices is established. Finally, an application is provided demonstrating the use of the Sombor index of a fuzzy graph to identify the country with the optimal case of human trafficking.

Keyword: Fuzzy graph, Molecular topology, Sombor index, Topological indices.

2020 Subject Classification: 05C09, 05C20, 05C38, 05C72, 05C90, 05C92.

## 1. Introduction

Graph theory is crucial for linking specific values to define parameters across various fields such as computer science, operation researches, network routing, wireless sensor networks, engineering, and medical science. However, uncertainty plays a significant role in many situations that impact our daily lives. In 1965 Zadeh's [51] fuzzy set theory was crucial in solving these kinds of issues. This turned out to be the motivation for Rosenfeld's [43] development of the fuzzy graph (FG) idea in 1975. This concept also encompassed fuzzy relations, fuzzy bridges, fuzzy blocks, and fuzzy distances for a fuzzy graph. Fuzzy graphs including strong arcs and different kinds of arcs have been explained by Mordeson and Peng [37] and by Bhutani et al. [6]. Samanta and Pal [47, 48] have discovered a fuzzy graph-based social network system and a fuzzy graph-based telephony system. In 2018, Ghorai and Pal [13] presented a novel idea concerning Regular fuzzy

[1]Department of Mathematics, National Institute of Technology, Durgapur, West Bengal-713209, India. e-mail:biswajitsome.6@gmail.com ORCID ID- 0009-0004-0677-0548.

[2]Department of Mathematics, National Institute of Technology, Durgapur, West Bengal-713209, India. e-mail:anita.pal@maths.nitdgp.ac.in ORCID ID 0000-0002-2514-5463.

∗ Corresponding author.

§ Manuscript received: March 03, 2024; accepted: July 08, 2024.

graphs within a bipolar fuzzy framework. Akram and his research team explored various concepts in fuzzy hyper-graph theory, including bipolar fuzzy soft hyper-graphs [46], soft fuzzy hyper-graphs [2], generalized m-polar fuzzy hyper-graphs [49], and connections within rough fuzzy hyper-graphs [4]. Pal et al. [42] analyzed the vertex degree, strong degree, and strong neighbor characteristics of a graph. Further developments and related findings of a FG were discussed in [35].

Chemical graph theory, an essential aspect of mathematical chemistry, employs topological indices (TIs). Topological indices in fuzzy graphs are numerical parameters that characterize the structure and properties of the graph in a way that incorporates the fuzziness or uncertainty inherent in the data. These indices have various uses in different fields, particularly in areas where data may be imprecise or uncertain. In a chemical structure, atoms are the vertices and bonds are the edges. In 1947, Harold Wiener [50] originally created the Wiener index (WI), a topological measure used to determine the boiling point of paraffin. A distinct degree-based topological index, the forgotten topological index (F-index), was introduced by Fortula and Gutman [11] in 2015. The subdivision graph and line graphs of the F-index and F-coindex were studied in 2018 by Amin and Nayeem [1]. In 2017, Abdo et al. [5] explored the F-index concerning extremal trees within crisp graphs. Their investigation involved the hyper-Wiener index (HWI), an extension of the Wiener index, from both theoretical and practical viewpoints. The HWI has applications in chemical graph theory, spectral graph theory, and biochemistry. The hyper-Wiener index was first introduced by Randic [45]. Mondal et al. [33, 34] introduced Neighborhood-ZI additionally explored Neighborhood-ZI in QSPR studies. A recently created molecular structure descriptor by Gutman [14] is the Sombor index, which is based on vertex degrees. This article states that the problem of determining which graph has the highest/lowest Sombor index among all trees, graphs, and connected graphs with a given order has been resolved. Gutman and his research team [15, 16, 17, 18] have investigated various aspects of the Sombor indices, including their basic properties, the KG - Sombor index of Kragujevac trees, the relationship of Sombor indices with geometry, and the product of Sombor and modified Sombor indices.

Motivated by this, Milovanovic et al. [36] demonstrate some notable characteristics of Sombor indices. A brief discussion of molecular trees with exceptionally high Sombor index values is provided by Deng et al. [9], while Cruz et al. [7] address the Sombor index of trees with three or fewer branch vertices. Extreme values on the Sombor index of trees are discussed by Chen and colleagues [8] in 2022. Jamila and their research team have made advancements in the field of topological numbers of fuzzy soft graphs and their applications [3, 19, 28] and they have also explored innovative single-value neutrosophic fuzzy topological graph parameters and introduced a novel approach for improving the performance of vaccination centers using intuitionistic fuzzy Sombor indices. Several other Sombor index forms are described [10, 29, 38]

1.1. **Motivation and significance of the article.** Topological indices for crisp graphs have practical applications, but understanding their characteristics often requires laboratory research on chemicals. To address this, theoretical chemistry has devised many topological indices. The Wiener index and the Connectivity index (CI) for graphs were recently introduced by Binu and colleagues [30, 31]. Recent advancements in this field, as introduced by Islam and Pal [20, 21, 22, 23] include the concepts of hyper-WI, hyper-CI, first ZI, and the F-index in fuzzy graphs. They have further developed the F-index for fuzzy graphs [24] and explored its application in analyzing Indian railway crime. Additionally, they have worked on the multiplicative version of the first Zagreb index [25] in fuzzy graphs and its application in crime analysis, as well as investigating the edge F-index [26] on fuzzy graphs with applications in molecular chemistry. Poulik and his research group [39, 40, 41] were the first to invent the Wiener absolute index in bipolar fuzzy graphs. They also introduced specific graph indices within bipolar fuzzy settings and discussed their applications. Additionally, they provided a brief explanation of how the Randić index can be applied to bipolar fuzzy graphs in network systems. Recent work by Mathew and collaborators [27] introduced the neighborhood connectivity index for fuzzy graphs and explored its application to human trafficking. Applications for topological indices in fuzzy graphs include bio informatics and drug design, chemical reactivity, material science, network analysis, pharmacology, image processing, data mining, machine learning, and environmental studies,cryptography, and pattern recognition. Inspired by the aforementioned publications and application this study examines the Sombor index for fuzzy graphs. We addressed the following research questions : What is the Sombor index in fuzzy graphs ? What are the bounds for the Sombor index of fuzzy graphs in terms of the order and size of G ? What are the bounds of different types of graphs ? What is the value of the Sombor index of fuzzy digraphs, regular fuzzy graphs, and fuzzy cycles ? And how does the Sombor index relate to other indices of fuzzy graphs ? What are the applications of this index ?

1.2. **Frame work of the article.** The article is organized as follows : section-2 offers some basic concepts that are required to construct our main results. In section-3, the definition of the Sombor index is provided and some theoretical developments of the Sombor index are also studied, and the boundaries of this index are determined for various types of fuzzy graphs, including the cycle, star, and complete fuzzy graph, etc. Section-4, defines and examines the Sombor index of fuzzy directed graphs (FDG), regular FGs, and fuzzy cycles. In section-5, relationship between Sombor index and other indices are established. In section-6, Utilizing the Sombor index for fuzzy graphs to pinpoint the country with the highest rates of human trafficking compared to others. Section-7, the concluding section provides insights and recommendations for future considerations.

## 2. Preliminaries

In this segment, we provide fundamental definitions crucial for enhancing our understanding. A mapping function $\xi : X \to [0,1]$ characterizes a fuzzy set, represented as $\widetilde{S} = (U, \xi)$ on $X$, where $U$ is a Universal set. As the membership function, the fuzzy set $\widetilde{S}$ is connected to the function $\xi$.

**Definition 2.1.** *The subdivision graph S(G) is the graph attained from G by replacing each of its edges by a path of length 2. The line graph L(G) of a graph is the graph derived from G in such a w ay that the edges in G are replaced by vertices in L(G) and two vertices in L(G) are connected whenever the corresponding edges in G are adjacent.*

**Definition 2.2.** *On a universe $U$, the fuzzy graph is an object of the form $\widetilde{G} = (V, \xi, \Omega)$. The vertex set of the fuzzy network $V$ is represented by the vertex membership function $\xi : X \to [0,1]$ and the edge membership function $\Omega : V \times V \to [0,1]$, which satisfy $\forall\, (\omega_i, \omega_j) \in U$ and $\Omega(\omega_i, \omega_j) \leq min\,\{\Omega(\omega_i), \Omega(\omega_j)\}$, where the edge set is defined by $E = \{(\omega_i, \omega_j) : \Omega(\omega_i, \omega_j) > 0\}$.*

**Definition 2.3.** *For a vertex $\gamma \in V(\widetilde{G})$, degree is defined as $\Gamma_G(\gamma) \leqslant 2\times$ no of edges. In a fuzzy graph, each edge has a weight (or membership value) between 0 and 1, inclusive, and the degree of a vertex is defined as the sum of the weights of all edges incident to it. If we consider a vertex $v$ in a fuzzy graph and each edge incident to $v$ has a certain weight.lets denote the weights of edges incident to $v$ as $w_1, w_2, w_3, \cdots, w_k$. Then $\Gamma_G(\gamma) = w_1 + w_2 + w_3 + \cdots + w_k$, where, $k$ is the no of edges incident to $v$. In the entire fuzzy graph, consider the sum of weights of all edges, each edge has two end points, thus contributing to the degree of two vertices.If $E$ is the set of edges and $w(e)$ is the weight of edge e, the total sum of weights of all edges is $\sum\limits_{e\in E} w(e)$. Since each edge e is counted in the degree of both its end points, the total degree in the graph is $\Gamma_G(\gamma) = 2\sum\limits_{e\in E} w(e)$. The degree of any single vertex v cannot exceed the total weight of edges incident to it; since each edge is counted twice in the total degree calculation, the degree of a vertex v must be less than or equal to twice the sum of weights of edges incident to v.Hence $\Gamma_G(\gamma) \leqslant 2\sum\limits_{e\in E} w(e)$.*

*Here $\Delta(G)$ and $\delta(G)$ represents the maximum degree and minimum degree of $G$. Where $\Delta(G) = \vee_{\gamma\in V}\Gamma(\gamma)$ and $\delta(G) = \wedge \underset{\gamma\in V}{\Gamma}(\gamma)$. The total degree of $G$ denoted by $T(G)$ or simply $T = \sum\limits_{\gamma\in V}\Gamma(\gamma) = \sum\limits_{\omega\gamma\in E}\Omega(\omega\gamma)$. Throughout this article, we consider the fuzzy graph, $G_1 = (V_1, \xi_1, \Omega_1)$ has $n_1$ - vertices, $m_1$ - edges, edge set $E_1$ and $G_2 = (V_2, \xi_2, \Omega_2)$ has $n_2$ - vertices, $m_2$ - edges, edge set $E_2$ and $\triangle_1 = \triangle(G_1)$, $\triangle_2 = \triangle(G_2)$, $\delta_1 = \delta(G_1)$, $\delta_2 = \delta(G_2)$.*

**Definition 2.4.** *A fuzzy graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is considered, and it is associated with the graph $G^* = (V', E')$. A bijective mapping $\psi : \widetilde{V} \to V'$ is said to be isomorphic if it has the following properties : $\xi(\omega_i) = \xi'(\psi(\omega_i))$, $\forall\ \omega_i \in \widetilde{V}$ and $\xi(\omega_i\omega_j) = \xi'\{\psi(\omega_i), \psi(\omega_j)\}$, $\forall\ \omega_i\omega_j \in \widetilde{E}$ respectively.*

**Definition 2.5.** *Suppose we have a fuzzy graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ related to the graph $G^* = (V', E')$. If $\xi(\omega_i\omega_j) = \xi'\{\psi(\omega_i), \psi(\omega_j)\}$, $\forall\ \omega_i\omega_j \in \widetilde{E}$, then $\psi$ is described as a co-weak isomorphism.*

**Definition 2.6.** *Take $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ represent a fuzzy graph associated with $G^* = (V', E')$. If $G^*$ is a cycle and there is no single edge $\omega_i\omega_j \in \widetilde{G}$ for which $\xi(\omega_i\omega_j)$ is the minimum, $\forall\ \omega_i\omega_j \in \widetilde{E}$, then a fuzzy cycle is defined as the fuzzy graph $\widetilde{G}$.*

**Definition 2.7.** *$\widetilde{G}' = (\widetilde{V}', \widetilde{E}')$ is a fuzzy graph. Given another fuzzy graph, is considered its fuzzy sub-graph (FSG). $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, $\xi(\omega_i) = \xi'(\omega_i)$, $\forall\ \omega_i \in \widetilde{V}'$, and $\xi(\omega_i\omega_j) = \xi'(\omega_i\omega_j)$, $\forall\ \omega_i\omega_j \in \widetilde{E}'$, if $\widetilde{V}' \subseteq \widetilde{V}$ and $\widetilde{E}' \subseteq \widetilde{E}$ satisfy these conditions.*

**Definition 2.8.** *In a fuzzy graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, the open neighborhood degree, or simply the degree of a vertex $\omega_i$ is denoted $deg(\omega_i) = \Gamma(\omega_i)$, where $deg(\omega_i) = \sum\limits_{\substack{\omega_i\neq\omega_j \\ \omega_i\omega_j\in\widetilde{E}(\widetilde{G})}} \xi(\omega_i\omega_j)$. If $deg(\omega_i) = r_1$, $\forall\ \omega_i \in \widetilde{V}$, then it is called $r_1$-regular in $\widetilde{G}$. If $\xi(\omega_i\omega_j) = \min\{\xi(\omega_i), \xi(\omega_j)\}$, $\forall\ \omega_i, \omega_j \in \widetilde{V}$, then $\widetilde{G}$ is said to be complete fuzzy graph.*

**Definition 2.9.** *In the fuzzy graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, the closed neighborhood degree (CND) of a node $\omega_i$ is defined as $deg[\omega_i] = deg(\omega_i)$, which can also be expressed as $deg^P[\omega_i] = deg(\omega_i) + \xi(\omega_i)$. $\Gamma_1$ is considered totally regular under $\widetilde{G}$ if, for every $\omega_i \in \widetilde{V}$, if $\Gamma_1 = deg[\omega_i]$.*

**Theorem 2.1.** *Consider an odd cycle* $\widetilde{G}^* = (\widetilde{V}, \widetilde{E})$. *A fuzzy graph* $\widetilde{G} = (\widetilde{V}', \widetilde{E}')$ *is termed a regular fuzzy graph if and only if* $\xi(\omega_i)$ *is constant,* $\forall\ \omega_i \in \widetilde{V}'$.

**Definition 2.10.** *Let* $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ *be a fuzzy graph. In the fuzzy graph* $\widetilde{G}$, *the modified first Zagreb index* ($ZF^1$) *is represented as* $ZF^1(\widetilde{G})$ *and described as follows :* $ZF^1(\widetilde{G}) = \sum_{\omega \in \widetilde{V}(\widetilde{G})} [\xi(\omega)\, \Gamma(\omega)]^2$.

**Definition 2.11.** *Given a fuzzy graph* $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, *The second Zagreb index* ($ZF^2$) *of the fuzzy graph* $\widetilde{G}$ *is represented as* $ZF^2(\widetilde{G})$ *and is defined by the following expression*
$ZF^2(\widetilde{G}) = \sum_{\substack{i \neq j \\ \omega\eta \in \widetilde{E}(\widetilde{G})}} \{\xi(\omega)\, \Gamma(\omega)\}\{\xi(\eta)\, \Gamma(\eta)\}$.

**Definition 2.12.** *Let* $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ *be a fuzzy graph. For the fuzzy graph* $\widetilde{G}$, *the hyper Zagreb index (HZI) is represented as* $HZI(\widetilde{G})$. *It is defined as follows :* $HZI(\widetilde{G}) = \sum_{\substack{i \neq j \\ \omega\eta \in \widetilde{E}(\widetilde{G})}} \{\xi(\omega)\Gamma(\omega) + \xi(\eta)\Gamma(\eta)\}^2$.

**Definition 2.13.** *Let* $\widetilde{G} = (\widetilde{V},\ \widetilde{E})$ *be a fuzzy graph. The Edge F-index (EFI) of the fuzzy graph* $\widetilde{G}$ *is symbolized by* $EFI(\widetilde{G})$ *and is defined by the following expression :*
$EFI(\widetilde{G}) = \sum_{\substack{i \neq j \\ \omega\eta \in \widetilde{E}(\widetilde{G})}} [\{\xi(\omega)\, \Gamma(\omega)\}^2 + \{\xi(\eta)\, \Gamma(\eta)\}^2]$.

**Definition 2.14.** *The Randic index (RI) of a fuzzy graph* $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ *is denoted by* $RI(\widetilde{G})$ *and is defined by the expression* $RI(\widetilde{G}) = \sum_{\substack{i \neq j \\ \omega\eta \in \widetilde{E}(\widetilde{G})}} [\{\xi(\omega)\Gamma(\omega)\}\{\xi(\eta)\Gamma(\eta)\}]^{-\frac{1}{2}}$

## 3. Sombor Index on fuzzy graphs

In several domains such as network theory, spectral graph theory, chemical graph theory, molecular chemistry, and fuzzy graph theory, topological indices play a crucial role.

The "Sombor index" for crisp graphs is a novel molecular descriptor based on vertex degree that was recently introduced by Gutman [14].

**Definition 3.1.** [14] *Assume that a crisp graph* $G = (V, E)$. *The concept of the Sombor index for* $G$ *is defined as follows :*

$$SO(G) = \sum_{v_i v_j \in E(G)} \sqrt{\{d_G(v_i)\}^2 + \{d_G(v_j)\}^2}.$$

*In this context,* $d(v_i)$ *and* $d(v_j)$ *denote the degrees of vertices* $v_i$ *and* $v_j$ *respectively.*

**Definition 3.2.** *Let* $\widetilde{G} = (\widetilde{V}, \xi, \Omega)$ *be a fuzzy graph. The definition of the Sombor index in fuzzy graph theory is:*

$$SOF(\widetilde{G}) = \sum_{\omega_i \omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\, \Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\, \Gamma(\omega_j)\}^2}\,.$$

*Here,* $\xi : \widetilde{V} \to [0, 1]$ *represents the vertex membership degree, and* $\Omega : \widetilde{V} \times \widetilde{V} \to [0, 1]$ *is the membership degree of edges. Additionally,* $\Gamma(\omega_i)$ *and* $\Gamma(\omega_j)$ *denote the degrees of vertices* $\omega_i$ *and* $\omega_j$ *respectively.*

The following theorem provides an upper bound for the Sombor index in fuzzy graphs.

**Theorem 3.1.** *If $n$ vertices and $m$ edges constitute a fuzzy graph $\widetilde{G} = (\widetilde{V},\ \widetilde{E})$, then $SOF(\widetilde{G}) \leqslant \sqrt{2}m(n-1)$.*

*Proof.* From the definition of Sombor index we get,

$$\begin{aligned} SOF(\widetilde{G}) &= \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \\ &\leqslant \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\Gamma(\omega_i)\}^2 + \{\Gamma(\omega_j)\}^2}\ , since\ \xi(\omega_i) \leqslant 1, \xi(\omega_j) \leqslant 1 \\ &\leqslant \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\triangle^2 + \triangle^2} \leqslant \sqrt{2}m(n-1) \end{aligned}$$

Therefore, $SOF(\widetilde{G}) \leqslant \sqrt{2}m(n-1)$. □

**Example 3.1.** *As an example, let us use the fuzzy graph $\widetilde{G}$ displayed in Figure -1. We choose a vertex set, $V = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ such that $\xi(\omega_1) = 0.5$, $\xi(\omega_2) = 0.7$, $\xi(\omega_3) = 0.6$, $\xi(\omega_4) = 0.9$, and $\xi(\omega_5) = 0.8$. Within this fuzzy graph, $\Omega(\omega_1, \omega_2) = 0.4$, $\Omega(\omega_2, \omega_3) = 0.3$, $\Omega(\omega_3, \omega_4) = 0.5$, $\Omega(\omega_4, \omega_5) = 0.7$, $\Omega(\omega_5, \omega_1) = 0.2$, $\Omega(\omega_2, \omega_4) = 0.6$, $\Omega(\omega_1, \omega_4) = 0.3$. Following that, $\Gamma(\omega_1) = 0.9$, $\Gamma(\omega_2) = 1.3$, $\Gamma(\omega_3) = 0.8$, $\Gamma(\omega_4) = 2.1$, and $\Gamma(\omega_5) = 0.9$. Therefore*

$$SOF(\widetilde{G}) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} = 39.5980.$$

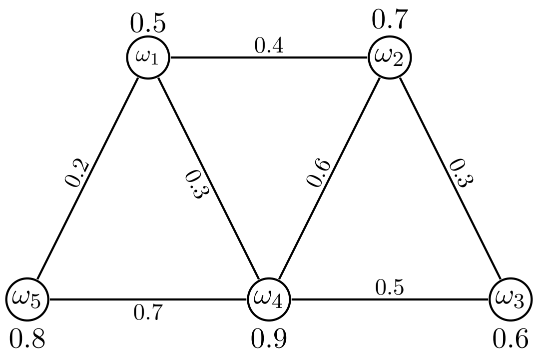



FIGURE 1. A fuzzy graph with Sombor index $SOF(\widetilde{G}) = 35.5980$.

**Example 3.2.** *Suppose $\widetilde{G}$ is a fuzzy graph. Its fuzzy subgraph, $\widetilde{G}'$, is obtained by deleting the vertices $\omega_3$ from it, and it is shown in Figure -2. Then $\Gamma(\omega_1) = 0.9$, $\Gamma(\omega_2) = 1.0$, $\Gamma(\omega_4) = 1.6$ and $\Gamma(\omega_5) = 0.9$.*

*Now,*

$$SOF(\widetilde{G}) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} = 6.4008.$$

*Therefore, $SOF(\widetilde{G}') \leqslant SOF(\widetilde{G})$.*

**Proposition 3.1.** *Let $\widetilde{G}' = (\widetilde{V}',\ \xi',\ \Omega')$ be a fuzzy subgraph of the fuzzy graph $\widetilde{G} = (\widetilde{V}, \xi, \Omega)$. Consequently, $SOF(\widetilde{G}') \leqslant SOF(\widetilde{G})$.*

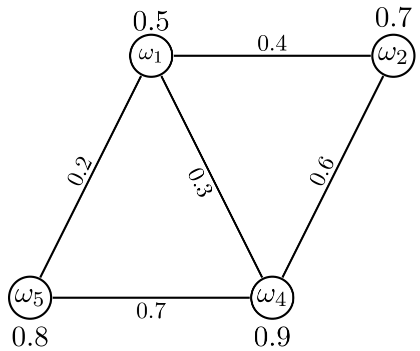



FIGURE 2. A FSG $\widetilde{G}'$ of the FG $\widetilde{G}$ in Fig-1 with Sombor index $SOF(\widetilde{G}')=6.4008$.

*Proof.* If $\omega_i,\omega_j\in\widetilde{V}'$, then $\xi'(\omega_i)\leqslant\xi(\omega_i)$ and $\Omega'(\omega_i\omega_j)\leqslant\Omega(\omega_i\omega_j)$.
So, $\Gamma'_{\widetilde{G}}(\omega_i)\leqslant\sum\limits_{\omega_j\in\widetilde{V}'(\widetilde{G}')}\Omega(\omega_i\omega_j)\leqslant\Gamma_{\widetilde{G}}(\omega_i)$.
Therefore,

$$\begin{aligned}SOF(\widetilde{G}') &= \sum_{\omega_i\omega_j\in E'(\widetilde{G}')}\sqrt{\{\xi'(\omega_i)\Gamma'_{\widetilde{G}}(\omega_i)\}^2+\{\xi'(\omega_j)\Gamma'_{\widetilde{G}}(\omega_j)\}^2}\\ &\leqslant \sum_{\omega_i\omega_j\in\widetilde{E}(\widetilde{G})}\sqrt{\{\xi(\omega_i)\Gamma_{\widetilde{G}}(\omega_i)\}^2+\{\xi(\omega_j)\Gamma_{\widetilde{G}}(\omega_j)\}^2}=SOF(\widetilde{G}).\end{aligned}$$

Therefore, $SOF(\widetilde{G}')\leqslant SOF(\widetilde{G})$. Hence the results follows. □

Let $0\leqslant r\leqslant 1$, the fuzzy graph $(\widetilde{G}^r)=(\widetilde{V}',\xi',\Omega')$ is a fuzzy subgraph of the fuzzy graph $\widetilde{G}=(\widetilde{V},\xi,\Omega)$ and is defined as $\widetilde{V}'=\{\omega_i\in\widetilde{V}:\xi(\omega_i)\leqslant r\}$ and $\xi'(\omega_i)=\xi(\omega_j)$, $\Omega'(\omega_i\omega_j)=\Omega(\omega_i\omega_j)$, $\forall\ \omega_i,\omega_j\in\widetilde{V}'$.

**Theorem 3.2.** *Let us assume that $\widetilde{G}=(\widetilde{V},\ \xi,\ \Omega)$ is a fuzzy graph and let $0\leqslant r_1\leqslant r_2\leqslant 1$. Then $SOF(\widetilde{G}^{r_2})\leqslant SOF(\widetilde{G}^{r_1})$.*

**Corollary 3.1.** *$\widetilde{G}=(\widetilde{V},\ \xi,\ \Omega)$ be an fuzzy graph and $0\leqslant r_1\leqslant r_2\leqslant r_3\leqslant r_4\cdots\leqslant r_n\leqslant 1$. Then, $SOF(\widetilde{G}^{r_n})\leqslant SOF(\widetilde{G}^{r_{n-1}})\leqslant SOF(\widetilde{G}^{r_{n-2}})\leqslant SOF(\widetilde{G}^{r_{n-3}})\leqslant SOF(\widetilde{G}^{r_{n-4}})\leq\cdots\leqslant SOF(\widetilde{G}^{r_1})$.*

**Theorem 3.3.** *Given a cycle fuzzy graph with $n$ vertices denoted by $C_n$. Then $SOF(C_n)\leqslant\sqrt{8}(n-1)$.*

*Proof.* As $C = \{\omega_1, \omega_2, \omega_3, \cdots, \omega_n\}$ is a n-vertex fuzzy cycle, then $\Gamma(\omega_1) = \Omega_1 + \Omega_n$ and $\Gamma(\omega_i) = \Omega_{i-1} + \Omega_i$, for $i = 1, 2, 3, 4, \cdots, n$. Therefore,

$$\begin{aligned} SOF(C_n) &= \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \\ &= \sqrt{\xi_1^2(\Omega_1 + \Omega_2)^2 + \xi_2^2(\Omega_1 + \Omega_n)^2} \\ &\quad + \sqrt{\xi_1^2(\Omega_1 + \Omega_n)^2 + \xi_n^2(\Omega_{n-1} + \Omega_n)^2} + \cdots \\ &\quad + \sum_{i=2} \sqrt{\xi_i^2(\Omega_{i-1} + \Omega_i)^2 + \xi_{i+1}^2(\Omega_i + \Omega_{i+1})^2} \\ &\leqslant \sqrt{2^2 + 2^2} + \sqrt{2^2 + 2^2} + \cdots + \sum_{i=2}^{n-2} \sqrt{2^2 + 2^2} \\ &= \sqrt{8} + \sqrt{8} + \sqrt{8} + \cdots + \sqrt{8}(n-3) \\ &= \sqrt{8}(2 + n - 3) \\ &= \sqrt{8}(n-1). \end{aligned}$$

Therefore, $SOF(C_n) \leqslant \sqrt{8}(n-1)$. Hence the result follows.
Equality occurs in this bound when all the edges of the fuzzy cycle graph have the same weight 1 leading to each vertex having a uniform degree of 2.

□

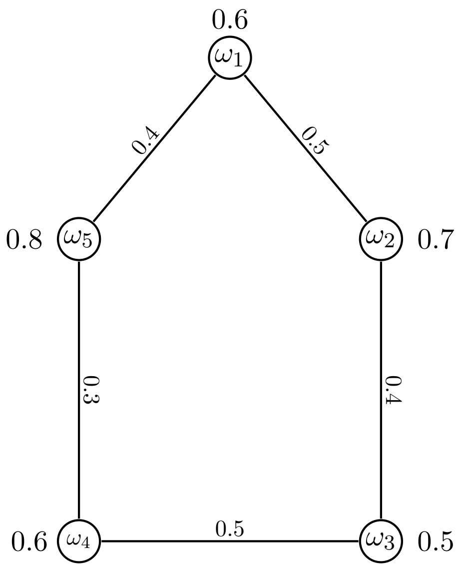



FIGURE 3. A cycle fuzzy graph with $SOF(G) = 3.7774$

**Example 3.3.** *Figure 3 illustrates the cycle fuzzy graph* $\widetilde{G} = (\widetilde{V},\ \widetilde{E})$. $\widetilde{V} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ *Fuzzy values are allocated in the following way, with serving as its vertex set: The values that were obtained are* $\xi(\omega_1) = 0.6$, $\xi(\omega_2) = 0.7$, $\xi(\omega_3) = 0.5$, $\xi(\omega_4) = 0.6$, *and* $\xi(\omega_5) = 0.8$. *Consequently, the degrees of the vertices are* $\Gamma(\omega_1) = 0.9$, $\Gamma(\omega_2) = 0.9$, $\Gamma(\omega_3) = 0.9$, *and* $\Gamma(\omega_4) = 0.8$, $\Gamma(\omega_5) = 0.7$. *Thus,* $SOF(C_5) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} =$

3.7774. *Alternatively,* $\sqrt{8}(5-1) = 4\sqrt{8} = 4 \times 2.8284 = 11.3136$. *Consequently, we can observe that* $SOF(C_5) = 3.7774 \leqslant 11.3136$. *As a result, theorem 3.3 is confirmed.*

Now, the boundaries of the star graph for the sombor index is being explored.

**Theorem 3.4.** *Assume that $S$ is a star fuzzy network with $(n+1)$ vertices.* $SOF(S_n) \leqslant n\sqrt{n^2+1}$.

*Proof.* As $S = \{\omega_0, \omega_1, \omega_2, \omega_3, \cdots, \omega_n\}$ is a star fuzzy graph having $(n+1)$ vertices, then $\Gamma(\omega_0) = \sum_{i=1}^{n} \Omega_i$ and $\Gamma(\omega_i) = \Omega_i$, for $i = 0, 1, 2, 3, \cdots, n$.
Thus we get,

$$\begin{aligned} SOF(S_n) &= \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \\ &= \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\xi_0^2(\sum_{j=1}^{n}\Omega_i)^2 + \xi_j^2(\Omega_j)^2} \\ &\leqslant n\sqrt{n^2+1}. \end{aligned}$$

Hence, $SOF(S_n) \leqslant n\sqrt{n^2+1}$. □

Equality occurs when all the edges in the fuzzy star graph have the same weight 1, leading to a uniform degree for the peripheral vertices and a degree of $(n+1)$ for the central vertex.

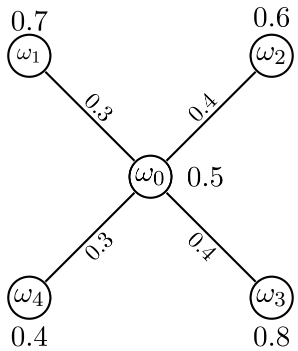
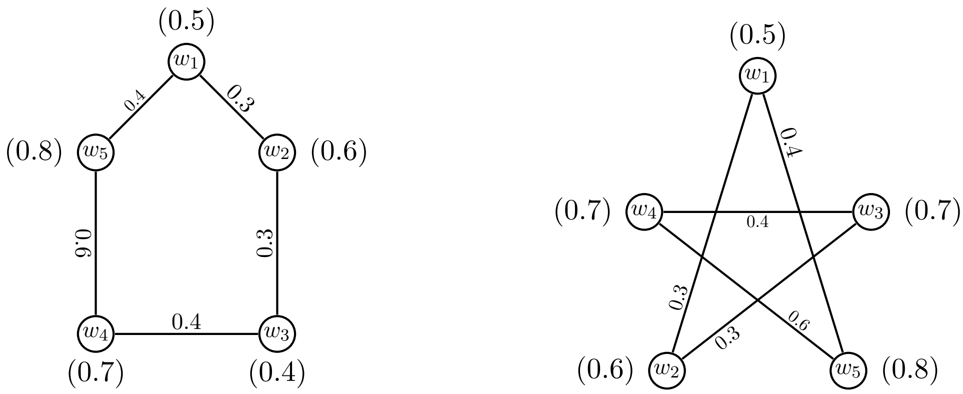



FIGURE 4. A star fuzzy graph with $SOF(S_5) = 2.9506$

**Example 3.4.** *Assume that $S_5$ is the star fuzzy graph shown in Figure-4, with the vertex set $\{\omega_0, \omega_1, \omega_2, \omega_3, \omega_4\}$ with the following values :* $\xi(\omega_0) = 0.5$, $\xi(\omega_1) = 0.7$, $\xi(\omega_2) = 0.6$, $\xi(\omega_3) = 0.8$, *and* $\xi(\omega_4) = 0.4$. *Then,* $\Gamma(\omega_0) = 1.4$, $\Gamma(\omega_1) = 0.3$, $\Gamma(\omega_2) = 0.4$, *and* $\Gamma(\omega_3) = 0.4$, $\Gamma(\omega_4) = 0.3$ *are the degrees of the vertices. As a result, for graph $S_5$,* $SOF(S_5) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} = 2.9506$.

On the other hand $SOF(S_5) = 5\sqrt{5^2+1} = 25.495$. Therefore, $SOF(S_5) = 2.9506 \leqslant 25.495$. Hence the theorem 3.4 is verified. The following theorem deals with the Sombor index for isomorphic fuzzy graph

**Theorem 3.5.** *Assuming $\widetilde{G}_1$ and $\widetilde{G}_2$ are isomorphic fuzzy graphs, it follows that* $SOF(\widetilde{G}_1) = SOF(\widetilde{G}_2)$.




(A) $\widetilde{G}$ (B) $\widetilde{G'}$

FIGURE 5. Two isomorphic FGs $\widetilde{G}$ and $\widetilde{G'}$

*Proof.* Since $\widetilde{G}_1$ and $\widetilde{G}_2$ are isomorphic, there exists a bijective mapping $\psi : \widetilde{V}_1 \to \widetilde{V}_2$ and $\forall \omega_i, \omega_j \in \widetilde{V}_1$. In addition, $\Omega_1(\omega_j) = \Omega_2(\psi(\omega_i), \psi(\omega_j))$. Also $\Gamma_{\widetilde{G}_1}(\omega_j) = \sum_{\omega_i \in \widetilde{V}_1(\widetilde{G}_1)} \Omega_1(\omega_i, \omega_j)$ $= \sum_{\omega_i \in \widetilde{V}_1(\widetilde{G}_1)} \Omega_2(\psi(\omega_i), \psi(\omega_j)) = \sum_{\xi(\omega_i) \in \widetilde{V}_2(\widetilde{G}_2)} \Omega_2(\psi(\omega_i), \psi(\omega_j)) = \Gamma_{\widetilde{G}_2}(\psi(\omega_j))$.
Therefore,

$$\begin{aligned} SOF(\widetilde{G}_1) &= \sum_{\omega_i\omega_j \in \widetilde{E}_1(\widetilde{G}_1)} \sqrt{\{\xi_1(\omega_i)\Gamma_{\widetilde{G}_1}(\omega_i)\}^2 + \{\xi_1(\omega_j)\Gamma_{\widetilde{G}_1}(\omega_j)\}^2} \\ &= \sum_{\omega_i\omega_j \in \widetilde{E}_1(\widetilde{G}_1)} \sqrt{\{\xi_2(\psi(\omega_i))\Gamma_{\widetilde{G}_2}(\psi(\omega_i))\}^2 + \{\xi_2(\psi(\omega_j))\Gamma_{\widetilde{G}_2}(\psi(\omega_j))\}^2} \\ &= \sum_{\psi(\omega_i)\psi(\omega_j) \in \widetilde{E}_2(\widetilde{G}_2)} \sqrt{\{\xi_2(\psi(\omega_i))\Gamma_{\widetilde{G}_2}(\psi(\omega_i))\}^2 + \{\xi_2(\psi(\omega_j))\Gamma_{\widetilde{G}_2}(\psi(\omega_j))\}^2} \\ &= SOF(\widetilde{G}_2). \end{aligned}$$

This completes the proof. □

**Example 3.5.** *Consider the FGs $\widetilde{G}_1$ and $\widetilde{G}_2$ in Figure -5, here $\xi(\psi(\omega_i)) = \xi(\omega_i)$, $\xi(\psi(\omega_i)) = \xi(\omega_i)$ and $\xi(\omega_i\omega_j) = \xi(\psi(\omega_i), \psi(\omega_j)) = \xi(\omega_i v_j)$, $\xi(\omega_i\omega_j) = \xi(\psi(\omega_i), \psi(\omega_j)) = \xi(\omega_i\omega_j)$, $\forall 1 \le i \le 5$. $SOF(\widetilde{G}_1) = 0.5021 + 0.4561 + 0.7540 + 1.0630 + 0.8732 = 3.6484 = SOF(\widetilde{G}_2)$. Thus we see that $SOF(\widetilde{G}_1) = SOF(\widetilde{G}_2)$. Hence the Theorem 3.5 is verified.*

**Corollary 3.2.** *If two fuzzy graphs $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ and $\widetilde{G}' = (\widetilde{V}', \widetilde{E}')$ exhibit co-weak isomorphism, and $\xi(\omega_i) \le \xi(\omega_i')$ holds for all $\omega_i \in V$ and $\omega_i' \in V'$, then it follows that $SOF(\widetilde{G}) \ge SOF(\widetilde{G}')$.*

The upper and lower bounds of the Sombor index can be determined using various methods, especially when the maximum and minimum values of vertex degrees in a fuzzy network are known. These different strategies are further elaborated upon in the following theorems.

**Theorem 3.6.** *Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be a FG such that $\delta = min\{\Gamma(\omega_i)\}$ and $\Delta = max\{\Gamma(\omega_i)\}$ $\forall \omega_i \in \widetilde{V}$. Then $\delta\lambda \leq SOF(\widetilde{G}) \leq \Delta\lambda$, where $\lambda = \sum_{\substack{i \neq j \\ \omega_i\omega_j \in \widetilde{E}(\widetilde{G})}} \sqrt{[\{\xi(\omega_i)\}^2 + \{\xi(\omega_j)\}^2]}$.*

*Proof.* Since $\omega_i \in \widetilde{V}$ and $\widetilde{G}$ is a FG, we may obtain the following from definition 2.8 we have $\Gamma(\omega_i) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \xi(\omega_i\omega_j)$, $\delta = min\{\Gamma(\omega_i)\}$ and $\Delta = max\{\Gamma(\omega_i)\}$, $\forall \omega_i \in \widetilde{V}$. So we have $\delta \leq \Gamma(\omega_i) \leq \Delta$.
Therefore,

$$\Rightarrow \delta\{\xi(\omega_i)\} \leq \{\xi(\omega_i)\Gamma(\omega_i\} \leq \Delta\{\xi(\omega_i)\}$$
$$\Rightarrow \delta^2\{\xi(\omega_i)\}^2 \leq \{\xi(\omega_i)\Gamma(\omega_i\}^2 \leq \Delta^2\{\xi(\omega_i)\}^2$$

and

$$\delta^2\{\xi(\omega_j)\}^2 \leq \{\xi(\omega_j)\Gamma(\omega_j\}^2 \leq \Delta^2\{\xi(\omega_j)\}^2$$

Therefore,

$$\delta^2\{\xi(\omega_i)\}^2 + \delta^2\{\xi(\omega_j)\}^2 \leq \{\xi(v_i)\, d(\omega_i)\}^2 + \{\xi(\omega_j)\Gamma(\omega_j\}^2 \leq \Delta^2\{\xi(\omega_i)\}^2 + \Delta^2\{\xi(\omega_j)\}^2$$
$$\Rightarrow \delta^2[\{\xi(\omega_i)\}^2 + \{\xi(\omega_j)\}^2] \leq \{\xi(v_i)\, d(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2 \leq \Delta^2[\{\xi(\omega_i)\}^2 + \{\xi(\omega_j)\}^2].$$
$$\Rightarrow \delta\lambda \leq SOF(\widetilde{G}) \leq \Delta\lambda, \text{ where } \lambda = \sum_{\substack{i \neq j \\ \omega_i\omega_j \in \widetilde{E}(\widetilde{G})}} \sqrt{[\{\xi(\omega_i)\}^2 + \{\xi(\omega_j)\}^2]}.$$

□

Moreover, the equality herein holds if and only if $\Gamma_G(v_i) = \Delta$, for any $v_i \in V(G)$, i.e., $G$ is regular. Similarly, we get the lower bound on Sombor index and equality holds if and only if $G$ becomes regular.

**Example 3.6.** *For this instance, we consider in Figure 5, $\widetilde{E} = \{\omega_1\omega_2,\ \omega_2\omega_3,\ \omega_4\omega_5,\ \omega_5\omega_1\}$ and $\widetilde{V} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ also $\Gamma(\omega_1) = 0.7$, $\Gamma(\omega_2) = 0.6$, $\Gamma(\omega_3) = 0.7$, $\Gamma(\omega_4) = 1.0$ $\Gamma(\omega_5) = 1.0$. As a result, $SOF(\widetilde{G}) = 3.6484$, $\delta = 0.6$, and $\Delta = 1.0$. The expression becomes $\lambda = \sum_{\substack{i \neq j \\ \omega_i\omega_j \in \widetilde{E}(\widetilde{G})}} \sqrt{[\{\xi(w_i)\}^2 + \{\xi(w_j)\}^2]} = 4.3146$. Then, $\Delta \times \lambda = 1.0 \times 4.3146 = 4.3146$ and $\delta \times \lambda = 0.6 \times 4.3146 = 2.5888$. Therefore, in this case, $\delta\lambda \leq SOF(\widetilde{G}) \leq \Delta\lambda \Rightarrow 2.5887 \leq 3.6484 \leq 4.3146$. The validity of Theorem 3.6 is thus established.*

## 4. Sombor index of fuzzy digraph, regular fuzzy graph and fuzzy cycle

**Definition 4.1.** *The degree of a vertex $\omega_i$ in an FDG $\vec{\widetilde{G}} = \left(\widetilde{V}, \vec{E}\right)$ is given by $\Gamma(\omega_i)$ and $\Gamma(\omega_i) = \sum_{i \neq j} [\xi(\overrightarrow{\omega_i\omega_j}) + \xi(\overrightarrow{\omega_j\omega_i})]$.*

**Example 4.1.** *Considering the fuzzy di-graph (FDG) $\vec{G}$ illustrated in Figure-6, where a vertex set $V = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\Gamma(\omega_1) = 1.4$, $\Gamma(\omega_2) = 1.8$, $\Gamma(\omega_3) = 2.2$ and $\Gamma(\omega_4) = 1.8$.*

**Definition 4.2.** *The fuzzy di-graph $\vec{G} = \left(V, \vec{E}\right)$ and its Sombor index is denoted by $SOFD(\vec{G})$ and defined as*

$$SOFD(\vec{G}) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{G})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2}$$

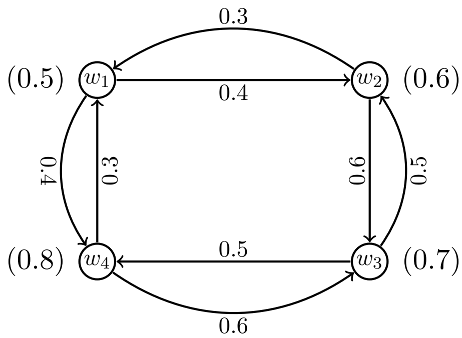



FIGURE 6. A FDG ($\vec{G}$).

Consider the FDG $\vec{G}$ of Figure - 6, $\Gamma(\omega_1) = 1.4$, similarly $\Gamma(\omega_2) = 1.8$, $\Gamma(\omega_3) = 2.2$, $\Gamma(\omega_4) = 1.8$. Therefore $SOFD(\vec{G}) = 13.753$.

**Theorem 4.1.** *Let $\vec{G} = \left(\widetilde{V},\ \vec{E}\right)$ be an FDG of order $n$ with $|E| = m$. Assume that each vertex of $\vec{G}$ is constant. If $\omega = \xi(\omega_i)$ and $\Gamma(\omega_i) = \Gamma$, then $SOF(\vec{G}) = \sqrt{2} \times m \times \omega \times \Gamma$.*

*Proof.* Suppose we have the FDG $\vec{G} = \left(\widetilde{V}, \vec{E}\right)$, where all vertices are constant. Given that $|V| = n$, and $m$ undirected edges exist. Again, $\Gamma(\omega_i) = \Gamma$ and $\omega = \xi(\omega_i)\ \forall \omega_i \in \widetilde{V}$. Therefore

$$\begin{aligned} SOFD(\vec{G}) &= \sum_{\substack{i \neq j \\ \omega_i\omega_j \in E(\widetilde{G})}} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \\ &= m\sqrt{\{\omega\Gamma\}^2 + \{\omega\Gamma\}^2} = \sqrt{2} \times m \times \omega \times \Gamma. \end{aligned}$$

Hence, $SOFD(\vec{G}) = \sqrt{2} \times m \times \omega \times \Gamma$. Thus the results follows.

□

**Example 4.2.** *Consider the FDG $\vec{G}$ of Figure-7. Given that $|E| = m = 4$ and that each vertex's degree is $\Gamma(\omega_i) = 0.6$, membership values of each vertex $\xi(\omega_i) = 0.3$. Therefore $SOFD(\vec{G}) = 1.0181$. Now $\sqrt{2} \times m \times \omega \times \Gamma = 1.414 \times 4 \times 0.3 \times 0.6 = 1.0181$. Thus the theorem 4.1 is verified.*

**Corollary 4.1.** *If two FDGs $\vec{G}_1$, $\vec{G}_2$ are isomorphic to each other, then $SOFD(\vec{G}_1) = SOFD(\vec{G}_2)$.*

**Theorem 4.2.** *An edge connects each pair of vertices in the regular and entirely regular FG $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, where $O(V) = n$. Then $SOF(\widetilde{G}) = \dfrac{n(n-1)}{\sqrt{2}} \times \omega \times \Gamma$, where $\Gamma = \Gamma(\omega_i)$, $\omega = \xi(\omega_i)\ ,\ \forall\ \omega_i \in \widetilde{V}$.*

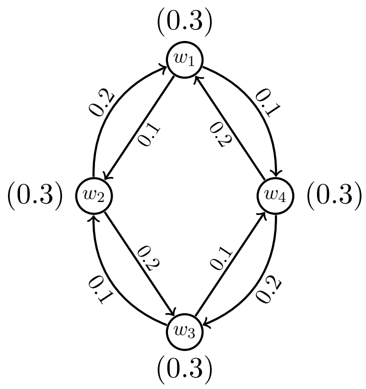



FIGURE 7. A FDG ($\vec{G}$) with membership values of the vertices are 0.3.

*Proof.* Given that $\widetilde{G}$ is regular, $\Gamma = \Gamma(\omega_i)$, $\forall \omega_i \in V$. Additionally, $\widetilde{G}$ is totally regular FG, so $\Gamma[\omega_i] = \lambda_1$(say) and $\forall \omega_i \in \widetilde{V}$. We know that $\Gamma[\omega_i] = \Gamma(\omega_i) + \xi(\omega_i) \Rightarrow \lambda = \Gamma + \xi(\omega_i) \Rightarrow$ $\xi(\omega_i) = \lambda - \omega \Rightarrow \omega = \lambda - \Gamma$, $\forall\ \omega_i \in \widetilde{V}$. As a result vertex of each vertices is constant and $\omega = \xi(\omega_i)$, $\forall \omega_i \in \widetilde{\widetilde{G}}$. Given that $|V| = n$ and that each pair of vertices in $\widetilde{G}$ has an edge, and it has $\dfrac{n(n-1)}{2}$ edges.
Therefore,

$$SOF(\widetilde{G}) = \sum_{\substack{i \neq j \\ \omega_i\omega_j \in \widetilde{E}(\widetilde{G})}} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2}$$

$$= \frac{n(n-1)}{2}\sqrt{\{\omega\Gamma\}^2 + \{\omega\Gamma\}^2} = \frac{n(n-1)}{\sqrt{2}} \times \omega \times \Gamma.$$

This completes the proof. □

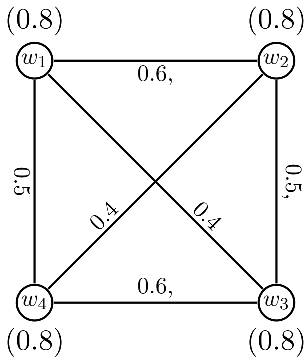



FIGURE 8. A regular and totally regular FG $\widetilde{G}$ with membership values of the vertices are 0.8.

**Example 4.3.** *Consider the FG $\widetilde{G}$ of Figure - 8, and here $V = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $E = \{\omega_1\omega_2, \omega_1\omega_3, \omega_1\omega_4, \omega_2\omega_3, \omega_2\omega_4, \omega_3\omega_4\}$ and $\xi(v_i) = 0.8$, $i = 1, 2, 3, 4$. Hence, $n = |V| = 4$ and each pair of vertices has an edge connecting them. Now $\Gamma = \Gamma(\omega_i) = 1.5$,*

$\Gamma[\omega_i] = 0.8 + 1.5 = 2.3$, $\forall i = 1, 2, 3, 4$. *As a result,* $\widetilde{G}$ *is a regular and totally regular FG. Therefore* $SOF(\widetilde{G}) = 6\sqrt{(0.8 \times 1.5)^2 + (0.8 \times 1.5)^2} = 10.182$. *Hence* $SOF(\widetilde{G}) = 10.182$. *Now applying the theorem 4.2 we have* $SOF(\widetilde{G}) = \dfrac{n(n-1)}{\sqrt{2}} \times \omega \times \Gamma = \dfrac{4(4-1)}{\sqrt{2}} \times 0.8 \times 1.5 = 10.182$. *Thus the theorem 4.2 is verified.*

**Theorem 4.3.** [13] *Let us consider an odd cycle* $G^* = (U,\ E)$. *A BFGs* $G = (U,\ S,\ T)$ *is called a regular BFGs if and only if* $T = (\mu_T^P,\ \mu_T^N) = constant$.

**Theorem 4.4.** *Let* $\widetilde{G} = (V, E)$ *be a regular fuzzy graph of the crisp graph* $G^*$ *, where* $G^*$ *is an odd cycle and* $|V| = n$. *The vertex set is* $V = \{w_1, w_2, w_3, w_4, \cdots, w_n\}$ *with membership values of vertices and edges are* $\xi(w_i) = w_{1i}$, $1 \leqslant i \leqslant n$ *and* $\xi(w_iw_j) = e_1$, $\forall w_iw_j \in E$. *Then* $SOF(\widetilde{G}) = 2\sqrt{2}e_1(w_{11}w_{12} + w_{12}w_{13} + \cdots + w_{1n}w_{11})$.

*Proof.* Since $G^*$ is an odd cycle of a regular fuzzy graph, by theorem 4.3, each vertex in $\widetilde{G}$ has exactly two incident edges because $G^*$ is a cycle. Additionally, since $\widetilde{G}$ is regular, the degree of all the vertices is the same. Therefore, for every vertex $w_i \in V$, we have $deg(w_i) = \Gamma(w_i) = 2e_1$. Given that $|V| = n$, and $G^*$ is a cycle, the total number of edges in $\widetilde{G}$ is $n$. This is because, in a cycle graph, the number of edges equals the number of vertices. Thus, the total number of edges in $\widetilde{G}$ is $n$. Therefore,

$$\begin{aligned}
SOF(\widetilde{G}) &= \sum_{\substack{i \neq j \\ w_iw_j \in E(\widetilde{G})}} \sqrt{\{\xi(w_i)\,\Gamma(w_i)\}^2 + \{\xi(w_j)\,\Gamma(w_j)\}^2} \\
&= \sum_{\substack{i \neq j \\ w_iw_j \in E(\widetilde{G})}} \sqrt{\{\xi(w_i)\,2e_1\}^2 + \{\xi(w_j)\,2e_1\}^2} \\
&= \sum_{\substack{i \neq j \\ w_iw_j \in E(\widetilde{G})}} \sqrt{8e_1^2(w_{11}w_{12} + w_{12}w_{13} + \cdots w_{1n}w_{11})^2} \\
&= 2\sqrt{2}e_1(w_{11}w_{12} + w_{12}w_{13} + \cdots w_{1n}w_{11}).
\end{aligned}$$

□

**Corollary 4.2.** *Let* $\widetilde{G} = (V, E)$ *be a regular fuzzy graph of the crisp graph* $G^*$ *, where* $G^*$ *is an odd cycle of order n. The vertex set is* $V = \{w_1, w_2, w_3, w_4, \cdots, w_n\}$ *with membership values of vertices and edges are* $\xi(w_i) = w_1$, $1 \leqslant i \leqslant n$ *and* $\xi(w_iw_j) = e_1$, $\forall w_iw_j \in E$. *Then* $SOF(\widetilde{G}) = 2\sqrt{2}nw_1e_1$, *where* $w_1 = \xi(w_i)$ *and* $e_1 = \xi(w_iw_j)$, $\forall\ w_i \in V$ *and* $\forall\ w_iw_j \in E$.

## 5. The relationship between the Sombor index and other topological indices

The links between the Sombor index of fuzzy graphs and other topological indices are shown in this section in multiple instances. First, we looked into the relationship between the first Zagreb index and the Sombor index.

**Theorem 5.1.** *Let* $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ *be a fuzzy graph. Then* $SOF(\widetilde{G}) = \frac{1}{\sqrt{2}}ZF^1(\widetilde{G})$

*Proof.* Any two real positive numbers, $\lambda$ and $\mu$, we have

$$\lambda^2 + \mu^2 \geq \frac{1}{2}(\lambda + \mu)^2. \tag{1}$$

In Eq. (1) put $\lambda = \xi(\omega_i)\Gamma(\omega_i)$ and $\mu = \xi(\omega_j)\Gamma(\omega_j)$, we get

$$\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2 \geq \frac{1}{2}[\{\xi(\omega_i)\,\Gamma(\omega_i)\} + \{\xi(\omega_j)\,\Gamma(\omega_j)\}]^2$$

$$\Rightarrow \sum_{\substack{i\neq j\\ \omega_i\omega_j \in E(\widetilde{G})}} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \geq \frac{1}{\sqrt{2}} \sum_{\substack{i\neq j\\ \omega_i\omega_j \in E(\widetilde{G})}} [\{\xi(\omega_i)\,\Gamma(\omega_i)\} + \{\xi(\omega_j)\,\Gamma(\omega_j)\}]$$

$$\Rightarrow SOF(\widetilde{G}) \geq \frac{1}{\sqrt{2}} \sum_{\omega \in V(\widetilde{G})} [\{\xi(\omega)\,\Gamma(\omega)\}]^2 = \frac{1}{\sqrt{2}} ZF^1(\widetilde{G}).$$

Therefore, $SOF(\widetilde{G}) = \frac{1}{\sqrt{2}} ZF^1(\widetilde{G})$. Hence the proof. □

The subsequent theorem elucidates the relationship between the Sombor index and the second Zagreb index for fuzzy graphs .

**Theorem 5.2.** *Suppose $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be a fuzzy graph. Then $[SOF(\widetilde{G})]^2 \geq 2[ZF^2(\widetilde{G})]$.*

*Proof.* Consider two positive real number $\lambda$ and $\mu$, and apply arithmetic mean $\geq$ geometric mean we get,

$$\frac{\lambda^2 + \mu^2}{2} \geq \sqrt{\lambda\mu} \Rightarrow \sqrt{\lambda^2 + \mu^2} \geq \sqrt{2}\sqrt{\lambda\mu} \tag{2}$$

In Eq. (2) put $\lambda = \xi(\omega_i)\Gamma(\omega_i)$ and $\mu = \xi(\omega_j)\Gamma(\omega_j)$ we get

$$\sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \geq \sqrt{2}\sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}\{\xi(\omega_j)\,\Gamma(\omega_j)\}}$$

$$\Rightarrow \sum_{\substack{i\neq j\\ \omega_i\omega_j \in E(\widetilde{G})}} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \geq \sqrt{2} \sum_{\substack{i\neq j\\ \omega_i\omega_j \in E(\widetilde{G})}} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}\{\xi(\omega_j)\,\Gamma(\omega_j)\}}$$

$$\Rightarrow [SOF(\widetilde{G})]^2 \geq 2[ZF^2(\widetilde{G})].$$

Thus $[SOF(\widetilde{G})]^2 \geq 2[ZF^2(\widetilde{G})]$. Hence the proof. □

The upcoming theorem establishes a link between the Sombor index and the hyper Zagreb index for fuzzy graphs.

**Theorem 5.3.** *Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be a fuzzy graph. Then $2[SOF(\widetilde{G})]^2 \geq [HZI(\widetilde{G})]$.*

*Proof.* We consider two positive real number $\lambda$ and $\mu$ we have

$$2(\lambda^2 + \mu^2) = (\lambda + \mu)^2 + (\lambda - \mu)^2$$
$$\Rightarrow 2[\sqrt{(\lambda^2 + \mu^2)}]^2 \geq (\lambda + \mu)^2. \tag{3}$$

Using Eq. (3) put $\lambda = \xi(\omega_i)\Gamma(\omega_i)$ and $\mu = \xi(\omega_j)\Gamma(\omega_j)$ we get,

$$2[\sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2}]^2 \geq \{\xi(\omega_i)\Gamma(\omega_i) + \xi(\omega_j)\Gamma(\omega_j)\}^2$$

$$\Rightarrow 2 \sum_{\substack{i\neq j\\ \omega_i\omega_j \in E(\widetilde{G})}} [\sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2}]^2 \geq \sum_{\substack{i\neq j\\ \omega_i\omega_j \in E(\widetilde{G})}} \{\xi(\omega_i)\Gamma(\omega_i) + \xi(\omega_j)\Gamma(\omega_j)\}^2$$

$$\Rightarrow 2[SOF(\widetilde{G})]^2 \geq HZI(\widetilde{G}).$$

Therefore, $2[SOF(\widetilde{G})]^2 \geq [HZI(\widetilde{G})]$. Thus, the proof. □

The subsequent theorem delineates the correlation between the Sombor index and the Edge Forgotten index (EFI) for fuzzy graphs.

**Theorem 5.4.** *Consider a fuzzy graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$. Then $[SOF(\widetilde{G})] \leq [EFI(\widetilde{G})]$.*

*Proof.* For any pair of positive real numbers $\lambda$ and $\mu$ we have :

$$\sqrt{\lambda^2 + \mu^2} \leq \lambda^2 + \mu^2. \tag{4}$$

Using Eq. (4) put $\lambda = \xi(\omega_i)\Gamma(\omega_i)$ and $\mu = \xi(\omega_j)\Gamma(\omega_j)$ we get,

$$\sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \leq \{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2$$

$$\Rightarrow \sum_{\substack{i \neq j \\ \omega_i\omega_j \in E(\widetilde{G})}} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2} \leq \sum_{\substack{i \neq j \\ \omega_i\omega_j \in E(\widetilde{G})}} \{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2$$

$$\Rightarrow [SOF(\widetilde{G})] \leq [EFI(\widetilde{G})].$$

Therefore, $[SOF(\widetilde{G})] \leq [EFI(\widetilde{G})]$. Thus, the outcome is evident. □

The following theorem illustrates the relationship between Sombor index and the Randic index for FGs.

**Theorem 5.5.** *Let us consider a fuzzy graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$. Then $[SOF(\widetilde{G})]^2 \times [RI(\widetilde{G})] \geq 2$.*

*Proof.* For any two positive real number $\lambda$ and $\mu$ we have

$$\frac{\lambda^2 + \mu^2}{2} \geq \sqrt{\lambda\mu}$$

$$\Rightarrow [\sqrt{(\lambda^2 + \mu^2)}]^2 (\lambda\mu)^{-\frac{1}{2}} \geq 2. \tag{5}$$

Substitute $\lambda = \xi(\omega_i)\Gamma(\omega_i)$ and $\mu = \xi(\omega_j)\Gamma(\omega_j)$ and taking sum in Eq. (5) we get

$$\sum_{\substack{i \neq j \\ \omega_i\omega_j \in \widetilde{E}(\widetilde{G})}} [\sqrt{\{\xi(\omega_i)\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\Gamma(\omega_j)\}^2}] \sum_{\substack{i \neq j \\ \omega_i\omega_j \in \widetilde{E}(\widetilde{G})}} [\{\xi(\omega_i)d(\omega_i)\}\{\xi(\omega_j)\Gamma(\omega_j)\}]^{-\frac{1}{2}} \geq 2.$$

Therefore, $[SOF(\widetilde{G})]^2 \times [RI(\widetilde{G})] \geq 2$. Hence the proof. □

## 6. Applying the Sombor index for fuzzy graphs to identify the country with the highest human trafficking rates relative to other countries

**Model construction.** Human trafficking is a form of modern-day slavery involving the illegal transport of individuals through force or deception for labor, sexual exploitation, or financial gain. There are various categories of human trafficking.

1. Sex Trafficking : Sex trafficking involves compelling a person to engage in the sex trade through force, fraud, or coercion.
2. Child Trafficking : Exploitation of children for labor, sexual exploitation, or use in illicit activities like drug trafficking or as child soldiers.
3. Labor Trafficking : Labor trafficking involves obtaining a person for labor services, often under exploitative conditions.
4. Organ Removal : Organ removal is a component of human trafficking, even though this is not widely accepted.

There are many reasons for human trafficking (HT), including poverty, lack of education, demand for cheap labor and sex, lack of human rights protections, lack of legitimate economic opportunities, cultural factors, conflict and natural disasters, lack of safe migration

TABLE 1. Data on human trafficking,

| Countries | Human Trafficking | | Political leadership | | GDP | Total |
|---|---|---|---|---|---|---|
| | 2023 | 2021 | 2023 | 2021 | *in USD* | *Population* |
| YAMEN | 9.00 | 8.50 | 2.00 | 1.50 | 9412.03 | 32,981,641 |
| ETHIOPIA | 8.00 | 6.00 | 5.00 | 4.00 | 111,271.00 | 120,283,026 |
| IRAN | 8.00 | 7.50 | 3.00 | 2.50 | 359713.03 | 87,923,432 |
| NEPAL | 8.00 | 7.00 | 3.50 | 2.50 | 36288.83 | 30,034,989 |
| B. FASO | 7.00 | 6.00 | 3.50 | 2.50 | 19,737.62 | 22,100,683 |
| NICARGUA | 7.00 | 6.00 | 2.00 | 1.50 | 14,013.02 | 6,850,540 |
| U. K | 7.00 | 6.00 | 8.00 | 7.50 | 3,131,380.00 | 67,326,569 |
| SRI LANKA | 6.50 | 5.50 | 4.00 | 2.50 | 88,927.06 | 22,156,000 |
| INDIA | 8.00 | 7.00 | 5.42 | 5.25 | 3176,300.00 | 1407,563,842 |
| BHUTAN | 6.00 | 5.50 | 6.50 | 6.00 | 2539.55 | 777486 |
| SWEDEN | 5.00 | 4.50 | 7.00 | 6.50 | 635,664.00 | 10425,811 |
| ARGINTINA | 5.00 | 4.00 | 7.00 | 6.50 | 487,227.00 | 45,808,747 |

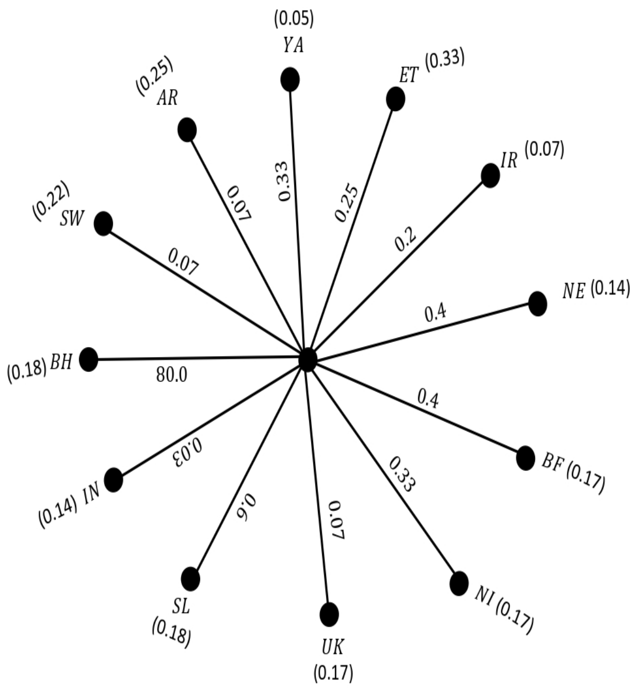



FIGURE 9. Fuzzy graph of human trafficking

options, deception and intimidation, and the population size of the country. Two of the most important factors contributing to human trafficking are the growth rate of Gross domestic product (GDP) and the total population of a country, as well as the level of control exercised by political leadership and governance (PLG). Human trafficking begins and spreads through various routes. Recruiters seek migrants via the internet, employment agencies, and local contacts in countries of origin such as Southeast Asia, Eastern Europe, and sub-Saharan Africa. International organizations, governments, and NGOs work together to raise awareness, prevent trafficking, and assist victims. Their efforts include implementing legal frameworks, providing victim support programs, and conducting public education campaigns. Therefore, to effectively combat human trafficking, extensive collaboration and unity are required. The objective of this article is to decrease human trafficking on a country-by-country basis, by using Sombor index in fuzzy graph and utilizing the data presented in Table 1. The data presented here are extracted from the "Global Organized Crime Index 2023" specifically from the rankings on human trafficking available at **https://ocindex.net/rankings/humantrafficking**. As illustrated in Figure 10, the data highlight the top twelve countries with the most significant increases in human trafficking rates between 2021 and 2023, alongside changes in political leadership and governance during the same period.

**Representation of membership values.** Now, the vertex membership value (MV) is calculated by the formula : $\frac{HT(2023)-HT(2021)}{HT(2021)}$. Similarly, the edge membership value (MV) is calculated using the formula : $\frac{PLG(2023)-PLG(2021)}{PLG(2021)}$. Here, vertex membership value (MV) and edge membership value range between 0 and 1. The membership values for all vertices and edges are provided in Table 2.

Now, $SOF^{\mathcal{E}}(\mathcal{G}) = \sum_{\omega_i\omega_j \in \widetilde{E}(\widetilde{\mathcal{G}})} \sqrt{\{\xi(\omega_i)\,\Gamma(\omega_i)\}^2 + \{\xi(\omega_j)\,\Gamma(\omega_j)\}^2}$

The Sombor index of countries (vertices) is presented in Table 3, calculated using the formula : $SOF^{\mathcal{E}}(Country) = SOF^{\mathcal{E}}(\mathcal{G}) - SOF^{\mathcal{E}}(\mathcal{G} - Country)$.

TABLE 2. Membership values of vertex and edges and degrees with respect to human trafficking

| Countries | Membership value of vertex | Membership value of an edge | Degree of vertex | Degrees of an edge |
|---|---|---|---|---|
| YAMEN(YA) | 0.05 | 0.33 | 0.05 | 2.83 |
| ETHIOPIA(ET) | 0.33 | 0.25 | 0.33 | 2.83 |
| IRAN (IR) | 0.07 | 0.2 | 0.07 | 2.83 |
| NEPAL(NE) | 0.14 | 0.4 | 0.14 | 2.83 |
| BURKINA FASO(BF) | 0.17 | 0.4 | 0.17 | 2.83 |
| NICARGUA(NI) | 0.17 | 0.33 | 0.17 | 2.83 |
| UNITED KINGDOM(UK) | 0.17 | 0.07 | 0.17 | 2.83 |
| SRI LANKA(SL) | 0.18 | 0.6 | 0.18 | 2.83 |
| INDIA(IN) | 0.14 | 0.03 | 0.14 | 2.83 |
| BHUTAN(BH) | 0.18 | 0.08 | 0.18 | 2.83 |
| SWEDEN (SW) | 0.22 | 0.07 | 0.22 | 2.83 |
| ARGINTINA (AR) | 0.25 | 0.07 | 0.25 | 2.83 |

**Decision-Making.** Our results depend on several parameters, including :
1. The total population and GDP of a country.
2. The total number of human trafficking cases in 2023.
3. The total number of human trafficking cases in 2021.

TABLE 3. Membership value and degrees of the fuzzy graph of Fig 9

| Countries | $SOF^{\mathcal{E}}(\mathcal{G})$ | $SOF^{\mathcal{E}}(\mathcal{G}-v)$ | $SOF^{\mathcal{E}}(Country)$ |
|---|---|---|---|
| YAMEN (YA) | 8.0237 | 7.0897 | 0.9340 |
| ETHIOPIA(ET) | 8.0237 | 7.3115 | 0.7122 |
| IRAN(IR) | 8.0237 | 7.4576 | 0.5661 |
| NEPAL(NE) | 8.0237 | 6.8904 | 1.1333 |
| BURKINA FASO(BF) | 8.0237 | 6.8897 | 1.1640 |
| NICARGUA(NI) | 8.0237 | 7.0881 | 0.9356 |
| UNITED KINGDOM(UK) | 8.0237 | 7.8255 | 0.1982 |
| SRI LANKA (SL) | 8.0237 | 6.3223 | 1.7014 |
| INDIA(IN) | 8.0237 | 7.9388 | 0.0849 |
| BHUTAN(BH) | 8.0237 | 7.7970 | 0.2267 |
| SWEDEN (SW) | 8.0237 | 7.8251 | 0.1986 |
| ARGINTINA(AR) | 8.0237 | 7.8249 | 0.1987 |

TABLE 4. Sombor index values and order of the countries

| Countries | $SOF^{\mathcal{E}}(Country)$ |
|---|---|
| SRI LANKA (SL) | 1.7014 |
| BURKINA FASO(BF) | 1.1640 |
| NEPAL(NE) | 1.1333 |
| NICARGUA (NI) | 0.9356 |
| YAMEN(YA) | 0.9340 |
| ETHIOPIA(ET) | 0.7122 |
| IRAN(IR) | 0.5661 |
| BHUTAN(BH) | 0.2267 |
| ARGINTINA (AR) | 0.1987 |
| SWEDEN(SW) | 0.1985 |
| UNITED KINGDOM (UK) | 0.1982 |
| INDIA(IN) | 0.0849 |

Now Sombor index of a vertex indicates that the country(vertex) has realized a large amount of human trafficking with respect to their total population and GDP and human trafficking in the last year.
A minimum Sombor index of a vertex indicates that the corresponding country has experienced the lowest level of human trafficking globally. Next, we arrange the countries in descending order based on the prevalence of large-scale human trafficking worldwide, as presented in Table 4. Based on the findings outlined in Table 4, our study offers the subsequent insights.
**(i)** If we identify the countries with higher occurrences of human trafficking worldwide based on the Sombor index, then these countries need to take immediate action to mitigate human trafficking.

**(ii)** Countries with the highest increase in human trafficking are identified by high population, low GDP growth, and rising trafficking rates. According to Table 1, Yemen, Ethiopia, and Burkina Faso show significant increases, but our results highlight in Table 4, Sri Lanka, Burkina Faso, and Nepal as the top three in the previous year.
**(iii)** Countries at the bottom of our results have demonstrated a decrease in human trafficking compared to the previous year. This improvement indicates progress in combating this issue, reflecting effective measures and efforts in these regions. Continued monitoring and support will be essential to sustain and further this positive trend.

## 7. Conclusion

In mathematical chemistry, various topological indices have been studied to explain the physical, chemical, pharmacological, and other properties of molecules. Among these, the Sombor index, a unique topological index has recently gained attention.The research establishes upper and lower bounds for the Sombor index within the context of fuzzy graphs, demonstrating its versatility across different graph structures including cycles, stars, and isomorphic graphs. Additionally, Sombor index values were computed for regular fuzzy graphs, fuzzy cycles, and fuzzy directed graphs in our investigation. Furthermore, distinct correlations between the Sombor index of a fuzzy graph and other indices within the fuzzy graph framework were explored. Finally, we apply the Sombor index for fuzzy graphs to identify the country with the highest human trafficking rates relative to other countries.

It is anticipated that further studies in this domain will explore fuzzy soft graphs, picture fuzzy graphs, bipolar fuzzy graphs, m-polar fuzzy graphs, Pythagorean fuzzy graphs, spherical fuzzy graphs, and the use of the Sombor index on intuitionistic fuzzy graphs.

**Conflict of Interest.** There is no conflict of interest disclosed by the authors.

**Ethical approval.** The authors of this article have not conducted any studies involving humans or animals.

**Acknowledgments.** The authors express their sincere gratitude to the anonymous referees for their valuable comments that significantly enhanced the original work and confirm that no grants, funding, or other assistance were received for its preparation.

## References

[1] Amin, R., Nayeem, A., (2018), On the F-index and F-coindex of the line graphs of the subdivision graphs, Malaysian J. Math., 6, pp. 362–368.
[2] Akram, M., Nawaz, S., (2015), On fuzzy soft graphs, Ital. J. Pure Appl. Math., 34, pp. 497–514.
[3] Azeema, M., Anwara, S., Jamila, M. K., Saeedb, M., Deveci, M., (2024), Topological numbers of fuzzy soft graphs and their application, Inf. Sci, 667, pp. 120468.
[4] Akram, M., Zafar, F., (2019), A new approach to compute measures of connectivity in rough fuzzy network models, J. Intell. Fuzzy Syst., 36, pp. 449–465.
[5] Abdo, H., Dimitrov, D., Gutman, I., (2017), On extremal trees with respect to the F-index, Kuwait J. Sci., 44, pp. 1–8.
[6] Bhutani, K. R., Rosenfeld, A., (2013), Strong arc in fuzzy graphs, Inf. Sci., 152, pp. 319–322.
[7] Cruz, R., Rada, J., Sigarreta, J. M. (2021), Sombor index of trees with at most three branch vertices. Appl. Math. Comput., 409, pp. 126414.
[8] Chen, H., Li, W., Wang, J., (2022), Extremal Values on the Sombor Index of Trees, MATCH Commun. Math. Comput. Chem., 87, pp. 23–49.
[9] Deng, H., Tang, Z., Renfang, W., (2021), Molecular trees with extremal values of Sombor indices, Int. J. Quantum Chem., 121, https://api.semanticscholar.org/CorpusID:233946800.

[10] Du, Z., L. You., Liu, H., H. Yufei., (2023), The Sombor index and coindex of two-trees. AIMS Math., 8, pp. 18982–18994.
[11] Fortula, B., Gutman, I., (2015), A forgotten topological index, J. Math. Chem., 53, pp. 1184–1190.
[12] Gutman, I., Trinajstic, N., (1972), Graph theorem and moleculer orbitals, Total $\pi$-electron energy of alternate hydrocarbons, Chem. Phys. Lett., 17, pp. 535–538.
[13] Ghorai, G., Pal, M., (2018), A note on Regular bipolar fuzzy graphs, Neural Comput. Appl., 30, pp. 1569—1572.
[14] Gutman, I., (2021), Geometric approach to degree-based topological indices Sombor indices, MATCH Commun. Math. Comput. Chem., 86, pp. 11–16.
[15] Gutman, I., (2021), Some basic properties of Sombor indices. Open J. Discret. Appl. Math., 4, pp. 1–3.
[16] Gutman, I., Redžepovi´c, I., Kulli, V. R., (2022), KG-Sombor index of Kragujevac trees, Open J. Discret. Appl. Math., 5, pp. 19–25.
[17] Gutman, I., (2022), Sombor indices- back to geometry, Open J. Discret. Appl. Math., 5, pp. 1–5.
[18] Gutman, I., Redžepovi´c, I., Furtula, B., (2023), On the product of Sombor and modified Sombor indices, Open J. Discret. Appl. Math., 6, pp. 1–6.
[19] Imran, M., Azeem, M., Jamila, M. K., Deveci, M., (2024), Exploring innovative single-value neutrosophic fuzzy topological graph parameters, Granul. Comput, 9, 37 .
[20] Islam, Sk. R., Pal, M., (2021), Hyper-wiener index for fuzzy graph and its application in share market, J. Intell. Fuzzy Syst., 41, pp. 2073–2083.
[21] Islam, Sk. R., Pal, M., (2023), Hyper-connectivity index for fuzzy graph with application, TWMS J. of Apl. and Eng. Math., 13, pp. 920–936.
[22] Islam, Sk. R., Pal, M., (2021), First Zagreb index on a fuzzy graph and its application, J. Intell. Fuzzy Syst., 40, pp. 10575–10587.
[23] Islam, Sk. R., Pal, M., (2021), F-index for fuzzy graph with application, TWMS J. of Apl. and Eng. Math, 13, pp. 517-530.
[24] Islam, Sk. R., Pal, M., (2023), Further development of F-index for fuzzy graph and its application in Indian railway crime, J. Comput. Appl. Math., 69, pp. 321–353.
[25] Islam, Sk. R., Pal, M., (2024), Multiplicative Version of First Zagreb Index in Fuzzy Graph and its Application in Crime Analysis, P NATL A SCI INDIA A, 94, pp. 127–141.
[26] Islam, Sk. R., Pal, M., (2022), An investigation of edge F-index on fuzzy graphs and application in molecular chemistry, COMPLEX INTELL SYST, 9, pp. 2043–2063.
[27] Josy, A., Mathew, S., Mordeson, J. N., (2022), Neighborhood connectivity index of a fuzzy graph and its application to human trafficking, Iran. J. Fuzzy Syst., 19, pp. 139–154.
[28] Jamila, M. K., Anwar, S., Azeem, M., Gutman, I., (2024), Intuitionistic fuzzy Sombor indices : A novel approach for improving the performance of vaccination centers, CCO, 9, pp. 1–31.
[29] Liu, H., You, L., Huang, Y., (2023), Sombor Index of C-Cyclic Chemical Graphs, MATCH Commun. Math. Comput. Chem., 90, pp. 495-504.
[30] Mothirakkattil, B., Mathew, S., Mordeson, J. N., (2020), Wiener index of a fuzzy graph and application to illegal immigration networks, Fuzzy Sets Syst., 384, pp. 132–147.
[31] Mothirakkattil, B., Mathew, S., Mordeson, J. N., (2019), Connectivity index of a fuzzy graph and its application to human trafficking, Fuzzy Sets Syst., 360, pp. 117–136.
[32] Mathew, S., Mordeson, J. N., Yang, H. L., (2019), Incidence cuts and connectivity in fuzzy incidence graphs, Iran. J. Fuzzy Syst., 16, pp. 31-43.
[33] Mondal, S., De, N., Pal, A., (2021), On neighbourhood Zagreb index of product graphs, J. Mol. Struct., 1223, pp. 129210.
[34] Mondal, S., De, N., Pal, A., (2021), QSPR analysis of some novel neighbourhood degree-based topological descriptors, COMPLEX INTELL SYST., 6, pp. 977–996.
[35] Mathew, S., Mordeson, J. N., Yang, H. L., (2019), Incidence cuts and connectivity in fuzzy incidence graphs, Iran. J. Fuzzy Syst., 16, pp. 31-43.
[36] Milovanovic, I., Milovanovic, E., MarjanMateji, C., (2021), On some mathematical properties of Sombor indices, Bull. Inter. Math., 11, pp. 341–353.
[37] Mordeson, J. N., Peng, C. S., (1994), Operation on fuzzy graphs, Inf. Sci., 793, pp. 159—170.
[38] Phanjoubam, C., Mawiong, S., Buhphang, A. M., (2022), On general Sombor index of graphs, Asian-Eur. J. Math., 16, pp. 751–757.
[39] Poulik, S., Ghorai, G.,(2021), Determination of journeys order based on graph's Wiener absolute index with bipolar fuzzy information, Inf. Sci., 545, pp. 608–619.

[40] Poulik, S., Ghorai, G.,(2020), Certain indices of graphs under bipolar fuzzy environment with applications, Soft Comput., 24, pp. 5119–5131.
[41] Poulik, S., Ghorai, G., (2021), Randic index of bipolar fuzzy graphs and its application in network systems, J. Comput. Appl. Math., 68, pp. 2317–2341.
[42] Pal, M., Samanta, S., Ghorai, G., (2020), Modern trends in fuzzy graph theory, Springer Sci. Rev., DOI:10.1007/978-981-15-8803-7.
[43] Rosenfeld, A., (1975), Fuzzy Sets and their Applications to Cognitive and Decision Processes, Academic Press, NY, USA pp. 77–95.
[44] Talali Ali Al-Hawary, Sumaya H. Al-Shalaldeh and Muhammad Akram, (2023), Certain Matrices and Energiesof Fuzzy Graphs, TWMS JPAM 14, No.1, pp.50-68.
[45] Randic, M. (1993), Novel molecular descriptor for structure property studies, Chem. Phys. Lett., 211, pp. 478–483.
[46] Sarwar, M., Akram, M., Shahzadi, S., (2021), Bipolar fuzzy soft information applied to hyper-graphs, Soft Comput., 25, pp. 3417–3439.
[47] Samanta, S., Pal, M., (2013), Telecommunication system based on fuzzy graphs, Telecommun. Manag., 3, pp. 1–6.
[48] Samanta, S., Pal, M., (2014), A new approach to social networks based on fuzzy graphs, Turk. J. Fuzzy Syst., 5 , pp 78–99.
[49] Siddique, S., Ahmad, U., Akram, M., (2021), A decision-making analysis with generalized m-Polar fuzzy graphs, J. Mult.-Valued Log. Soft Comput., 37, pp. 409–436.
[50] Wiener, H., (1947), Structural determination of paraffin boiling points, J. Am. Chem. Soc., 69, pp. 17-20.
[51] Zadeh, L. A., (1965), Fuzzy Sets, Inf. Control., 8, pp. 338–353.

**Biswajit Some** is a senior research scholar in the Department of Mathematics at the National Institute of Technology Durgapur, West Bengal, India. He has published two papers in reputed SCI-indexed journals and one paper in a SCOPUS-indexed journal. His research interests encompass fuzzy sets, fuzzy graphs, intuitionistic fuzzy sets, intuitionistic fuzzy graphs, picture fuzzy graphs, spherical fuzzy graphs, chemical graph theory, and spectral graph theory.

**Dr. Anita Pal** is an Associate Professor in the Department of Mathematics at the National Institute of Technology Durgapur, West Bengal, India. She has published more than 100 research articles in various international and national journals and has successfully guided 16 Ph.D. scholars. Her specialization includes computational and fuzzy graph theory, as well as chemical graph theory. She has written books that have been published in India for undergraduate and postgraduate students, as well as other professionals.